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AN EXPLICIT VOLUME FORMULA FOR THE LINK $7_3^2(\alpha,\alpha)$ CONE-MANIFOLDS

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Abstract. We calculate the volume of the 7_3^2 link cone-manifolds using the Schläfli formula. As an application, we give the volume of the cyclic coverings branched over the link.

Keywords: hyperbolic orbifold, hyperbolic cone-manifold, volume, link 7_3^2, orbifold covering, Riley-Mednykh polynomial.

1. Introduction

Let us denote the link complement of 7_3^2 in Rolfsen's link table by X. Note that it is a hyperbolic knot. Hence by Mostow-Prasad rigidity theorem, X has a unique hyperbolic structure. Let ρ_∞ be the holonomy representation from $\pi_1(X)$ to $\mathrm{PSL}(2,\mathbb{C})$ and denote $\rho_\infty(\pi_1(X))$ by Γ, a Kleinian group. X is a $(\mathrm{PSL}(2,\mathbb{C}),\mathbb{H}^3)$-manifold and can be identified with $\mathbb{H}^3/\Gamma$. Thurston's orbifold theorem guarantees an orbifold, $X(\alpha)=X(\alpha,\alpha)$, with underlying space S^3 and with the link 7_3^2 as the singular locus of the cone-angle $\alpha=2\pi/k$ for some nonzero integer k, can be identified with $\mathbb{H}^3/\Gamma'$ for some $\Gamma'\in\mathrm{PSL}(2,\mathbb{C})$; the hyperbolic structure of X is deformed to the hyperbolic structure of $X(\alpha)$. For the intermediate angles whose multiples are not 2π and not bigger than π, Kojima [10] showed that the hyperbolic structure of $X(\alpha)$ can be obtained uniquely by deforming nearby orbifold structures. Note that there exists an angle $\alpha_0\in[\frac{2\pi}{3},\pi)$ for the link 7_3^2 such that $X(\alpha)$ is hyperbolic for $\alpha\in(0,\alpha_0)$, Euclidean for $\alpha=\alpha_0$, and spherical for $\alpha\in(\alpha_0,\pi]$ [8, 10, 19, 20]. For further knowledge of cone-manifolds a reader can consult [1, 7].

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Even though we have wide discussions on orbifolds, it seems to us we have a little in regard to cone-manifolds. Explicit volume formulae for hyperbolic cone-manifolds of knots and links are known a little. The volume formulae for hyperbolic cone-manifolds of the knot 4_1 [8, 10, 11, 15], the knot 5_2 [13], the link 5_1^2 [16], the link 6_2^2 [17], and the link 6_3^2 [2] have been computed. In [9] a method of calculating the volumes of two-bridge knot cone-manifolds was introduced but without explicit formulae. In [6, 7], explicit volume formulae of cone-manifolds for the hyperbolic twist knot and for the knot with Conway notation $C(2n,3)$ are computed. Similar methods are used for computing Chern-Simons invariants of orbifolds for the twist knot and $C(2n,3)$ knot in [4, 5].

The main purpose of the paper is to find an explicit and efficient volume formula of hyperbolic cone-manifolds for the link 7_3^2. The following theorem gives the volume formula for $X(\alpha)$.

Theorem 1. *Let $X(\alpha)$, $0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space S^3 and with singular set the link 7_3^2 of cone-angle α. $X(0)$ denotes X. Then the volume of $X(\alpha)$ is given by the following formula*

$$\mathrm{Vol}\,(X(\alpha)) = \int_{\alpha}^{\pi} 2\log\left|\frac{A - iV}{A + iV}\right|\, d\alpha,$$

where for $A = \cot\frac{\alpha}{2}$, V ($\mathrm{Re}(V) \leq 0$ and $\mathrm{Im}(V) \geq 0$ is the largest) is a zero of the Riley-Mednykh polynomial $P = P(V,A)$ for the link 7_3^2 given below.

$$\begin{aligned} P = {} & 8V^5 + 8A^2V^4 + \left(8A^4 + 16A^2 - 8\right)V^3 + \left(4A^6 + 8A^4 - 12A^2\right)V^2 \\ & + \left(A^8 + 4A^6 - 2A^4 - 12A^2 + 1\right)V - 4A^6 - 8A^4 + 4A^2. \end{aligned}$$

The following corollary gives the hyperbolic volume of the k-fold strictly-cyclic covering [12, 18] over the link 7_3^2, $M_k(X)$, for $k \geq 3$.

Corollary 1. *The volume of $M_k(X)$ is given by the following formula*

$$\mathrm{Vol}\,(M_k(X))) = k\int_{\frac{2\pi}{k}}^{\pi} 2\log\left|\frac{A - iV}{A + iV}\right|\, d\alpha,$$

where for $A = \cot\frac{\alpha}{2}$, V ($\mathrm{Re}(V) \leq 0$ and $\mathrm{Im}(V) \geq 0$ is the largest) is a zero of the Riley-Mednykh polynomial $P = P(V,A)$ for the link 7_3^2.

In Section 2, we present the fundamental group $\pi_1(X)$ of X with slope 9/16. In Section 3, we give the defining equation of the representation variety of $\pi_1(X)$. In Section 4, we compute the longitude of the link 7_3^2 using the Pythagorean theorem. And in Section 5, we give the Theorem 1 using the Schläfli formula.

2. Link 7_3^2

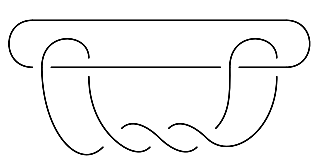

Figure 1. Link 7_3^2 in Rolfsen's link table.

Link 7_3^2 is presented in Figure 1. It is the same as W_3 from [2]. The slope of this link is 7/16. The link with slope 9/16 is the mirror of the link 7_3^2. Since the volume of the link with slope 7/16 is the same as the volume of link with slope 9/16, in the rest of the paper, the link with slope 9/16 is used.

The following fundamental group of X is stated in [2] with slope 7/16.

Proposition 1. $\pi_1(X) = \left\langle s,t \mid sws^{-1}w^{-1} = 1\right\rangle$, *where* $w = s^{-1}[s,t]^2[s,t^{-1}]^2$.

3. $\left(\mathrm{PSL}(2,\mathbb{C}),\mathbb{H}^3\right)$ STRUCTURE OF $X(\alpha)$

Let $R = \mathrm{Hom}(\pi_1(X), \mathrm{SL}(2,\mathbb{C}))$. Given a set of generators, s, t, of the fundamental group for $\pi_1(X)$, we define a set $R(\pi_1(X)) \subset \mathrm{SL}(2,\mathbb{C})^2 \subset \mathbb{C}^8$ to be the set of all points $(h(s), h(t))$, where h is a representaion of $\pi_1(X)$ into $\mathrm{SL}(2,\mathbb{C})$. Since the defining relation of $\pi_1(X)$ gives the defining equation of $R(\pi_1(X))$ [21], $R(\pi_1(X))$ is an affine algebraic set in $\mathbb{C}^8$. $R(\pi_1(X))$ is well-defined up to isomorphisms which arise from changing the set of generators. We say elements in R which differ by conjugations in $\mathrm{SL}(2,\mathbb{C})$ are *equivalent*. A point on the variety gives the $\left(\mathrm{PSL}(2,\mathbb{C}),\mathbb{H}^3\right)$ structure of $X(\alpha)$.

Let

$$h(s) = \begin{bmatrix} \cos\frac{\alpha}{2} & ie^{\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ ie^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}, \quad h(t) = \begin{bmatrix} \cos\frac{\alpha}{2} & ie^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ ie^{\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}.$$

Then h becomes a representation if and only if $A = \cot\frac{\alpha}{2}$ and $V = \cosh\rho$ satisfies a polynomial equation [14, 21]. We call the defining polynomial of the algebraic set $\{(V, A)\}$ as the *Riley-Mednykh polynomial* for the link 7_3^2. Thoughout the paper, h can be sometimes any representation and sometimes the unique hyperbolic representation.

Given the fundamental group of X,

$$\pi_1(X) = \left\langle s,t \mid sws^{-1}w^{-1} = 1\right\rangle,$$

where $w = s^{-1}[s,t]^2[s,t^{-1}]^2$, let $S = h(s),\ T = h(t)$ and $W = h(w)$. Then the trace of S and the trace of T are both $2\cos\frac{\alpha}{2}$.

Lemma 1. *For $n \in \mathrm{SL}(2,\mathbb{C})$ which satisfies $nS = S^{-1}n$, $nT = T^{-1}n$, and $n^2 = -I$,*

$$SWS^{-1}W^{-1} = -(SWn)^2.$$

Proof.

$$\begin{aligned}(SWn)^2 &= SWnSWn = SWS^{-1}n(S^{-1}(STS^{-1}T^{-1})^2(ST^{-1}S^{-1}T)^2)n \\ &= SWS^{-1}(S(S^{-1}T^{-1}ST)^2(S^{-1}TST^{-1})^2)n^2 = -SWS^{-1}W^{-1}.\end{aligned}$$

□

From the structure of the algebraic set of $R(\pi_1(X))$ with coordinates $h(s)$ and $h(t)$ we have the defining equation of $R(\pi_1(X))$. The following theorem is stated in [2, Proposition 4] with slope 7/16.

Theorem 2. *h is a representation of $\pi_1(X)$ if V is a root of the following Riley-Mednykh polynomial $P = P(V, A)$ which is given below.*

$$P = 8V^5 + 8A^2V^4 + \left(8A^4 + 16A^2 - 8\right)V^3 + \left(4A^6 + 8A^4 - 12A^2\right)V^2 + \left(A^8 + 4A^6 - 2A^4 - 12A^2 + 1\right)V - 4A^6 - 8A^4 + 4A^2.$$

Proof. Note that $SWS^{-1}W^{-1} = I$, which gives the defining equations of $R(\pi_1(X))$, is equivalent to $(SWn)^2 = -I$ in $\mathrm{SL}(2,\mathbb{C})$ by Lemma 1 and $(SWn)^2 = -I$ in $\mathrm{SL}(2,\mathbb{C})$ is equivalent to $\mathrm{tr}(SWn) = 0$.

We can find two n's in $\mathrm{SL}(2,\mathbb{C})$ which satisfies $nS = S^{-1}n$ and $n^2 = -I$ by direct computations. The existence and the uniqueness of the isometry (the involution) which is represented by n are shown in [3, p. 46]. Since two n's give the same element in $\mathrm{PSL}(2,\mathbb{C})$, we use one of them. Hence, we may assume

$$n = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$S = \begin{bmatrix} \cos\frac{\alpha}{2} & ie^{\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ ie^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}, \quad T = \begin{bmatrix} \cos\frac{\alpha}{2} & ie^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ ie^{\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}.$$

Recall that P is the defining polynomial of the algebraic set $\{(V, A)\}$ and the defining polynomial of $R(\pi_1(X))$ corresponding to our choice of $h(s)$ and $h(t)$. By direct computation P is a factor of $\mathrm{tr}(SWn) = -4i\sinh\rho(2V^2 + A^4 + 2A^2 - 1)P$. As in [2], P can not be $\sinh\rho$ or have only real roots. Also, P can not have only purely imaginary roots similarly. P in the theorem is the only factor of $\mathrm{tr}(SWn)$ which is different from $\sinh\rho$ and has roots which are not real or purely imaginary. P is the Riley-Mednykh polynomial. □

4. Longitude

Let $l_s = ws$ and $l_t = (t^{-1}[t, s]^2[t, s^{-1}]^2)t$. Then l_s and l_t are the longitudes which are null-homologus in X. Let $L_S = h(l_s)$ and Let $L_T = h(l_t)$.

Lemma 2. *$tr(S^{-1}L_T) = tr(S)$ and $tr(T^{-1}L_S) = tr(T)$.*

Proof. Since

$$\begin{aligned} S^{-1}L_T &= S^{-1}(T^{-1}(TST^{-1}S^{-1}TST^{-1}S^{-1} \cdot TS^{-1}T^{-1}STS^{-1}T^{-1}S)T) \\ &= (T^{-1}S^{-1}TST^{-1}S^{-1}T)(S^{-1})(T^{-1}S^{-1}TST^{-1}S^{-1}T)^{-1}, \end{aligned}$$

$$tr(S^{-1}L_T) = tr(S^{-1}) = tr(S).$$

The second statement can be obtained in a similar way. □

Definition 1. *The* complex length *of the longitude l (l_s or l_t) of the link 7_3^2 is the complex number γ_α modulo $4\pi\mathbb{Z}$ satisfying*

$$\mathrm{tr}(h(l)) = 2\cosh\frac{\gamma_\alpha}{2}.$$

Note that $l_\alpha = |Re(\gamma_\alpha)|$ is the real length of the longitude of the cone-manifold $X(\alpha)$.

By sending common fixed points of T and $L_T = h(l_t)$ to 0 and ∞, we have

$$T = \begin{bmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{bmatrix}, \quad L_T = \begin{bmatrix} e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha}{2}} \end{bmatrix},$$

and the following normalized line matrices of T (resp. L_T) which share the fixed points with T (resp. L_T).

$$\begin{aligned} l(T) &\equiv \frac{T - T^{-1}}{2i \sinh \frac{i\alpha}{2}} \\ &= \frac{1}{i(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}})} \begin{bmatrix} e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} - e^{\frac{i\alpha}{2}} \end{bmatrix} \\ &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} l(L_T) &\equiv \frac{L_T - L_T^{-1}}{2i \sinh \frac{\gamma_\alpha}{2}} \\ &= \frac{1}{i(e^{\frac{\gamma_\alpha}{2}} - e^{-\frac{\gamma_\alpha}{2}})} \begin{bmatrix} e^{\frac{\gamma_\alpha}{2}} - e^{-\frac{\gamma_\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha}{2}} - e^{\frac{\gamma_\alpha}{2}} \end{bmatrix} \\ &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \end{aligned}$$

which give the orientations of axes of T and L_T.

Now, we are ready to prove the following theorem which gives Theorem 4. Recall that γ_α modulo $4\pi\mathbb{Z}$ is the *complex length* of the longitude l_s or l_t of $X(\alpha)$. The following theorem is a particular case of Proposition 5 from [2].

Theorem 3. *(Pythagorean Theorem)* [2] *Let $X(\alpha)$ be a hyperbolic cone-manifold and let ρ be the complex distance between the oriented axes S and T. Then we have*

$$i \cosh \rho = \cot \frac{\alpha}{2} \coth(\frac{\gamma_\alpha}{4}).$$

Proof.

$$\begin{aligned}
\cosh\rho &= -\frac{\operatorname{tr}(l(S)l(T))}{2}\\
&= -\frac{\operatorname{tr}(l(S)l(L_T))}{2}\\
&= \frac{\operatorname{tr}((S-S^{-1})(L_T-L_T^{-1}))}{8\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= \frac{\operatorname{tr}(SL_T-S^{-1}L_T-SL_T^{-1}+(L_TS)^{-1}))}{8\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= \frac{2(\operatorname{tr}(SL_T)-\operatorname{tr}(S^{-1}L_T))}{8\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= \frac{\operatorname{tr}(S)\operatorname{tr}(L_T)-2\operatorname{tr}(S^{-1}L_T)}{4\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= \frac{\operatorname{tr}(S)\operatorname{tr}(L_T)-2\operatorname{tr}(S)}{4\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= \frac{\operatorname{tr}(S)(\operatorname{tr}(L_T)-2)}{4\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= \frac{2\cos\frac{\alpha}{2}(2\cosh\frac{\gamma_\alpha}{2}-2)}{4i\sin\frac{\alpha}{2}\sinh\frac{\gamma_\alpha}{2}}\\
&= -i\cot\frac{\alpha}{2}\tanh(\frac{\gamma_\alpha}{4}).
\end{aligned}$$

where the first equality comes from [3, p. 68], the sixth equality comes from the Cayley-Hamilton theorem, and the seventh equality comes from Lemma 2. Therefore, we have

$$i\cosh\rho = \cot\frac{\alpha}{2}\coth(\frac{\gamma_\alpha}{4}).$$

□

Pythagorean theorem 3 gives the following theorem which relates the eigenvalues of $h(l)$ and $V = \cosh\rho$ for $A = \cot\frac{\alpha}{2}$.

Theorem 4. *Recall that l is the longitude. By conjugating if necessary, we may assume $h(l)$ is upper triangular. Let $L = h(l)_{11}$. Let $A = \cot\frac{\alpha}{2}$. Then the following formulae show that there is a one to one correspondence between the the eigenvalues of $h(l)$ and $V = \cosh\rho$:*

$$iV = A\frac{L-1}{L+1} \text{ and } L = \frac{A-iV}{A+iV}.$$

Proof. By Theorem 3,

$$\begin{aligned} iV &= i\cosh\rho \\ &= \cot\frac{\alpha}{2}\tanh(\frac{\gamma_\alpha}{4}) \\ &= \cot\frac{\alpha}{2}\frac{\sinh(\frac{\gamma_\alpha}{4})}{\cosh(\frac{\gamma_\alpha}{4})} \\ &= \cot\frac{\alpha}{2}\frac{e^{\frac{\gamma_\alpha}{4}}-e^{-\frac{\gamma_\alpha}{4}}}{e^{\frac{\gamma_\alpha}{4}}+e^{-\frac{\gamma_\alpha}{4}}} \\ &= \cot\frac{\alpha}{2}\frac{e^{\frac{\gamma_\alpha}{2}}-1}{e^{\frac{\gamma_\alpha}{2}}+1} \\ &= A\frac{L-1}{L+1}. \end{aligned}$$

If we solve the above equation,

$$iV = A\frac{L-1}{L+1},$$

for L, we have

$$L = \frac{A-iV}{A+iV}.$$

□

5. Proof of Theorem 1

According to [8, 10, 19, 20], there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ such that $X(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$. Denote by $D(X(\alpha))$ the discriminant of $P(V, A)$ over V. Then α_0 is the only zero of $D(X(\alpha))$ in $[\frac{2\pi}{3}, \pi)$.

From Theorem 4, we have the following equality,

$$|L|^2 = \left|\frac{A-iV}{A+iV}\right|^2 = \frac{|A|^2+|V|^2+2AImV}{|A|^2+|V|^2-2AImV}. \tag{1}$$

For the volume, we choose L with $|L| \geq 1$ and hence we have $\mathrm{Im}(V) \geq 0$ by Equality (1). The component of V with $\mathrm{Im}(V) \geq 0$ which becomes real at α_0 has negative real part. On the geometric component which gives the unique hyperbolic structure, we have the volume of a hyperbolic cone-manifold $X(\alpha)$ for $0 \leq \alpha < \alpha_0$:

$$\begin{aligned} \mathrm{Vol}(X(\alpha)) &= -\int_{\alpha_0}^{\alpha} 2\left(\frac{l_\alpha}{2}\right) d\alpha \\ &= -\int_{\alpha_0}^{\alpha} 2\log|L|\, d\alpha \\ &= -\int_{\pi}^{\alpha} 2\log|L|\, d\alpha \\ &= \int_{\alpha}^{\pi} 2\log|L|\, d\alpha \\ &= \int_{\alpha}^{\pi} 2\log\left|\frac{A-iV}{A+iV}\right| d\alpha, \end{aligned}$$

where the first equality comes from the Schläfli formula for cone-manifolds (Theorem 3.20 of [1]), the second equality comes from the fact that $l_\alpha = |Re(\gamma_\alpha)|$ is the real length of the one longitude of $X(\alpha)$, the third equality comes from the fact that $\log|L| = 0$ for $\alpha_0 < \alpha \leq \pi$ by Equality (1) since all V's are real for $\alpha_0 < \alpha \leq \pi$, and $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ is the zero of the discriminant $D(X(\alpha))$. Numerical calculations give us the following value for α_0 : $\alpha_0 \approx 2.83003$.

References

[1] D. Cooper, C.D. Hodgson, and S.P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, MSJ Memoirs, **5**, Mathematical Society of Japan, Tokyo, 2000. Zbl 0955.57014

[2] D. Derevnin, A. Mednykh, and M. Mulazzani, *Volumes for twist link cone-manifolds*, Bol. Soc. Mat. Mexicana III, Ser.10 (Special Issue), (2004), 129–145. Zbl 1114.57017

[3] W. Fenchel, *Elementary geometry in hyperbolic space*, Volume 11, de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1989. Zbl 0674.51001

[4] J.-Y. Ham and J. Lee, *Explicit formulae for Chern-Simons invariants of the hyperbolic orbifolds of the knot with Conway's notation $C(2n,3)$*, www.math.snu.ac.kr/˜jyham, Preprint, 2016.

[5] J.-Y. Ham and J. Lee, *Explicit formulae for Chern-Simons invariants of the twist knot orbifolds and edge polynomials of twist knots*, Matematicheskii Sbornik, **207**:9 (2016).

[6] J.-Y. Ham and J. Lee, *The volume of hyperbolic cone-manifolds of the knot with Conway's notation $C(2n,3)$*, J. Knot Theory Ramifications, **25**:6 (2016) Article ID 1650030, 9 p. Zbl 1341.57002

[7] J.-Y. Ham, A. Mednykh, and V. Petrov, *Trigonometric identities and volumes of the hyperbolic twist knot cone-manifolds*, J. Knot Theory Ramifications, **23**:12 (2014) Article ID 1450064, 16 p. Zbl 06398737

[8] H.M. Hilden, M.T. Lozano, and J.M. Montesinos-Amilibia, *On a remarkable polyhedron geometrizing the figure eight knot cone manifolds*, J. Math. Sci. Univ. Tokyo, **2**:3 (1995), 501–561. Zbl 0856.57007

[9] H.M. Hilden, M.T. Lozano, and J.M. Montesinos-Amilibia, *Volumes and Chern-Simons invariants of cyclic coverings over rational knots*, In Topology and Teichmüller spaces (Katinkulta, 1995), World Sci. Publ., River Edge, NJ, (1996), 31–55.

[10] S. Kojima, *Deformations of hyperbolic 3-cone-manifolds*, J. Differential Geom., **49**:3 (1998), 469–516. Zbl 0990.57004

[11] S. Kojima, *Hyperbolic 3-manifolds singular along knots*, Knot theory and its applications Chaos Solitons Fractals, **9**:4–5 (1998), 765–777. Chaos Solitons Fractals 9, No.4-5, 765-777 (1998).

[12] J.P. Mayberry and K. Murasugi, *Torsion-groups of abelian coverings of links*, Trans. Amer. Math. Soc., **271**:1 (1982), 143–173. Zbl 0487.57001

[13] A. Mednykh, *The volumes of cone-manifolds and polyhedra*, Lecture Notes, Seoul National University, http://mathlab.snu.ac.kr/˜top/workshop01.pdf, 2007.

[14] A. Mednykh and A. Rasskazov, *On the structure of the canonical fundamental set for the 2-bridge link orbifolds*, Universität Bielefeld, Sonderforschungsbereich 343, Discrete Structuren in der Mathematik, Preprint 98-062, www.mathematik.uni-bielefeld.de/sfb343/preprints/pr98062.ps.gz, 1998.

[15] A. Mednykh and A. Rasskazov, *Volumes and degeneration of cone-structures on the figure-eight knot*, Tokyo J. Math., **29**:2 (2006), 445–464. Zbl 1124.57008

[16] A. Mednykh and A. Vesnin, *On the volume of hyperbolic Whitehead link cone-manifolds*, SCIENTIA, Series A: Sci. Ser. A Math. Sci. (N.S.), **8** (2002), 1–11. Zbl 1104.57300

[17] A.D. Mednykh, *Trigonometric identities and geometrical inequalities for links and knots*, In Proceedings of the Third Asian Mathematical Conference, 2000 (Diliman), World Sci. Publ., River Edge, NJ, (2002), 352–368. Zbl 1034.57007

[18] M. Mulazzani and A. Vesnin, *The many faces of cyclic branched coverings of 2-bridge knots and links*, Atti Sem. Mat. Fis. Univ. Modena, **49** (suppl.), (2001), 177–215. Zbl 1221.57009

[19] J. Porti, *Spherical cone structures on 2-bridge knots and links*, Kobe J. Math., 21 **(1-2)**, (2004), 61–70.

[20] J. Porti and H. Weiss, *Deforming Euclidean cone 3-manifolds*, Geom. Topol., **11** (2007), 1507–1538. Zbl 1159.57007

[21] R. Riley, *Nonabelian representations of* 2*-bridge knot groups*, Quart. J. Math. Oxford II, Ser. 35 **(138)** (1984), 191–208. Zbl 0549.57005

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