AN EXPLICIT VOLUME FORMULA FOR THE LINK $7^2_3(\alpha, \alpha)$ CONE-MANIFOLDS

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Abstract. We calculate the volume of the $7^2_3$ link cone-manifolds using the Schl"afli formula. As an application, we give the volume of the cyclic coverings branched over the link.

Keywords: hyperbolic orbifold, hyperbolic cone-manifold, volume, link $7^2_3$, orbifold covering, Riley-Mednykh polynomial.

1. Introduction

Let us denote the link complement of $7^2_3$ in Rolfsen’s link table by $X$. Note that it is a hyperbolic knot. Hence by Mostow-Prasad rigidity theorem, $X$ has a unique hyperbolic structure. Let $\rho_\infty$ be the holonomy representation from $\pi_1(X)$ to $\text{PSL}(2, \mathbb{C})$ and denote $\rho_\infty(\pi_1(X))$ by $\Gamma$, a Kleinian group. $X$ is a $(\text{PSL}(2, \mathbb{C}), \mathbb{H}^3)$-manifold and can be identified with $\mathbb{H}^3/\Gamma$. Thurston’s orbifold theorem guarantees an orbifold, $X(\alpha) = X(\alpha, \alpha)$, with underlying space $S^3$ and with the link $7^2_3$ as the singular locus of the cone-angle $\alpha = 2\pi/k$ for some nonzero integer $k$, can be identified with $\mathbb{H}^3/\Gamma'$ for some $\Gamma' \in \text{PSL}(2, \mathbb{C})$; the hyperbolic structure of $X$ is deformed to the hyperbolic structure of $X(\alpha)$. For the intermediate angles whose multiples are not $2\pi$ and not bigger than $\pi$, Kojima [10] showed that the hyperbolic structure of $X(\alpha)$ can be obtained uniquely by deforming nearby orbifold structures. Note that there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi]$ for the link $7^2_3$ such that $X(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [8, 10, 19, 20]. For further knowledge of cone-manifolds a reader can consult [1, 7].
Even though we have wide discussions on orbifolds, it seems to us we have a little in regard to cone-manifolds. Explicit volume formulae for hyperbolic cone-manifolds of knots and links are known a little. The volume formulae for hyperbolic cone-manifolds of the knot $4_1 [8, 10, 11, 15]$, the knot $5_2 [13]$, the link $5_2^1 [16]$, and the link $6_2^3 [2]$ have been computed. In [9] a method of calculating the volumes of two-bridge knot cone-manifolds was introduced but without explicit formulae. In [6, 7], explicit volume formulae of cone-manifolds for the hyperbolic twist knot and for the knot with Conway notation $C(2n, 3)$ are computed. Similar methods are used for computing Chern-Simons invariants of orbifolds for the twist knot and $C(2n, 3)$ knot in [4, 5].

The main purpose of the paper is to find an explicit and efficient volume formula of hyperbolic cone-manifolds for the link $7_2^3$. The following theorem gives the volume formula for $X(\alpha)$.

**Theorem 1.** Let $X(\alpha), 0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space $S^3$ and with singular set the link $7_2^3$ of cone-angle $\alpha$. $X(0)$ denotes $X$. Then the volume of $X(\alpha)$ is given by the following formula

$$\text{Vol}(X(\alpha)) = \int_0^\pi 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,$$

where for $A = \cot \frac{\alpha}{2}$, $V$ ($\text{Re}(V) \leq 0$ and $\text{Im}(V) \geq 0$ is the largest) is a zero of the Riley-Mednykh polynomial $P = P(V, A)$ for the link $7_2^3$ given below.

$$P = 8V^5 + 8A^2V^4 + (8A^4 + 16A^2 - 8)V^3 + (4A^6 + 8A^4 - 12A^2)V^2$$
$$+ (4A^8 + 4A^6 - 2A^4 - 12A^2 + 1)V - 4A^6 - 8A^4 + 4A^2.$$

The following corollary gives the hyperbolic volume of the $k$-fold strictly-cyclic covering [12, 18] over the link $7_2^3$, $M_k(X)$, for $k \geq 3$.

**Corollary 1.** The volume of $M_k(X)$ is given by the following formula

$$\text{Vol}(M_k(X)) = k \int_0^\pi 2 \log \left| \frac{A - iV}{A + iV} \right| d\alpha,$$

where for $A = \cot \frac{\alpha}{2}$, $V$ ($\text{Re}(V) \leq 0$ and $\text{Im}(V) \geq 0$ is the largest) is a zero of the Riley-Mednykh polynomial $P = P(V, A)$ for the link $7_2^3$.

In Section 2, we present the fundamental group $\pi_1(X)$ of $X$ with slope $9/16$. In Section 3, we give the defining equation of the representation variety of $\pi_1(X)$. In Section 4, we compute the longitude of the link $7_2^3$ using the Pythagorean theorem. And in Section 5, we give the Theorem 1 using the Schlafli formula.

2. Link $7_2^3$
Figure 1. Link \(7^2_4\) in Rolfsen’s link table.

Link \(7^2_4\) is presented in Figure 1. It is the same as \(W_3\) from [2]. The slope of this link is \(7/16\). The link with slope \(9/16\) is the mirror of the link \(7^2_4\). Since the volume of the link with slope \(7/16\) is the same as the volume of link with slope \(9/16\), in the rest of the paper, the link with slope \(9/16\) is used.

The following fundamental group of \(X\) is stated in [2] with slope \(7/16\).

**Proposition 1.** \(\pi_1(X) = \langle s, t \mid s w s^{-1} w^{-1} = 1 \rangle\), where \(w = s^{-1}[s, t]^2[s, t^{-1}]^2\).

3. \((\text{PSL}(2, \mathbb{C}), \mathbb{H}^3)\) structure of \(X(\alpha)\)

Let \(R = \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))\). Given a set of generators, \(s, t\), of the fundamental group for \(\pi_1(X)\), we define a set \(R(\pi_1(X)) \subset \text{SL}(2, \mathbb{C})^2 \subset \mathbb{C}^8\) to be the set of all points \((h(s), h(t))\), where \(h\) is a representation of \(\pi_1(X)\) into \(\text{SL}(2, \mathbb{C})\). Since the defining relation of \(\pi_1(X)\) gives the defining equation of \(R(\pi_1(X))\) [21], \(R(\pi_1(X))\) is an affine algebraic set in \(\mathbb{C}^8\). \(R(\pi_1(X))\) is well-defined up to isomorphisms which arise from changing the set of generators. We say elements in \(R\) which differ by conjugations in \(\text{SL}(2, \mathbb{C})\) are equivalent. A point on the variety gives the \((\text{PSL}(2, \mathbb{C}), \mathbb{H}^3)\) structure of \(X(\alpha)\).

Let

\[
h(s) = \begin{bmatrix}
\cos \frac{\alpha}{2} & ie^{\frac{\alpha}{2}} \sin \frac{\alpha}{2} \\
-ie^{-\frac{\alpha}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2}
\end{bmatrix}, \quad h(t) = \begin{bmatrix}
\cos \frac{\alpha}{2} & ie^{-\frac{\alpha}{2}} \sin \frac{\alpha}{2} \\
-ie^{\frac{\alpha}{2}} \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2}
\end{bmatrix}.
\]

Then \(h\) becomes a representation if and only if \(A = \cot \frac{\alpha}{2}\) and \(V = \cosh \rho\) satisfies a polynomial equation [14, 21]. We call the defining polynomial of the algebraic set \(\{(V, A)\}\) as the *Riley-Mednykh polynomial* for the link \(7^2_4\). Thoughout the paper, \(h\) can be sometimes any representation and sometimes the unique hyperbolic representation.

Given the fundamental group of \(X\),

\[
\pi_1(X) = \langle s, t \mid s w s^{-1} w^{-1} = 1 \rangle,
\]

where \(w = s^{-1}[s, t]^2[s, t^{-1}]^2\), let \(S = h(s), T = h(t)\) and \(W = h(w)\). Then the trace of \(S\) and the trace of \(T\) are both \(2 \cos \frac{\alpha}{2}\).

**Lemma 1.** For \(n \in \text{SL}(2, \mathbb{C})\) which satisfies \(nS = S^{-1}n, nT = T^{-1}n,\) and \(n^2 = -I\),

\[
SW S^{-1} W^{-1} = -(SWn)^2.
\]

**Proof.**

\[
(SWn)^2 = SWnSWn = SW S^{-1} n (SST^{-1}S^{-1}T^{-1}S^{-1}T) (ST^{-1}S^{-1}T)^2 n
\]

\[
= SW S^{-1} (S S^{-1} T^{-1} S T^{-1}) (S^{-1} T S T^{-1})^2 n^2 = -SW S^{-1} W^{-1}.
\]

\(\Box\)

From the structure of the algebraic set of \(R(\pi_1(X))\) with coordinates \(h(s)\) and \(h(t)\) we have the defining equation of \(R(\pi_1(X))\). The following theorem is stated in [2, Proposition 4] with slope \(7/16\).
Theorem 2. $h$ is a representation of $\pi_1(X)$ if $V$ is a root of the following Riley-Mednykh polynomial $P = P(V, A)$ which is given below.

$$
$$

Proof. Note that $SWS^{-1}W^{-1} = I$, which gives the defining equations of $R(\pi_1(X))$, is equivalent to $(SWn)^2 = -I$ in $\text{SL}(2, \mathbb{C})$ by Lemma 1 and $(SWn)^2 = -I$ in $\text{SL}(2, \mathbb{C})$ is equivalent to $\text{tr}(SWn) = 0$.

We can find two $n$’s in $\text{SL}(2, \mathbb{C})$ which satisfies $nS = S^{-1}n$ and $n^2 = -I$ by direct computations. The existence and the uniqueness of the isometry (the involution) which is represented by $n$ are shown in [3, p. 46]. Since two $n$’s give the same element in $\text{PSL}(2, \mathbb{C})$, we use one of them. Hence, we may assume

$$
n = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},
$$

$$
S = \begin{bmatrix} \cos \tfrac{\alpha}{2} & i e^{\frac{\pi}{2}} \sin \tfrac{\alpha}{2} \\ i e^{-\frac{\pi}{2}} \sin \tfrac{\alpha}{2} & \cos \tfrac{\alpha}{2} \end{bmatrix}, \quad T = \begin{bmatrix} \cos \tfrac{\alpha}{2} & i e^{-\frac{\pi}{2}} \sin \tfrac{\alpha}{2} \\ i e^{\frac{\pi}{2}} \sin \tfrac{\alpha}{2} & \cos \tfrac{\alpha}{2} \end{bmatrix}.
$$

Recall that $P$ is the defining polynomial of the algebraic set $\{(V, A)\}$ and the defining polynomial of $R(\pi_1(X))$ corresponding to our choice of $h(s)$ and $h(t)$. By direct computation $P$ is a factor of $\text{tr}(SWn) = -4i \sin \rho(2V^2 + A^4 + 2A^2 - 1)P$. As in [2], $P$ can not be $\sinh \rho$ or have only real roots. Also, $P$ can have only purely imaginary roots similarly. $P$ in the theorem is the only factor of $\text{tr}(SWn)$ which is different from $\sinh \rho$ and has roots which are not real or purely imaginary. $P$ is the Riley-Mednykh polynomial.

4. Longitude

Let $l_s = ws$ and $l_t = (t^{-1}|t, s|^2[t, s^{-1}]^2)t$. Then $l_s$ and $l_t$ are the longitudes which are null-homologous in $X$. Let $L_S = h(l_s)$ and Let $L_T = h(l_t)$.

Lemma 2. $\text{tr}(S^{-1}L_T) = \text{tr}(S)$ and $\text{tr}(T^{-1}L_S) = \text{tr}(T)$.

Proof. Since

$$
S^{-1}L_T = S^{-1}(T^{-1}(TST^{-1}S^{-1}TST^{-1}S^{-1} \cdot TS^{-1}T^{-1}STT^{-1}S^{-1}T^{-1}S^{-1}T)T)
$$

$$
= (T^{-1}S^{-1}TST^{-1}S^{-1}T)(S^{-1})(T^{-1}S^{-1}TST^{-1}S^{-1}T^{-1}),
$$

$$
\text{tr}(S^{-1}L_T) = \text{tr}(S^{-1}) = \text{tr}(S).
$$

The second statement can be obtained in a similar way.

Definition 1. The complex length of the longitude $l$ ($l_s$ or $l_t$) of the link $\mathcal{T}_3^3$ is the complex number $\gamma_\alpha$ modulo $4\pi \mathbb{Z}$ satisfying

$$
\text{tr}(h(l)) = 2 \cosh \frac{\gamma_\alpha}{2}.
$$

Note that $l_\alpha = |\text{Re}(\gamma_\alpha)|$ is the real length of the longitude of the cone-manifold $X(\alpha)$. ”
By sending common fixed points of $T$ and $L_T = h(l_t)$ to 0 and $\infty$, we have

$$T = \begin{bmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{bmatrix}, \quad L_T = \begin{bmatrix} e^{\frac{\gamma\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma\alpha}{2}} \end{bmatrix},$$

and the following normalized line matrices of $T$ (resp. $L_T$) which share the fixed points with $T$ (resp. $L_T$).

$$l(T) = \frac{T - T^{-1}}{2i \sinh \frac{\alpha}{2}}$$

$$= \frac{1}{i(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}})} \begin{bmatrix} e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} - e^{\frac{i\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

$$l(L_T) = \frac{L_T - L_T^{-1}}{2i \sinh \frac{\gamma\alpha}{2}}$$

$$= \frac{1}{i(e^{\frac{i\gamma\alpha}{2}} - e^{-\frac{i\gamma\alpha}{2}})} \begin{bmatrix} e^{\frac{\gamma\alpha}{2}} - e^{-\frac{\gamma\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma\alpha}{2}} - e^{\frac{\gamma\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

which give the orientations of axes of $T$ and $L_T$.

Now, we are ready to prove the following theorem which gives Theorem 4. Recall that $\gamma_\alpha$ modulo $4\pi\mathbb{Z}$ is the complex length of the longitude $l_s$ or $l_t$ of $X(\alpha)$. The following theorem is a particular case of Proposition 5 from [2].

**Theorem 3. (Pythagorean Theorem) [2]** Let $X(\alpha)$ be a hyperbolic cone-manifold and let $\rho$ be the complex distance between the oriented axes $S$ and $T$. Then we have

$$i \cosh \rho = \cot \frac{\alpha}{2} \coth \left( \frac{\gamma_\alpha}{4} \right).$$
Proof.
\[
\cosh \rho = \frac{-\text{tr}(l(S)l(T))}{2} = \frac{-\text{tr}(l(S)l(L_T))}{2} = \frac{\text{tr}((S-S^{-1})(L_T-L_T^{-1}))}{8 \sinh \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(SL_T-S^{-1}L_T-1+(L_T S)^{-1})}{8 \sinh \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = \frac{2(\text{tr}(S L_T)-\text{tr}(S^{-1}L_T))}{8 \sinh \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(SL_T)-2\text{tr}(S^{-1}L_T)}{4 \sinh \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(S)(L_T)-2\text{tr}(S)}{4 \sinh \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = \frac{\text{tr}(S)(L_T)-2}{4 \sinh \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = \frac{2 \cos \frac{\alpha}{2} (2 \cosh \frac{\gamma}{4} - 2)}{4i \sin \frac{\alpha}{2} \sinh \frac{\gamma}{2}} = -i \cot \frac{\alpha}{2} \tanh \left( \frac{\gamma}{4} \right).
\]
where the first equality comes from [3, p. 68], the sixth equality comes from the Cayley-Hamilton theorem, and the seventh equality comes from Lemma 2. Therefore, we have

\[
i \cosh \rho = \cot \frac{\alpha}{2} \coth \left( \frac{\gamma}{4} \right).
\]

\[\square\]

Pythagorean theorem 3 gives the following theorem which relates the eigenvalues of \(h(l)\) and \(V = \cosh \rho\) for \(A = \cot \frac{\alpha}{2}\).

**Theorem 4.** Recall that \(l\) is the longitude. By conjugating if necessary, we may assume \(h(l)\) is upper triangular. Let \(L = h(l)_{11}\). Let \(A = \cot \frac{\alpha}{2}\). Then the following formulae show that there is a one to one correspondence between the eigenvalues of \(h(l)\) and \(V = \cosh \rho\):

\[
iV = A \frac{L - 1}{L + 1} \quad \text{and} \quad L = \frac{A - iV}{A + iV}.
\]
Proof. By Theorem 3,

\[ iV = i \cosh \rho \]
\[ = \cot \frac{\alpha}{2} \tanh \left( \frac{7\alpha}{4} \right) \]
\[ = \cot \frac{\alpha}{2} \sinh \left( \frac{7\alpha}{4} \right) \cosh \left( \frac{7\alpha}{4} \right) \]
\[ = \cot \frac{e^{\frac{7\alpha}{4}} - e^{-\frac{7\alpha}{4}}}{2 e^{\frac{7\alpha}{4}} + e^{-\frac{7\alpha}{4}}} \]
\[ = \cot \frac{e^{\frac{7\alpha}{4}} - 1}{2 e^{\frac{7\alpha}{4}} + 1} \]
\[ = A \frac{L - 1}{L + 1}. \]

If we solve the above equation,

\[ iV = A \frac{L - 1}{L + 1}, \]

for \( L \), we have

\[ L = \frac{A - iV}{A + iV}. \]

□

5. Proof of Theorem 1

According to [8, 10, 19, 20], there exists an angle \( \alpha_0 \in \left[ \frac{2\pi}{3}, \pi \right) \) such that \( X(\alpha) \) is hyperbolic for \( \alpha \in (0, \alpha_0) \), Euclidean for \( \alpha = \alpha_0 \), and spherical for \( \alpha \in (\alpha_0, \pi) \). Denote by \( D(X(\alpha)) \) the discriminant of \( P(V, A) \) over \( V \). Then \( \alpha_0 \) is the only zero of \( D(X(\alpha)) \) in \( \left[ \frac{2\pi}{3}, \pi \right) \).

From Theorem 4, we have the following equality,

\[ (1) \quad |L|^2 = \frac{|A - iV|^2}{A + iV} = \frac{|A|^2 + |V|^2 + 2A \text{Im} V}{|A|^2 + |V|^2 - 2A \text{Im} V}. \]

For the volume, we choose \( L \) with \( |L| \geq 1 \) and hence we have \( \text{Im}(V) \geq 0 \) by Equality (1). The component of \( V \) with \( \text{Im}(V) \geq 0 \) which becomes real at \( \alpha_0 \) has negative real part. On the geometric component which gives the unique hyperbolic structure, we have the volume of a hyperbolic cone-manifold \( X(\alpha) \) for \( 0 \leq \alpha < \alpha_0 \):

\[ \text{Vol}(X(\alpha)) = - \int_{\alpha_0}^{\alpha} 2 \left( \frac{1_{\alpha}}{2} \right) d\alpha \]
\[ = - \int_{\alpha_0}^{\alpha} 2 \log |L| d\alpha \]
\[ = - \int_{\alpha_0}^{\alpha} 2 \log |L| d\alpha \]
\[ = \int_{\alpha}^{\pi} 2 \log |A - iV| d\alpha, \]
where the first equality comes from the Schl"{a}fli formula for cone-manifolds (Theorem 3.20 of [1]), the second equality comes from the fact that $l_\alpha = |Re(\gamma_\alpha)|$ is the real length of the one longitude of $X(\alpha)$, the third equality comes from the fact that $\log |L| = 0$ for $\alpha_0 < \alpha \leq \pi$ by Equality (1) since all $V$’s are real for $\alpha_0 < \alpha \leq \pi$, and $\alpha_0 \in \left(\frac{2\pi}{3}, \pi\right)$ is the zero of the discriminant $D(X(\alpha))$. Numerical calculations give us the following value for $\alpha_0$: $\alpha_0 \approx 2.83003$.

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