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ERDÖS-KO-RADO PROPERTIES OF SOME FINITE GROUPS

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Abstract. Let G be a subgroup of the symmetric group $\mathrm{Sym}(n)$ and A be a subset of G. The subset A is said to be intersecting if for any pair of permutations $\sigma, \tau \in A$ there exists $i, 1 \leq i \leq n$, such that $\sigma(i) = \tau(i)$. The group G has Erdös-Ko-Rado (EKR) property, if the size of any intersecting subset of G is bounded above by the size of a point stabilizer in G. The group G has the strict EKR property if every intersecting set of maximum size is the coset of the stabilizer of a point. The aim of this paper is to investigate the EKR and strict EKR properties of the groups V_{8n}, U_{6n}, T_{4n} and SD_{8n}.

Keywords: Erdös-Ko-Rado property, finite group.

1. Introduction

In extremal set theory, there is a famous result named "Erdös–Ko–Rado theorem". This result states that if $n \geq 2r$ and A is a family of distinct subsets of $[n] = \{1, 2, ..., n\}$ such that each subset has size r and each pair of subsets has non-empty intersection, then the maximum size of A is the binomial coefficient $\binom{n-1}{r-1}$ [8]. In mathematics literature, there is an elegant proof of the Erdös–Ko–Rado theorem by Katona, where he discovered a new method, which is called Katona's cycle method [13]. In this paper, we consider an Erdös–Ko–Rado type theorem for finite permutation groups.

Suppose G is a subgroup of the symmetric group $\mathrm{Sym}(n)$. A subset A of G is said to be intersecting if for any pair of permutations $\sigma, \tau \in A$ there exists $i \in [n]$ such that $\sigma(i) = \tau(i)$. A group G has Erdös-Ko-Rado (EKR) property, if the size

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of any intersecting subset of G is bounded above by the size of a point stabilizer in G. The group G has the strict EKR property if every intersecting set of maximum size is the coset of the stabilizer of a point.

Frankl and Deza [9], proved that the symmetric group $\mathrm{Sym}(n)$ has EKR property and conjectured that it has strict EKR property. Cameron and Ku [4] proved this conjecture. Ku and Wong [14] proved that the alternating group $Alt(n)$ has strict EKR property. Wang and Zhang [18] established the EKR property of some Coxeter groups and Meagher and Spiga [15, 16] showed that the projective special groups $PGL_2(q)$ have strict EKR property, but the group $PGL_3(q)$ does not have this property. Ahmadi and Meagher [1] proved EKR property of cyclic, dihedral and Frobenius groups and characterized which ones have the strict EKR property. They also showed that if all the groups in an external direct sum or an internal direct sum have the EKR (or strict EKR) property, then the product does as well. We refer to [2] for a complete history of this problem for permutation groups. Ahmadi and Meagher [3], in a recent paper investigated the EKR property of Mathieu groups and all $2-$transitive groups with degree no more than 20.

Throughout this paper our notations are standard and mostly taken from [12] and [2]. For two positive integers n and m, (n, m) denotes the greatest common divisor of m and n. Suppose a finite group G acts on a set X. If $g \in G$ then $\mathrm{Fix}(g)$ denotes the set of all points $i \in X$ such that $i^g = i$ and the stabilizer subgroup G_i is the set of all elements $g \in G$ with $i^g = i$. The group action in our work is essential since it is possible a group has (strict) EKR property under one action while it fails to have this property under another action. Ahmadi and Meagher [1, Section 7] presented an example of such a group. So, by a permutation group we consider a subgroup of $\mathrm{Sym}(X)$ with its natural action on $[n]$. All calculations are done with the aid of GAP [17].

2. Main Results

Suppose G is a subgroup of the symmetric group $\mathrm{Sym}(n)$ and Δ is a simple graph. An independent set in Δ is a subset of $V(\Delta)$ without adjacent vertices. The independent number of G is the size of a maximum independent set in G. Following Ahmadi and Meagher [1], a fixed point free permutation of G is called a derangement and the derangement graph Γ_G is the graph with vertex set G in which two vertices are adjacent if and only if they do not intersect. We will also denote the set of all derangements of the group G by $D(G)$. In particular, $\alpha, \beta \in G$ are adjacent if and only if $\alpha\beta^{-1}$ is a derangement. Clearly, this is a Cayley graph on the set of all derangements of G. On the other hand, if σ_1 and σ_2 are two elements in a conjugacy class of G and σ_1 is derangement then σ_2 is also a derangement. This proves that the set of all derangements is a union of conjugacy classes and so Γ_G is a normal Cayley graph. Also, the group G has a transitive action on the elements of G by left multiplication.

The semi-dihedral group SD_{8n}, dicyclic group T_{4n} and the groups U_{6n} and V_{8n} have the following presentations, respectively:

$$\begin{aligned}
SD_{8n} &= \langle x, y \mid x^{4n} = y^2 = e,\ yxy = x^{2n-1} \rangle, \\
T_{4n} &= \langle x, y \mid x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1} \rangle, \\
U_{6n} &= \langle x, y \mid x^{2n} = y^3 = e,\ x^{-1}yx = y^{-1} \rangle, \\
V_{8n} &= \langle x, y \mid x^{2n} = y^4 = e,\ xyx = y^{-1},\ xy^{-1}x = y \rangle.
\end{aligned}$$

It is easy to see that the dicyclic group T_{4n} has order $4n$, The group U_{6n} is of order $6n$, the semi-dihedral group SD_{8n} and the group V_{8n} has order $8n$. The ratio bound for independent sets is an important result was proved by Delsarte [6]. We also refer to [10, Section 9.6] for another proof of this result.

Proposition 1 (Ratio Bound for Independent Sets). *Let X be a $k-$regular graph on n vertices with τ the least eigenvalue of X. For any independent set S we have $|S| \leq \frac{n}{1-\frac{k}{\tau}}$.*

Suppose S is a normal subset of a group G and $e \notin S$. To compute the eigenvalues of $Cay(G, S)$, we apply a result of Zieschang [7, Theorem 1] as follows:

Proposition 2. *Let $Cay(G,T)$ denote the Cayley graph of a finite group G with respect to a normal subset T of $G \setminus \{e\}$. Let further $\{\chi_1, \ldots, \chi_s\}$ be the set of all irreducible complex characters of G and define $\lambda_j = \frac{1}{\chi_j(1)} \sum_{t \in T} \chi_j(t)$, where $1 \leq j \leq s$. Then $\{\lambda_1, \ldots, \lambda_s\}$ is the set of all values of the spectrum of $Cay(G,T)$. Moreover, if m_i is the multiplicity of λ_j, then $m_j = \sum_{k=1,\ \lambda_k=\lambda_j}^{s} (\chi_k(e))^2$.*

Suppose G is finite group. Since the derangement set of G is a union of its conjugacy classes, we have to first compute the conjugacy classes of G. The conjugacy classes of semi-dihedral group SD_{8n} computed by Hormozi and Rodtes [11]. To describe their method, we need some notations taken from [11]. Define:

$$C^{even} = C_1 \cup C_2^{even} \cup C_3^{even} \text{ and } C^{odd} = C_1 \cup C_2^{odd} \cup C_3^{odd},$$

where

$$\begin{aligned}
C_1 &= \{0, 2, 4, \ldots, 2n\}, \\
C_2^{even} &= \{1, 3, 5, \ldots, n-1\}, \\
C_3^{even} &= \{2n+1, 2n+3, 2n+5, \ldots, 3n-1\}, \\
C_2^{odd} &= \{1, 3, 5, \ldots, n\}, \\
C_3^{odd} &= \{2n+1, 2n+3, 2n+5, \ldots, 3n\}, \\
C_{even}^{\dagger} &= C_1 \setminus \{0, 2n\}, \\
C_{odd}^{\dagger} &= C_2^{even} \cup C_3^{even}.
\end{aligned}$$

Moreover, we assume that $C_\star^{even} = C^{even} \setminus \{0, 2n\}$ and $C_\star^{odd} = C^{odd} \setminus \{0, n, 2n, 3n\}$. By [14, Proposition 2.2], the conjugacy classes of SD_{8n}, $n \geq 2$, can be computed as follows:

- If n is even, then $Z(SD_{8n}) = \{e, a^{2n}\}$ and there are $2n-1$ conjugacy classes of size two in the form of $(a^r)^{SD_{8n}} = \{a^r, a^{(2n-1)r}\}$, $r \in C_\star^{even}$. In this case, we have two other conjugacy classes of size $2n$ as $b^{SD_{8n}} = \{ba^{2t} \mid 0 \leq t \leq 2n-1\}$ and $(ba)^{SD_{8n}} = \{ba^{2t+1} \mid 0 \leq t \leq 2n-1\}$.
- If n is odd, then $Z(SD_{8n}) = \{e, a^n, a^{2n}, a^{3n}\}$ and there are $2n-2$ conjugacy classes of size two as $(a^r)^{SD_{8n}} = \{a^r, a^{(2n-1)r}\}$, $r \in C_\star^{odd}$. In this case, we have four other conjugacy classes of size n as $b^{SD_{8n}} = \{ba^{4t} \mid 0 \leq t \leq n-1\}$, $(ba)^{SD_{8n}} = \{ba^{4t+1} \mid 0 \leq t \leq n-1\}$, $(ba^2)^{SD_{8n}} = \{ba^{4t+2} \mid 0 \leq t \leq n-1\}$ and $(ba^3)^{SD_{8n}} = \{ba^{4t+3} \mid 0 \leq t \leq n-1\}$.

The character tables of the groups U_{6n} and V_{8n} (n is odd) are given in the book of James and Liebeck [12]. If n is even then the character table of the group V_{8n} computed by Darafsheh and Poursalavati [5]. The group V_{8n} has exactly $2n+3$

conjugacy classes, when n is odd and the number of conjugacy classes is $2n+6$, when n is even. The conjugacy classes of the group V_{8n}, n is odd, are $\{e\}$, $\{b^2\}$, $\{a^{2r+1}, a^{-2r-1}b^2\}(0 \le r \le n-1)$, $\{a^{2s}, a^{-2s}\}$, $\{a^{2s}b^2, a^{-2s}b^2\}$ $(1 \le s \le \frac{n-1}{2})$, $\{a^j b^k \mid j \textit{ is even } \& \ k = 1, 3\}$, $\{a^j b^k \mid j \textit{ is odd } \& \ k = 1, 3\}$. If n is even then the conjugacy classes of this group are $\{e\}$, $\{b^2\}$, $\{a^n\}$, $\{a^n b^2\}$, $\{a^{2r+1}, a^{-2r-1}b^2\}$ $(0 \le r \le n-1)$, $\{a^{2s}, a^{-2s}\}$, $\{a^{2s}b^2, a^{-2s}b^2\}$ $(1 \le s \le \frac{n}{2} - 1)$, $\{a^{2k}b^{(-1)^k} \mid 0 \le k \le n-1\}$, $\{a^{2k}b^{(-1)^{k+1}} \mid 0 \le k \le n-1\}$, $\{a^{2k+1}b^{(-1)^k} \mid 0 \le k \le n-1\}$, $\{a^{2k+1}b^{(-1)^{k+1}} \mid 0 \le k \le n-1\}$.

The group T_{4n} has exactly $n+3$ conjugacy classes as $\{e\}$, $\{a^n\}$, $\{a^r, a^{-r}\}$ $(1 \le r \le n-1)$, $\{a^{2j}b \mid 0 \le j \le n-1\}$ and $\{a^{2j+1}b \mid 0 \le j \le n-1\}$, and the conjugacy classes of U_{6n} are $\{e\}$, $\{a^{2r}\}$, $\{a^{2r}b, a^{2r}b^2\}$, $\{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\}$, $0 \le r \le n-1$, see [12] for details.

Following Ahmadi [2], we now present a generalization of the EKR and strict EKR property to the case where the set of all the derangements of the group is replaced by an arbitrary union C of the conjugacy classes of the derangements of the group. In this case, the EKR and the strict EKR property will be changed to the EKR and the strict EKR property with respect to C, respectively.

Theorem 1. *Suppose $C \subseteq D$. The following statements hold:*

(1) *If n is odd then*
  (a) $D(V_{8n}) = V_{8n} \setminus (ab)^{V_{8n}} \cup \{\mathrm{id}\}$ *and the group V_{8n} has strict EKR property,*
  (b) *if $C \ne D$ then the group V_{8n} does not have EKR property with respect to C.*

(2) *If n is even then*
  (a) $D(V_{8n}) = V_{8n} \setminus (ab^{-1})^{V_{8n}} \cup \{\mathrm{id}\}$ *and the group V_{8n} has strict EKR property,*
  (b) *if C is a union of a conjugacy class and its inverse then the group V_{8n} does not have EKR property with respect to C,*

(3) (a) $D(T_{4n}) = T_{4n} \setminus \{\mathrm{id}\}$ *and the group T_{4n} has EKR property,*
  (b) *the group T_{4n} does not have EKR property with respect to an arbitrary proper union of members in D.*

(4) (a) $D(U_{6n}) = \bigcup_{k=1}^{n-1}(a^{2k}b)^{U_{6n}}$ *and the group U_{6n} has strict EKR property if and only if $n \ne 2$,*
  (b) *if $F \subseteq \{1, 2, \ldots, n-1\}$ and $C = \bigcup_{k \in F}(a^{2k}b)^{U_{6n}}$ then $\langle C\rangle = \langle b, a^{2r}\rangle$ and $Cay(U_{6n}, C)$ has exactly $2(n, r)$ components, where r is the greatest common divisor of all numbers k and $n-k$, where $k \in F$,*
  (c) *if $C = (a^{2k}b)^{U_{6n}} \cup (a^{2n-2k}b)^{U_{6n}}$ then U_{6n} has strict EKR property with respect to C if and only if $3|n$ and $k = \frac{n}{3}$,*
  (d) *if $C = (a^{2k}b)^{U_{6n}} \cup (a^{2n-2k}b)^{U_{6n}}$, then U_{6n} has strict EKR property with respect to C if and only if $|\langle C\rangle| = 9$.*

(5) $D(SD_{8n}) = SD_{8n} \setminus \{\mathrm{id}\} \cup b^{SD_{8n}}$ *and the group SD_{8n} has strict EKR property.*

Proof. Our main proof will consider some separate cases as follows:

(1) Suppose n is odd and $G = V_{8n}$.

  (a) In this case, we use the permutation representation $a = (1, 2, \ldots, 2n)$ $(2n+1, 2n+2, \ldots, 4n)$ and $b = (1, 2, 2n+1, 2n+2)(3, 2n, 2n+3, 4n)(4, 4n-$

$1, 2n+4, 2n-1)\dots(n+1, 3n+2, 3n+1, n+2)$. By [5], the mapping $x \mapsto a$ and $y \mapsto b$, embeds the group G in S_{4n} and so we can assume that G is a subgroup of S_{4n}. We now compute $\mathrm{Fix}(a^r b^s)$, where $0 \leq r \leq 2n$ and $0 \leq s \leq 4$. It can easily see that when r or s is even, $a^r b^s$ does not have fixed point. Hence it is enough to assume that r and s are odd. In this case, $a^r b^s$ has exactly two fixed points l and $l+2n$, where $1 \leq l \leq 2n$. Thus $D(V_{8n}) = V_{8n} \setminus (ab)^{V_{8n}} \cup \{\mathrm{id}\}$ and since $\{\mathrm{id}, a^r b^s\}$ is an intersection set so $\alpha(\Gamma_{V_{8n}}) \geq 2$.

We claim that $\alpha(\Gamma_{V_{8n}}) = 2$. To prove, we assume that S is a maximal independent set in $\Gamma_{V_{8n}}$. Without loss of generality, we may assume that $\mathrm{id} \in S$ and by above discussion we can add $a^r b^s$ to S. If $a^o b^m$ is another element in S, m and o are odd, then there exists i such that $a^r b^s(i) = a^o b^m(i)$. This implies that $b^{-k+1}(b^{-1}a^{r-s})b^m(i) = i$. We now consider two cases that $r-s$ is even or odd. Notice that if m is even then $b^{-1}a^m = a^{-m}b^{-1}$ and otherwise $b^{-1}a^m = a^{-m}b$. If $r-s$ is even then $a^{s-r}b^{m-k}(i) = i$, a contradiction, and if $r-s$ is odd then $a^{s-r}b^{m+k}(i) = i$, which is impossible. Therefore, $\alpha(\Gamma_{V_{8n}}) = 2$ and $S = \{\mathrm{id}, a^r b^s\}$, for some $1 \leq r \leq 2n$ and $1 \leq s \leq 4$, where r and s are odd. A similar argument shows that every maximal independent set has size 2. Our argument given above shows that $G_i = \{\mathrm{id}, a^r b^s\}$, $1 \leq i \leq 4n$. This implies that G has EKR property and any maximal independent set in $\Gamma_{V_{8n}}$ is a coset of point stabilizer. Hence G has strict EKR property.

(b) Suppose $C \subseteq D$ and $(a^i b^j), (a^r b^s) \in V_{8n}$. $(a^i b^j)$ and $(a^r b^s)$ are independent if and only if $(a^i b^j)(a^r b^s)^{-1} \notin C$. If $s-j$ is even then

$$(a^i b^j)(a^r b^s)^{-1} = \begin{cases} a^{i-r}b^{j-s} & 2|r \\ a^{i-r}b^{s-j} & 2 \nmid r \end{cases}$$

and if $s-j$ is odd then

$$(a^i b^j)(a^r b^s)^{-1} = \begin{cases} a^{i+r}b^{j-s} & 2|r \\ a^{i+r}b^{s-j} & 2 \nmid r \end{cases}.$$

Define $T_1 = \{(b^2)^{V_{8n}}, b^{V_{8n}}, (a^{2s}b^2)^{V_{8n}}, (b^2)^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}}, (b^2)^{V_{8n}} \cup b^{V_{8n}}, (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}}, (b^2)^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}}; 1 \leq s \leq \frac{n-1}{2}\}$. Since for each i, $1 \leq i \leq 4n$, $|\mathrm{Stab}(i)| = 2$ and $A_1 = \{a^j; 0 \leq j \leq 2n-1\}$ is an independent set of size $2n$ with respect to $C \in T_1$, the group V_{8n} does not have EKR property with respect to C. In a similar way, we define:

$$\begin{aligned}
T_2 &= \{(a^{2r+1})^{V_{8n}}, \quad (b^2)^{V_{8n}} \cup (a^{2r+1})^{V_{8n}}, \\
&\quad (a^{2r+1})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}}, \quad (b^2)^{V_{8n}} \cup (a^{2r+1})^{V_{8n}} \cup b^{V_{8n}}, \\
&\quad (b^2)^{V_{8n}} \cup (a^{2r+1})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}}; \\
&\quad 1 \leq s \leq \tfrac{n-1}{2}\}, \\
A_2 &= \{a^i; \quad 0 \leq i \leq 2n-1, \ 2|i\}, \\
T_3 &= \{(a^{2s})^{V_{8n}}, \quad (a^{2r+1})^{V_{8n}} \cup (a^{2s})^{V_{8n}}, \\
&\quad (b^2)^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}}, \\
&\quad (a^{2r+1})^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}}; \quad 1 \leq s \leq \tfrac{n-1}{2}\}, \\
A_3 &= \{a^i, \ a^i b, \ a^i b^2\}, \quad \text{where } i \text{ is odd and } 1 \leq i \leq 2n-1,
\end{aligned}$$

$$\begin{aligned}
T_4 &= \{(b^2)^{V_{8n}} \cup (a^{2s})^{V_{8n}}, \quad (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}},\\
&\quad (b^2)^{V_{8n}} \cup (a^{2r+1})^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup b^{V_{8n}}; \quad 1 \le s \le \tfrac{n-1}{2}\},\\
A_4 &= \{a^i,\ a^ib,\ \mathrm{id};\quad 2 \nmid i\}, \qquad \text{where } i \text{ is odd and } 1 \le i \le 2n-1,\\
T_5 &= \{(a^{2s})^{V_{8n}} \cup b^{V_{8n}}, \quad (b^2)^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup b^{V_{8n}},\\
&\quad (a^{2r+1})^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup b^{V_{8n}},\\
&\quad (a^{2r+1})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}}, \quad (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}},\\
&\quad (b^2)^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}}; \quad 1 \le s \le \frac{n-1}{2}\},\\
A_5 &= \{a^i,\ a^jb,\ \mathrm{id}\}, \quad \text{where } 2 \nmid i \text{ and } 2 \mid j.
\end{aligned}$$

Then the group V_{8n} does not have EKR property with respect to $C \in T_i$, $i = 1, \ldots, 5$.

(2) (a) Suppose n is even and $G = V_{8n}$. In this case, we use the permutation representation $a = (1, 2, \ldots, 2n)(2n+1, 2n+2, \ldots, 4n)$ and $b = (1, 2, 2n+1, 2n+2)(3, 2n, 2n+3, 4n)(4, 4n-1, 2n+4, 2n-1)\ldots(n, 3n+3, 3n, n+3)(n+1, n+2, 3n+1, 3n+2)$. By [5], the mapping $x \mapsto a$ and $y \mapsto b$, embeds the group G in S_{4n} and so we can assume that G is a subgroup of S_{4n}. Since $b^2(j) = 2n+j$ and $b^2(2n+j) = j$, $1 \le j \le 2n$, $a^kb^2(i) \neq i$. Moreover, if r and k are odd then a^rb^k does not have fixed point if and only if $r+k$ is not divisible by 4. In the case that $4 \mid r+k$, a^rb^k has exactly four fixed points l, $l+n$, $l+2n$ and $l+3n$, $1 \le l \le n$. So $D(V_{8n}) = V_{8n} \backslash (ab^{-1})^{V_{8n}} \cup \{\mathrm{id}\}$. Now by a similar argument as the case that n is odd, one can prove $S = \{\mathrm{id}, a^rb^s\}$, r, s are odd and $4 | r+s$, is a maximum intersection set. Thus, $\alpha(\Gamma_{V_{8n}}) = 2$. On the other hand, for each $i, 1 \le i \le 4n$, there are odd integers r and s with $4 | r+s$ such that $G_i = S$, proving the EKR property of G. Finally, any maximal independent set in $\Gamma_{V_{8n}}$ is a coset of point stabilizer and so G has strict EKR property.

(b) Define $F_1 = \{(b^2)^{V_{8n}},\ (a^nb^2)^{V_{8n}},\ b^{V_{8n}} \cup (b^{-1})^{V_{8n}},\ (ab)^{V_{8n}},\ (a^{2s}b^2)^{V_{8n}}; 1 \le s \le \frac{n}{2} - 1\}$. Since for each i, $|G_i| = 2$ and $B_1 = \{a^i; 0 \le i \le 2n-1\}$ is an independent set of size $2n$, the group V_{8n} does not have EKR property with respect to $C \in F_1$. Similarly we define the sets $F_2 = \{(a^n)^{V_{8n}}\}$, $B_2 = \{b^i; 0 \le i \le 3\}$, $F_3 = \{(a^{2s})^{V_{8n}}; 1 \le s \le \frac{n}{2} - 1\}$, $B_3 = \{\mathrm{id}, ab^{-1}, a^ib^3\}$, $2 \nmid i$, $F_4 = \{(a^{2r+1})^{V_{8n}}; 1 \le s \le n-1\}$ and $B_4 = \{b^i; 0 \le i \le 3\}$. Hence the group V_{8n} does not have EKR property with respect to $C \in F_i$, $i = 1, \ldots, 4$.

(3) Suppose $G = T_{4n}$. Define $a = (1, 2, \ldots, 2n)(2n+1, \ldots, 4n)$, $\delta = (1, 2n+1, n+1, 3n+1)$ and $b = \delta \prod_{i=2}^{n}(i, 4n-i+2, n+i, 3n-i+2)$. Then $G \cong \langle a, b\rangle$.

(a) $C = D$. We note that a^rb^s, $0 \le r < 2n$ and $0 \le s \le 3$, does not have fixed points. Hence $G_i = \{\mathrm{id}\}$ where $i \in \{1, 2, \ldots, 4n\}$. So, $D(T_{4n}) = T_{4n} \setminus \{\mathrm{id}\}$. This implies that $\Gamma_{T_{4n}}$ is complete and $\mathrm{Spec}(\Gamma_{T_{4n}}) = \{-1^{(|G|-1 \text{ times})}, |G|-1\}$. Thus, -1 is the minimum eigenvalue of $\Gamma_{T_{4n}}$. By Proposition 2, for each independent set S, $|S| \le \frac{4n}{1 - \frac{4n-1}{-1}} = 1 = |G_i|$. Therefore, G has EKR property.

(b) $C \subsetneqq D$. Choose $g \in T_{4n} \setminus C$. Then $\{\mathrm{id}, g\}$ will be an independent subset of T_{4n} which shows that $\alpha(\Gamma_{T_{4n}}) \ge 2$. Since $G_i = \{\mathrm{id}\}$ where $i \in \{1, 2, \ldots, 4n\}$, therefore T_{4n} with respect to C does not have EKR property.

(4) Suppose $G = U_{6n}$. Define $a = (1, 2, \ldots, 2n)(2n+1, 2n+2)$ and $b = (2n+1, 2n+2, 2n+3)$. Following Darafsheh and Poursalavati [5], the mapping $x \mapsto a$ and $y \mapsto b$, embeds the group G in S_{2n+3}.

(a) Compute $\mathrm{Fix}(a^r b^s)$, where $0 \leq r \leq 2n$ and $0 \leq s \leq 3$. It can easily see that

$$\mathrm{Fix}(a^r b^s) = \begin{cases} \{2n+1, 2n+2, 2n+3\} & s = 0 \textit{ and } 2|r \\ \{2n+3\} & s = 0 \textit{ and } 2 \nmid r \\ \{2n+2\} & s = 1 \textit{ and } 2 \nmid r \\ \{2n+1\} & s = 2 \textit{ and } 2 \nmid r \\ \emptyset & \text{otherwise} \end{cases},$$

therefore $D(U_{6n}) = \bigcup_{r=1}^{n-1} \{a^{2r}b, a^{2r}b^2\}$. On the other hand,

$$\begin{aligned} G_1 &= G_2 = \ldots = G_{2n} = \{\mathrm{id}, b, b^2\}, \\ G_{2n+1} &= \{\mathrm{id}, a^m, a^k b^2;\ 2 \mid m,\ 2 \nmid k\}, \\ G_{2n+2} &= \{\mathrm{id}, a^m, a^k b;\ 2 \mid m,\ 2 \nmid k\}, \\ G_{2n+3} &= \{\mathrm{id}, a^m, a^k;\ 2 \mid m,\ 2 \nmid k\}. \end{aligned}$$

The maximum size of any point-stabilizer in U_{6n} is $2n$. Suppose $2|m$ and $2 \nmid k$. Then $\mathrm{id}(2n+2) = a^m(2n+2) = a^k b(2n+2) = 2n+2$ and so $S = \{\mathrm{id}, a^m, a^k b \mid 2 \leq m \leq 2n-2 \& 1 \leq k \leq 2n-1\}$ is a intersection set of size $2n$. By the character table of U_{6n} given in [12] and Proposition 2, the spectrum of $\Gamma(U_{6n})$ is

$$\mathrm{Spec}(\Gamma_{U_{6n}}) = \begin{pmatrix} 2n-2 & -n+1 & -2 & 1 \\ 2 & 4 & 2n-2 & 4n-4 \end{pmatrix}.$$

Apply Proposition 1 to deduce that the group U_{6n}, $n > 2$, has EKR property. If $n = 1$ then $U_6 \cong S_3$ and trivially S_3 has strict EKR property, but for $n = 2$ the group U_{12} does not have EKR property, since $S = \{\mathrm{id}, a, b, b^2, ab, ab^2\}$ is an independent set.

(b) In this case, we first prove that $\langle C \rangle = \langle b, a^{2r} \rangle$, where r is the greatest common divisor of all numbers k and $n-k$, where $k \in F$. Choose m such that $k = mr$. Thus, $a^{2k}b = a^{2mr}b = (a^{2r})^m b \in \langle b, a^{2r} \rangle$ which implies that $\langle C \rangle \subseteq \langle b, a^{2r} \rangle$. Conversely, since $Z(U_{6n}) = \langle a^2 \rangle$, we have $(a^{2k}b)(a^{2k}b^2)^{-1} = b^{-1}$. Hence $b \in \langle C \rangle$. On the other hand, by the form of elements in C, $a^{2k} \in \langle C \rangle$ and by definition of r, $a^{2r} \in \langle C \rangle$. This shows that $\langle C \rangle = \langle b, a^{2r} \rangle$.

Next $\langle a^{2r} \rangle \subseteq Z(U_{6n})$ and so $\langle a^{2r} \rangle \trianglelefteq U_{6n}$. This proves that $|\langle C \rangle| = |\langle b, a^{2r} \rangle| = \frac{3n}{(n,r)}$ and so $|U_{6n} : \langle C \rangle| = 2(n, r)$. Therefore, the number of connected components of $Cay(U_{6n}, C)$ is equal to $2(n, r)$.

(c) Suppose $a^i b^j$ and $a^r b^s$ are independent with respect to C. Thus $a^i b^j (a^r b^s)^{-1} \notin C$. Since

$$\begin{aligned} ba^{-r} &= \begin{cases} a^{-r}b^{-1} & r \text{ is odd} \\ a^{-r}b & r \text{ is even} \end{cases} \quad \text{and} \\ b^{-1}a^{-r} &= \begin{cases} a^{-r}b & r \text{ is odd} \\ a^{-r}b^{-1} & r \text{ is even} \end{cases}, \end{aligned}$$

we have

$$\begin{aligned} a^ib^j(a^rb^s)^{-1} &= \begin{cases} a^{i-r}b^{s-j} & r \text{ is odd} \\ a^{i-r}b^{j-s} & r \text{ is even} \end{cases} \quad \text{and} \\ a^r(a^sb)^{-1} &= a^r(a^sb^2)^{-1} = (a^rb)(a^sb^2)^{-1} \\ &= \begin{cases} a^{r-s}b & r \text{ is odd} \\ a^{r-s}b^{-1} & r \text{ is even} \end{cases}. \end{aligned}$$

Hence, if $C = (a^{2k}b)^{U_{6n}} \cup (a^{2n-2k}b)^{U_{6n}}$ then $A = \{a^i, a^ib, a^ib^2\}$ is a maximal independent set with respect to C if and only if $0 \leq i,j \leq 2n-1$ and $i-j \neq 2k, 2n-2k$. Define W to be a maximal set of non-negative integers i such that $0 \leq i \leq 2n-1$ and difference between two arbitrary members of W are different from $2k$ and $2n-2k$. Suppose r is a divisor of k. Then $|W| = n$ provided that $n \equiv 4j \pmod{2k}$, $4j \neq \frac{k}{r}$ and $0 \leq 4j \leq 2k-1$. In general, $|W| = n - \frac{k}{r}$ provided that $n \equiv \frac{ik}{r} \pmod{2k}$, i is odd, $\frac{ik}{r} \leq 2k-1$ and if $s < r$ is another divisor of k then $\frac{jk}{s} \neq \frac{ik}{r}$, for all odd j. If $3|n$ and $k = \frac{n}{3}$ then $|W| = n-k$ and so $|A| = 3|W| = 2n$ which shows that U_{6n} has EKR property with respect to C. Conversely, if U_{6n} has EKR property with respect to C then $|A| = 2n$ and so $|W| = \frac{2n}{3}$. Then by definition $n = 3k$.

(d) We first assume that U_{6n} has EKR property with respect to C. Now by Part 4(c) of this theorem we have $n = 3k$. So $r = (2k, 4k) = 2k$, $o(a^{4k}) = 3$ and by Part 4(b) we have $|\langle C\rangle| = 9$. Conversely, we assume that $|\langle C\rangle| = 9$. Then $o(a^{2r}) = \frac{n}{(n,r)} = 3$ and so $n = 3(n,r) = 3k$. We now apply 4(c) of this theorem to conclude that U_{6n} has EKR property with respect to C.

(5) $G = SD_{8n}$. We use a permutation representation for this group given by Hormozi and Rodtes [11]. Suppose $\bar{m} = m \pmod{4n}$, $a = (1, 2, \ldots, 4n)$ and $b = \prod_{i \in C_*^{even}}(i, \overline{(2n-1)i})$, when n is even, and if n is odd, then we define $b = \prod_{i \in C_*^{odd}}(i, \overline{(2n-1)i})$. If n is even then each element $a^{2l}b$, $1 \leq l \leq n-1$, has exactly two fixed points in the set $Z_1 = \{1, 2, \ldots, 4n\} \setminus \{2n, 4n\}$. Conversely, each point of Z_1 is a fixed point of a unique element in the form of $a^{2l}b$, $1 \leq l \leq n-1$. Thus, $|G_{2n}| = |G_{4n}| = 1$ and for any element in Z_1, $|G_1| = |G_2| = \ldots = |G_{2n-1}| = |G_{2n+1}| = \ldots = |G_{4n-1}| = 2$. Obviously, $\{\mathrm{id}, a^{2s}b\}$ is an independent set in Γ_G and so $\alpha(\Gamma_G) \geq 2$. Suppose S is a maximal independent set and without lose of generality we can assume that $\mathrm{id} \in S$. It is clear that $a^{2s}b \in S$. Since $b(2n) = 2n$ and $b(4n) = 4n$, $b \notin S$. We now assume that $a^{2l}b \in S$. Then there exists a point m such that $a^{2s}b(m) = a^{2l}b(m)$ and so $a^{2(s-l)}(b(m)) = b(m)$, which is impossible. A similar argument shows that $a^{2s+1}b \notin S$. Therefore, $\alpha(\Gamma_G) = |S| = 2$. If n is odd then each element $a^{4l}b$, $1 \leq l \leq n-1$, has exactly four fixed points in the set $Z_2 = \{1, 2, \ldots, 4n\} \setminus \{n, 2n, 3n, 4n\}$. Conversely, each point of Z_2 is a fixed point of a unique element in the form of $a^{4l}b$, $1 \leq l \leq n-1$. Thus, $|G_n| = |G_{2n}| = |G_{3n}| = |G_{4n}| = 1$ and for any element $i \in Z_2$, $|G_i| = 2$. By a similar argument as in the case that n is even, one can prove $\alpha(\Gamma_G) = 2$. This shows that H has EKR property.

This completes the proof. □

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