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ON WEAK SEPARATION PROPERTY FOR AFFINE FRACTAL FUNCTIONS

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Abstract. We show that a fractal affine function $f(x)$ defined by a system $\mathcal{S}$ which does not satisfy weak separation property is a quadratic function.

Keywords: self-similar set, fractal, weak separation property, affine FIF.

Introduction

Weak separation property (WSP), which was developed since 90-s in papers of C. Bandt [2], K.-S.Lau and S.-M.Ngai [5] and M. Zerner [9] remains one of the main tools of analyzing dimension problems. In recent years it proved to be useful for the study of geometric structure of self-similar sets [7] and rigidity of self-similar structures [8].

In this short note we apply this notion to the theory of affine fractal interpolation functions.

Standard definition (see [3], [4], [6]) of affine fractal function $f : [a, b] \to \mathbb{R}$ deals with a partition $a = x_0 < x_1 < ... < x_m = b$ of the interval $[a, b]$ and a system $\mathcal{S} = \{S_1, ..., S_m\}$ of affine transformations

$$S_i(x, y) = \begin{pmatrix} p_i & 0 \\ r_i & q_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_i \\ s_i \end{pmatrix}, \quad |p_i| < 1, |q_i| < 1,$$

which send vertical strip $a \le x \le b$ to vertical strips $L_i = \{(x, y) : x_{i-1} \le x \le x_i\}$. These strips divide the graph $\Gamma(f)$ to **non-overlapping** pieces $\Gamma_i = S_i(\Gamma(f)) = \Gamma(f) \cup L_i$ whose union is $\Gamma(f)$.

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But a more general approach must take into account the possibility of overlaps. For example, a system $\mathcal{S}$ consisting of 4 maps

$$S_1 : \begin{pmatrix} 1/5 & 0 \\ 1/5 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad S_2 : \begin{pmatrix} 1/3 & 0 \\ -1/5 & -1/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix},$$

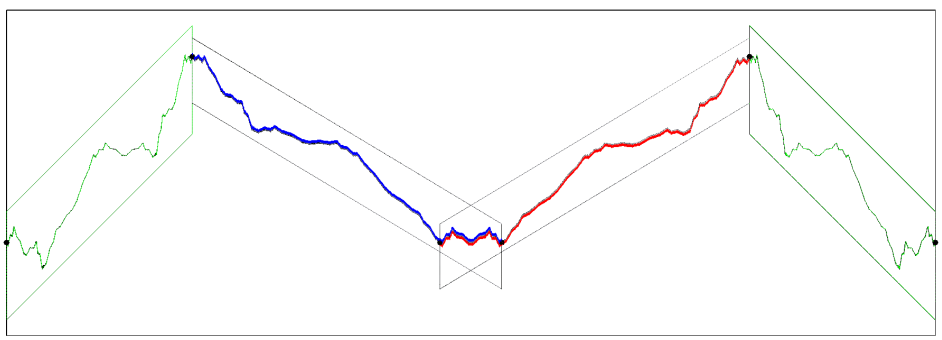
$$S_3 : \begin{pmatrix} 1/3 & 0 \\ 1/5 & -1/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7/15 \\ 0 \end{pmatrix}, \quad S_4 : \begin{pmatrix} 1/5 & 0 \\ -1/5 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix}$$

defines a self-affine function whose graph passes through the points

$$(0,0), \quad (1/5,1/5), \quad (7/15,0), \quad (8/15,0), \quad (4/5,1/5), \quad (1,0)$$

and has overlapping pieces

$$S_2(\Gamma) \cap S_3(\Gamma) = S_2S_4(\Gamma) = S_3S_1(\Gamma) = \Gamma\left(f|_{[\frac{7}{15},\frac{8}{15}]}\right).$$



The graph of $f(x)$: overlapping pieces are blue and red.

In view of the above argument, we use the following definition which allows the overlaps.

Definition 1. *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a system of affine maps*

$$S_i(x,y) = (p_i x + h_i, q_i y + r_i x + s_i), \quad |p_i|, |q_i| < 1.$$

A function $f(x)$ is affine fractal function *on $[a,b]$ defined by the system $\mathcal{S}$, if its graph $\Gamma(f) = \{(x, f(x)), x \in [a,b]\}$ is the attractor of the system $\mathcal{S}$.*

To formulate the main Theorem 1 we recall some definitions and notations.

We denote the projections of S_i to $\mathbb{R}$ by $S_i^\diamond(x) = p_i x + h_i$ and we denote $\mathcal{S}^\diamond = \{S_1^\diamond, \ldots, S_m^\diamond\}$. G denotes the semigroup generated by $\mathcal{S}$ and $G^\diamond$ denotes the semigroup generated by $\mathcal{S}^\diamond$.

Observe that each element $g_{\mathbf{i}} = S_{i_1}S_{i_2}\ldots S_{i_k}$ of the semigroup G is a map of the form $g_{\mathbf{i}}(x,y) = (p_{\mathbf{i}}x + h_{\mathbf{i}}, q_{\mathbf{i}}y + r_{\mathbf{i}}y + s_{\mathbf{i}})$; $|p_{\mathbf{i}}|, |q_{\mathbf{i}}| < 1$ where $p_{\mathbf{i}} = p_{i_1}p_{i_2}\ldots p_{i_k}$, $q_{\mathbf{i}} = q_{i_1}q_{i_2}\ldots q_{i_k}$, while $g_{\mathbf{i}}^\diamond(x) = S_{i_1}^\diamond S_{i_2}^\diamond \ldots S_{i_k}^\diamond(x) = (p_{\mathbf{i}}x + h_{\mathbf{i}})$.

We define *associated families* $\mathcal{F} = G^{-1} \circ G$ and $\mathcal{F}^\diamond = G^{\diamond-1} \circ G^\diamond$ for the system $\mathcal{S}$ (resp. $\mathcal{S}^\diamond$). Each element of the family $\mathcal{F}$ is a composition $g = g_{\mathbf{j}}^{-1}g_{\mathbf{i}}$ and also has the form $g(x, y) = (px + h, qy + rx + s)$, while its projection $g^\diamond = g_{\mathbf{j}}^{\diamond -1} g_{\mathbf{i}}^\diamond$ satisfies $g^\diamond(x) = px + h$.

Definition 2. *The system* $\mathcal{S}$ *satisfies* weak separation property (WSP) *if* Id *is an isolated point in the associated family* $\mathcal{F}$.

So, if the system $\mathcal{S}$ does not satisfy the weak separation property, there is a sequence $g_n \in \mathcal{F}$ which converges to Id.

1. The Main Theorem

In this paper we prove the following

Theorem 1. *Let* $f(x)$ *be the affine fractal function defined by a system* $\mathcal{S}$ *on the segment* $[a, b]$. *If* $\mathcal{S}^\diamond$ *does not satisfy weak separation property, then the graph* $\Gamma(f)$ *is a parabolic arc.*

First of all, it follows from the definition that each fractal affine function is continuous, because its graph $\Gamma(f)$ is a compact set.

Second, a remarkable property of the maps $g \in \mathcal{F}$ is that these maps move the points of $\Gamma(f)$ along $\Gamma(f)$.

Lemma 1. *If* $g \in \mathcal{F}$ *and for some* $x \in [a, b]$, $g^\diamond(x) \in [a, b]$, *then*

$$g(x, f(x)) = (g^\diamond(x), f(g^\diamond(x)).$$

Proof. Let $g \in \mathcal{F}$, so $g = g_{\mathbf{j}}^{-1} g_{\mathbf{i}}$. If $(x, y) \in \Gamma(f)$, then $g_{\mathbf{i}}(x, y) \in \Gamma(f)$. Suppose $(u, v) \in \Gamma(f)$ and $g_{\mathbf{j}}^\diamond(u) = g_{\mathbf{i}}^\diamond(x)$. Since $g_{\mathbf{j}}(u, v) \in \Gamma(f)$, $g_{\mathbf{j}}(u, v) = g_{\mathbf{i}}(x, y)$, therefore $g_{\mathbf{j}}^{-1} g_{\mathbf{i}}(x, y) = (u, v)$, so $g(x, f(x)) = (g^\diamond(x), f(g^\diamond(x))$. □

These facts imply that weak separation property holds for both systems $\mathcal{S}^\diamond$ and $\mathcal{S}$ simultaneously.

Lemma 2. *Let* $f(x)$ *be the affine fractal function defined by a system* $\mathcal{S}$ *on the segment* $[a, b]$ *whose graph is not a straight line segment. The system* $\mathcal{S}$ *satisfies WSP iff* $\mathcal{S}^\diamond$ *satisfies WSP.*

Proof. Suppose that WSP does not hold for $\mathcal{F}^\diamond$. Take three points (x_i, y_i), $i = 1, 2, 3$, on $\Gamma(f)$ which do not lie on a line. If $g_n^\diamond \to \mathrm{Id}$ then for each i, $g_n^\diamond(x_i) \to x_i$. Since f is continuous, $g_n(x_i, y_i) \to (x_i, y_i)$. This means that g_n converges to Id and WSP does not hold for $\mathcal{F}$.

Suppose now that WSP does not hold for $\mathcal{F}$ and there is a sequence $g_n \in \mathcal{F}$ which converges to Id. Consider the coefficients of $g_n(x, y) = (p_n x + h_n, q_n y + r_n x + s_n)$. The coefficients p_n and q_n converge to 1, while h_n, r_n and s_n converge to 0. Therefore $g_n^\diamond(x) = p_n x + h_n$ also converges to Id. □

Lemma 3. *Suppose* U *is a family of functions* $\varphi(x) \in C^3[a, b]$, *satisfying inequality* $|\varphi(x)| \le M$. *If for any* $\varphi(x) \in U$, $\varphi''(x)$ *and* $\varphi'''(x)$ *are monotonous and do not change the sign on* $[a, b]$, *then for any segment* $[a', b'] \subset (a, b)$, *the family* $U' = \{\varphi|_{[a', b']}, \varphi \in U\}$ *is bounded in* $C^3([a', b'])$.

Proof. Without loss of generality, we suppose $[a, b] = [0, 1]$ and $\varphi''(x) > 0$ on $[0, 1]$. Take some $\lambda \in (2^{1/3}, 1)$.

Since $\varphi(1) \le M$ and $\varphi(\lambda) \ge -M$, $\varphi'(\lambda) < \dfrac{2M}{1-\lambda}$. Similarly, we get

$$\varphi'(1-\lambda) > \frac{-2M}{1-\lambda}.$$

So $\varphi'(x) < \left|\dfrac{2M}{1-\lambda}\right|$ on $[1-\lambda, \lambda]$.

Repeating the same step for φ' we get $0 < \varphi''(\lambda^2) < \dfrac{4M}{\lambda(1-\lambda)^2}$ if φ'' increases and $\varphi''(1-\lambda^2) > \dfrac{4M}{\lambda(1-\lambda)^2}$ if φ'' decreases, so $\varphi''(x) < \dfrac{4M}{\lambda(1-\lambda)^2}$ on $[1-\lambda^2, \lambda^2]$.

The same way, we have $|\varphi'''(x)| < \dfrac{8M}{\lambda^3(1-\lambda)^3}$ on $[1-\lambda^3, \lambda^3]$. Taking such λ, that $[a', b'] \subset [1-\lambda^3, \lambda^3]$, we obtain the statement for the segment $[a', b']$. □

Lemma 4. *Let $g \in \mathcal{F}$ and $\mathrm{fix}(g^\diamond) \notin [a,b]$. Suppose that*
(i) if $x_1, x_2 \in [a,b]$ and $|x_1 - x_2| < \delta$, then $\|(x_1, f(x_1)) - (x_2, f(x_2))\| < \varepsilon$,
(ii) $\|g(x,y) - (x,y)\| < \delta$ for any point $(x,y) \in \Gamma(f)$.
Then for some $M \in \mathbb{N}$ either

$$\{g^n(a, f(a)), n = 0, \ldots, M\} \quad \text{or} \quad \{g^n(b, f(b)), n = 0, \ldots, M\}$$

is an ε-net in $\Gamma(f)$.

Proof. The condition (i) implies that if $\{x_1, \ldots, x_k\}$ is a δ-net in $[a,b]$, then

$$\{(x_1, f(x_1)), \ldots, (x_k, f(x_k))\}$$

is an ε-net in $\Gamma(f)$. So we have to show that $g^{\diamond n}(a)$ or $g^{\diamond n}(b)$ form a δ-net in $[a,b]$.

Since $\mathrm{fix}(g^\diamond) \notin [a,b]$, we have either $g^\diamond(x) > x$ for any $x \in [a,b]$ or $g^\diamond(x) < x$ for any $x \in [a,b]$.

Suppose $g^\diamond(a) > a$. Then for any point $x \in [a,b]$, $g^\diamond(x) > x$ and $g^\diamond(x) - x < \delta$. Since the limit point of the sequence $g^{\diamond n}(a)$ is outside $[a,b]$, there is such $M \in \mathbb{N}$ for which $g^{\diamond M}(a) < b < g^{\diamond M+1}(a)$, so for any $n = 1, \ldots, M$, $g^{\diamond n}(a) - g^{\diamond n-1}(a) < \delta$ and $b - g^{\diamond M}(a) < \delta$. Therefore $\{g^n(a, f(a)), n = 0, 1, \ldots, M\}$ is an ε-net in $\Gamma(f)$. The proof in the case $g^\diamond(b) < b$ is similar. □

Lemma 5. *Suppose $g(x,y) \in \mathcal{F}$, $\mathrm{fix(g^\diamond)} \notin \mathrm{[a,b]}$ and $g^\diamond(x) > x$ on $[a,b]$. Let $g^{\diamond T}(a) = b$. Then the set $\{g^t(a, f(a)), t \in [0,T]\}$ is a graph of one of the following functions on $[a,b]$:*
1. $y = Ax^2 + Bx + C$,
2. $y = Ax + Be^{Kx} + C$,
3. $y = Ax + B(\log(x - C)) + D$, $C \notin [a,b]$,
4. $y = Ax + B(x - C)^K + D$, $C \notin [a,b]$,
5. $y = A(x - C)\log(x - C) + Dx + E$, $C \notin [a,b]$.

Proof. It is sufficient to check the statement in the case $a = 0$, $b = 1$, $f(0) = 0$ and $p > 1$. Since g is close to Id, p and q are close to 1 and therefore positive.

The five types of functions arise from direct solution of recurrence equations.

1. If $g(x,y) = (x + h, y + rx + s)$, then the points $g^n(0,0)$ lie on a parabola $y = Ax^2 + Bx$, where $A = \dfrac{r}{2h}$ and $B = \dfrac{2s - hr}{2h}$.

2. If $g(x,y)=(x+h, qy+rx+s)$, $q\neq 1$, then the points $g^n(0,0)$ lie on a graph of a function $y=Ax+B(e^{Kx}-1)$, where

$$K=\frac{\log q}{h},\quad A=\frac{r}{q-1},\quad B=\frac{hr+(q-1)s}{(q-1)^2}.$$

3. If $g(x,y)=(px+h, y+rx+s)$, then the points $g^n(0,0)$ lie on a graph of $y=Ax+B(\log(1+x/C))$, where

$$C=\frac{h}{p-1},\quad A=\frac{r}{p-1},\quad B=\frac{hr+(1-p)s}{(1-p)\log p}.$$

4. If $g(x,y)=(px+h, qy+rx+s)$, then the points $g^n(0,0)$ lie on a graph of a function $y=Ax+B(x/C+1)^K-B$, where

$$A=\frac{r}{p-q},\quad C=\frac{h}{p-1},\quad B=\frac{hr+s(q-p)}{(q-1)(q-p)},\quad K=\frac{\log q}{\log p}.$$

5. If $g(x,y)=(px+h, py+rx+s)$, then the points $g^n(0,0)$ lie on a graph of a function $y=A(x/C+1)\log(x/C+1)+Bx$, where

$$C=\frac{h}{p-1},\quad A=\frac{rC}{p\log p},\quad B=\frac{Cr-s}{C-Cp}.$$

Applying to x coordinate a linear transformation which sends [a,b] to [0,1], we get the formulas 1–5 of the statement. □

Proof of the Theorem 1. Take such sequence $g_n\to \mathrm{Id}$, $g_n\in\mathcal{F}$ and such segment $[a_1,b_1]\subset(a,b)$, that for any n, $\mathrm{fix}(g_n^\diamond)\notin[a_1,b_1]$.

Since g_n^{-1} also converge to Id, we may suppose that for any n, $p_n\geq 1$.

Without loss of generality we may suppose that for any n, $g_n^\diamond(a_1)>a_1$. Let T_n be such number, that $g_n^{\diamond T_n}(a_1)=b_1$. Each curve $\{g_n^t(a_1,f(a_1)), t\in[0,T_n]\}$ is a graph of a function $\varphi_n(x)$ on the segment $[a_1,b_1]$.

It follows from Lemma 4 that $\varphi_n(x)$ uniformly converges to $f(x)$ on $[a_1,b_1]$.

By Lemma 5, each of these functions is of one of 5 types, indicated by the Lemma 5. Therefore the functions $\varphi_n(x)$ have monotonous derivatives $\varphi_n''(x)$, $\varphi_n'''(x)$, which do not change their sign on $[a_1,b_1]$. By Lemma 3, for any $[a_2,b_2]\in(a_1,b_1)$, the family $\{\varphi_n(x)|_{[a_2,b_2]}\}$ is a bounded subset of $C^3([a_2,b_2])$. Therefore some subsequence of $\varphi_n(x)$ converges in $C^2([a_2,b_2])$, which implies that $f(x)$ is twice differentiable on $[a_2,b_2]$. This means that $f(x)\in C^2([a,b])$. As it was proved in [1, Theorem 3] this implies that $\Gamma(f)$ is a parabolic arc. □

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