**S(e)MR** ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

*Том 13, стр. 1258–1270 (2016)* УДК 519.1

DOI 10.17377/semi.2016.13.098 MSC 05C25, 05E10, 05C50, 05E15

Special issue: Graphs and Groups, Spectra and Symmetries — G2S2 2016

# MULTIPLICITIES OF EIGENVALUES OF THE STAR GRAPH

S.V. AVGUSTINOVICH, E.N. KHOMYAKOVA, E.V. KONSTANTINOVA

Abstract. The Star graph $S_n$, $n \geqslant 2$, is the Cayley graph on the symmetric group $\mathrm{Sym}_n$ generated by the set of transpositions $\{(1\ 2), (1\ 3), \ldots, (1\ n)\}$. We consider the spectrum of the Star graph as the spectrum of its adjacency matrix. It is known that the spectrum of $S_n$ is integral. Analytic formulas for multiplicities of eigenvalues $\pm(n-k)$ for $k = 2, 3, 4, 5$ in the Star graph are given in this paper. We also prove that any fixed integer has multiplicity at least $2^{\frac{1}{2} n \log n (1-o(1))}$ as an eigenvalue of $S_n$.

**Keywords:** Cayley graph, Star graph, symmetric group, graph spectrum, eigenvalues, multiplicity

## 1. Introduction

*The Star graph* $S_n = \mathrm{Cay}(\mathrm{Sym}_n, t)$, $n \geqslant 2$, is the Cayley graph on the symmetric group $\mathrm{Sym}_n$ of permutations $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ with the generating set $t = \{(1\ i) \in \mathrm{Sym}_n : 2 \leqslant i \leqslant n\}$ of all transpositions $(1\ i)$ swapping the 1st and $i$th elements of a permutation $\pi$.

It is a connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ [2]. Since this graph is bipartite it does not contain odd cycles but it does contain $l$–cycles for all even l, where $6 \leqslant l \leqslant n!$ (with the sole

Avgustinovich, S.V., Khomyakova, E.N., Konstantinova, E.V., Multiplicities of eigenvalues of the Star graph.



The second author is supported by RFBR Grant 15-01-05867; the third author is supported by RFBR Grant 15-01-07906 and International Mathematical Center of Novosibirsk State University.

*Received September, 28, 2016, published December, 23, 2016.*

exception when $l = 4$) [6]. The hamiltonicity of this graph follows from results by V. Kompel'makher and V. Liskovets [8] and by P. J. Slater [11].

We consider the spectrum of the Star graph as the spectrum of its adjacency matrix. In 2009 A. Abdollahi and E. Vatandoost conjectured [1] that the spectrum of $S_n$ is integral, moreover it contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an eigenvalue of $S_n$). For $n \leqslant 6$ they verified this conjecture numerically using GAP.

In 2012 R. Krakovski and B. Mohar [9] proved that the spectrum of $S_n$ is integral. Moreover, they showed that for $n \geqslant 2$ and for each integer $1 \leqslant k \leqslant n-1$ the values $\pm(n-k)$ are eigenvalues of the Star graph $S_n$ with multiplicity at least $\binom{n-2}{k-1}$. If $n \geqslant 4$, then 0 is an eigenvalue of $S_n$ with multiplicity at least $\binom{n-1}{2}$. Since the Star graph is bipartite, the spectrum of the Star graph is symmetric and $\mathrm{mul}(n-k) = \mathrm{mul}(-n+k)$ for each integer $1 \leqslant k \leqslant n$ [3]. Let us also note that $\pm(n-1)$ is a simple eigenvalue of $S_n$.

At the same time, G. Chapuy and V. Feray [4] showed another approach to obtain the exact values of multiplicities of eigenvalues of $S_n$. Their combinatorial approach is based on the Jucys–Murphy elements and the standard Young tableaux. In particular, they gave the following lower bound on multiplicities of eigenvalues of the Star graph:

$$\mathrm{mul}(n-k) \geqslant \binom{n-2}{n-k-1}\binom{n-1}{n-k}. \tag{1}$$

In 2015 this approach was used to obtain the exact values of multiplicities of eigenvalues of $S_n$ for $n \leqslant 10$ [7].

In this paper we present analytic formulas to calculate the multiplicities of eigenvalues of the Star graph. Hereinafter, we consider the asymptotic behavior of the function $\mathrm{mul}(n-k)$ for sufficiently large $n$.

**Theorem 1.** *The multiplicities* $\mathrm{mul}(n-k)$*, where* $k = 2, 3, 4, 5$ *and* $n \geqslant 2k-1$*, of the eigenvalues* $(n-k)$ *of the Star graph* $S_n$ *are given by the following formulas:*

$$\mathrm{mul}(n-2) = (n-1)(n-2); \tag{2}$$

$$\mathrm{mul}(n-3) = \frac{(n-3)(n-1)}{2}(n^2-4n+2); \tag{3}$$

$$\mathrm{mul}(n-4) = \frac{(n-2)(n-1)}{6}(n^4-12n^3+47n^2-62n+12); \tag{4}$$

$$\mathrm{mul}(n-5) = \frac{(n-2)(n-1)}{24}(n^6-21n^5+169n^4-647n^3+1174n^2-820n+60). \tag{5}$$

From Theorem 1, we immediately have that the Chapuy-Feray bound (1) achieved for $k = 2$.

The paper is organized as follows. Section 2 contains two subsections. First we give basic knowledge on the representation theory [10]. Then relationships between group representation and spectra of the Star graph are presented. We also show that the formula given by G. Chapuy and V. Feray for multiplicities of eigenvalues of $S_n$ can be rewritten using the Hook formula [5]. This new formula is used to prove Theorem 1 in Section 3. We give an improved lower bound on multiplicities of eigenvalues of the Star graph in Section 4.

## 2. Preliminaries

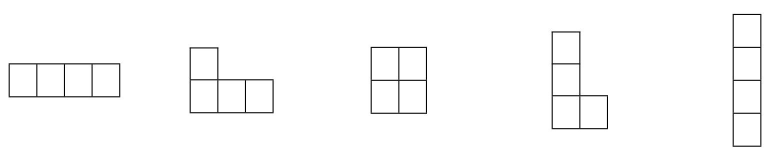
2.1. **Partitions and standard Young tableaux.** The symmetric group $\mathrm{Sym}_n$ consists of all bijections from $\{1, 2, \ldots, n\}$ to itself using compositions as the multiplication. Any permutation $\pi \in \mathrm{Sym}_n$ has the cycle type defined as the unordered list of the sizes of the cycles in the cycle decomposition of $\pi$. In this paper we consider a cycle type as a partition of $n$.

A *partition of a positive integer* $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \leqslant n$, of positive integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_l$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_l = n$. We write $\lambda \vdash n$ to denote that $\lambda$ is a partition of $n$.

As example, the number 4 has five partitions: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$. Partitions are represented by Young diagrams as follows[1].

A *Young diagram* is a finite collection of $n$ boxes arranged in left-justified rows, with the row lengths in non-increasing order. The Young diagram $[\lambda]$ associated to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is the one that has $l$ rows and $\lambda_i$ boxes on the $i$th row. Let us set $[\lambda] = \{(i, j) : 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant \lambda_i\}$, where the index $i$ gives the row and the index $j$ gives the column of a box with numbering rows from bottom to top, and columns from left to right.

As example, the Young diagrams corresponding to the partition of 4 are:



(4) (3, 1) (2, 2) (2, 1, 1) (1, 1, 1, 1)

Let $\lambda$ is a partition of $n$. Then a *Young tableau of shape* $\lambda$ is obtained by filling in the boxes of a Young diagram of $\lambda$ with $1, 2, \ldots, n$, where each number occurs exactly once. Thus, each box $(i, j) \in [\lambda]$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant \lambda_i$, is labeled by the unique integer $m \in \{1, 2, \ldots, n\}$ and we put $c(m) = i - j$.

A *standard Young tableau* is a Young tableau in which the numbers appear in ascending order within each row and each column from left to right and bottom to top.

Let $[\lambda]$ be a Young diagram. For a box $(i, j) \in [\lambda]$, we define the *hook* of $(i, j)$ to be the set of all boxes directly to the right of $(i, j)$ and directly above $(i, j)$, including $(i, j)$ itself. The number of boxes in the hook is called the *hook length of* $(i, j)$ and is denoted by $h_{ij}$.

We write $\lambda'$ for the *conjugate partition* of $\lambda$ defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{l'})$, where $l' = \lambda_1$, $\lambda'_j = max\{i : (i, j) \in [\lambda]\}$, $1 \leqslant j \leqslant l'$. In other words, $(i, j) \in [\lambda]$ if and only if $(j, i) \in [\lambda']$. Then, the hook length $h_{ij}$ is defined by the following formula [5]:

$$h_{ij} = \lambda_i - j + \lambda'_j - i + 1. \tag{6}$$

[1] The notation used here is known as the *French notation*. There is also the *English notation*, which is the upside-down form of the French notation.

A *Hook tableau* $H_\lambda$ is a tableau listing the hook length of each box in the Young diagram of shape $\lambda$.

2.2. **Group representations and multiplicities of eigenvalues of $S_n$.** Let $G$ be a group and $V$ be a vector space over the complex numbers and of finite dimension. Let $GL(V)$ stands for the set of all invertible linear transformations of $V$ to itself. Then a *representation* of a group $G$ on a vector space $V$ is a group homomorphism $\rho : G \to GL(V)$. Usually, $V$ is called the *representation space* and the dimension of $V$ is called the dimension $\dim(V)$ of the representation. Moreover, it is referred to $V$ itself as the representation if the group is clear. An *irreducible representation* of a group is a group representation that has no nontrivial invariant subspaces. Given two vector spaces $V_1$ and $V_2$, two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are *equivalent* if there exists a vector space isomorphism $M : V_1 \to V_2$ so that $M\rho_2(g) = \rho_1(g)M$ for all $g \in G$.

The general theory of group representation is applied to the symmetric group $\mathrm{Sym}_n$. The inequivalent irreducible representations of $\mathrm{Sym}_n$ are conveniently indexed by the partitions of $n$. We denote by $V_\lambda$ the irreducible representation associated with the partition $\lambda \vdash n$.

The following result associates the representation of the symmetric group with the multiplicities of eigenvalues of the Star graph $S_n$.

**Theorem 2.** [4] *In the Star graph $S_n$ the multiplicity $\mathrm{mul}(n-k)$, $1 \leqslant k \leqslant n-1$, is given by the following formula:*

$$\mathrm{mul}(n-k) = \sum_{\lambda \vdash n} \dim(V_\lambda) I_\lambda(n-k), \tag{7}$$

*where $\dim(V_\lambda)$ is the dimension of an irreducible representation, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape $\lambda$, satisfying $c(n) = n-k$.*

Let us note that the dimension of the irreducible representation $V_\lambda$ of the symmetric group $\mathrm{Sym}_n$ corresponding to a partition $\lambda$ of $n$ is equal to the number of different standard Young tableaux. This number can be calculated by the Hook Formula.

**Theorem 3. (Hook Formula)** [5] *Let $\lambda \vdash n$. Then,*

$$\dim(V_\lambda) = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}}. \tag{8}$$

Moreover, the number of standard Young tableaux of shape $\lambda$ such that $c(n) = n-k$ is also calculated by the Hook Formula:

$$I_\lambda(n-k) = \frac{(n-1)!}{\prod_{(i,j)\in[\lambda]} \hat{h_{ij}}}, \tag{9}$$

where $\hat{h_{ij}} = \begin{cases} \lambda_i - j + (\lambda'_j - 1) - i + 1, & j = 1,\ 1 \leqslant i \leqslant l-1; \\ \lambda_i - j + \lambda'_j - i + 1, & 1 < j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l-1. \end{cases}$

From (7), (8) and (9), we immediately get the following lemma.

**Lemma 1.** *In the Star graph $S_n$ the multiplicity* $\mathrm{mul}(n-k)$, $1 \leqslant k \leqslant n$, *is given by the following formula:*

$$\mathrm{mul}(n-k) = \sum_{\lambda \vdash n} \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}} \cdot \frac{(n-1)!}{\prod_{(i,j)\in[\lambda]} \hat{h_{ij}}}, \tag{10}$$

*where* $h_{ij} = \lambda_i - j + \lambda'_j - i + 1$ *and*

$$\hat{h_{ij}} = \begin{cases} \lambda_i - j + (\lambda'_j - 1) - i + 1, & j = 1,\ 1 \leqslant i \leqslant l-1; \\ \lambda_i - j + \lambda'_j - i + 1, & 1 < j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l-1. \end{cases}$$

Let us note that computing multiplicities of eigenvalues of the Star graph by formula (10) is highly simplified in comparison with formula (7), which requires the calculation of the irreducible representations $V_\lambda$ of the symmetric group $\mathrm{Sym}_n$ with calculating the number of partitions of $n$, enumerating all shapes of partition and processing each shapes individually. Formula (10) eliminates these steps, offering only to calculate the hook length of boxes.

## 3. Proof of Theorem 1

We use formula (10) and the following general approach to prove formulas (2) – (5) in Theorem 1 for multiplicities $\mathrm{mul}(n-k)$, where $k = 2, 3, 4, 5$, of the eigenvalues $(n-k)$. By theorem 2, we have $c(n) = n-k$ for the eigenvalues $(n-k)$ of the Star graph $S_n$. Thus, we need to get Young diagrams $[\lambda_i]$ of the shapes $\lambda_i$ for which standard Young tableaux satisfy $c(n) = n-k$.

**Proposition 1.** *For eigenvalues $(n-k)$, $n \geqslant 2k-1$, of the Star graph the number $N$ of Young diagrams $[\lambda_i]$, $1 \leqslant i \leqslant N$, is equal to the number of partitions of a positive integer $(k-1)$ for $k \geqslant 2$.*

*Proof.* The proof is a straightforward. For eigenvalues $(n-k)$ we have $c(n) = n-k$. For $n \geqslant 2k-1$, such Young diagrams contain $n-k+1$ boxes in the first column. The remaining $k-1$ boxes are placed to the right of the first column and form a Young diagram of volume $k-1$. Since $k-1 \leqslant n-k+1$, every such diagram can appear in the right part; their number is the number of partitions of $k-1$. □

Thus, starting with $k = 2$ we have the following first values for $N$: 1, 2, 3, 5, 7, 11, 15, ... (sequence A000041 in the OEIS [12]). This gives us the numbers $N = 1, 2, 3, 5$ of Young diagrams for $k = 2, 3, 4, 5$, correspondingly. The structure of these Young diagrams in each of the cases comes from the condition $c(n) = n-k$ for standard Young tableaux. In each of the cases we use formulas (8) and (9) so that we calculate the Hook tableaux $H_{\lambda_i}$ based on the Young diagrams $[\lambda_i]$ and the hook lengths of all boxes in the $[\lambda_i]$ without the topmost and the leftmost box. Finally, we get the multiplicities of eigenvalues $(n-k)$ of the Star graph $S_n$ by formula (10). Now let us consider the cases.

Case 1: $\mathrm{mul}(n-2)$

Let us prove formula (2), considering the eigenvalue $(n-2)$ of the Star graph $S_n$. In this case $c(n) = n-2$. By Proposition 1 there is the only Young diagram $[\lambda]$ of the shape $\lambda = (2, 1, 1, \dots, 1)$ for which standard Young tableaux satisfy $c(n) = n-2$

(i.e. the number $n$ appears in the topmost and the leftmost box). Such Young diagram $[\lambda]$ has two columns, where the first column contains $(n-1)$ boxes, and the remaining box is placed in the second column (see Figure 1(a)).

To use formula (8) and (9), we need to get the Hook tableau $H_\lambda$ based on the Young diagram $[\lambda]$, where $\lambda = (2, 1, 1, \dots, 1)$. By formula (6), the hook length of a box $(i, j)$ is the number of boxes that are in the same row $i$ to the right of it plus the number of boxes in the same column $j$ above it, plus one (for the box itself). From the definition of the hook length, we immediately get that for 1-row and 2-column box we have $h_{12} = 1$. The lengths of the hooks for the boxes in the first column are calculated as follows:

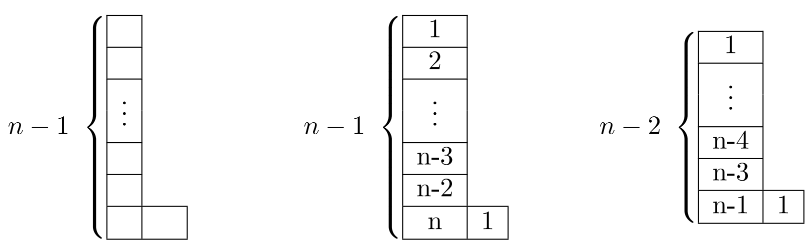
$$\begin{aligned}
&h_{11} = 1 + (n-2) + 1 = n,\\
&h_{21} = 0 + (n-3) + 1 = n-2,\\
&h_{31} = 0 + (n-4) + 1 = n-3,\\
&\dots\\
&h_{(n-1)1} = 0 + 0 + 1 = 1.
\end{aligned}$$

The Hook tableau $H_\lambda$ is presented in Figure 1(b), where each box contains the corresponding hook length of $(i, j)$. Thus,

$$\prod_{(i,j)\in[\lambda]} h_{ij} = 1 \cdot n(n-2)(n-3) \cdot \ldots \cdot 1 = n(n-2)!$$

and from (8) we get

$$\dim(V_\lambda) = \frac{n!}{n(n-2)!} = \frac{n(n-1)(n-2)!}{n(n-2)!} = (n-1).$$




(a) $[\lambda]$ (b) $H_\lambda$ (c) $\bar{H}_\lambda$

Figure 1. $\lambda = (2, 1, 1, \dots, 1)$

To obtain $I_\lambda(n-2)$, we need to calculate the hook length of all boxes in the Young diagram of the shape $\lambda = (2, 1, 1, \dots, 1)$ without the topmost and the leftmost box containing $n$. This diagram contains $(n-2)$ boxes in the first column, and it is unchanged in the second column. The corresponding Hook tableau $\bar{H}_\lambda$ for the Young diagram $[\lambda]$ without the topmost and the leftmost box is presented in Figure 1(c). Hence, from (9) we have:

$$I_\lambda(n-2) = \frac{(n-1)!}{1 \cdot (n-1)(n-3)!} = (n-2),$$

and from (10) we get:

$$\mathrm{mul}(n-2) = (n-1)(n-2)$$

for $n \geqslant 2k-1$, which gives us (2) in Theorem 1.

Case 2: $\mathrm{mul}(n-3)$

Now let us prove (3), considering the eigenvalue $(n-3)$ of the Star graph $S_n$. In this case $c(n) = n-3$. By Proposition 1 there are two Young diagrams $[\lambda_1]$ and $[\lambda_2]$ of the shapes $\lambda_1 = (2,2,1,\dots,1)$ and $\lambda_2 = (3,1,1,\dots,1)$ for which standard Young tableaux satisfy $c(n) = n-3$ (i.e. the number $n$ appears in the topmost and the leftmost box in the both shapes $\lambda_1$ and $\lambda_2$). The Young diagrams $[\lambda_1]$ and $[\lambda_2]$ and the Hook tableaux $H_{\lambda_1}$ and $H_{\lambda_2}$ are presented in Figure 2(a, b) and Figure 3(a, b), correspondingly. Thus, by formulas (6) and (8) for $\lambda_1$ we have:

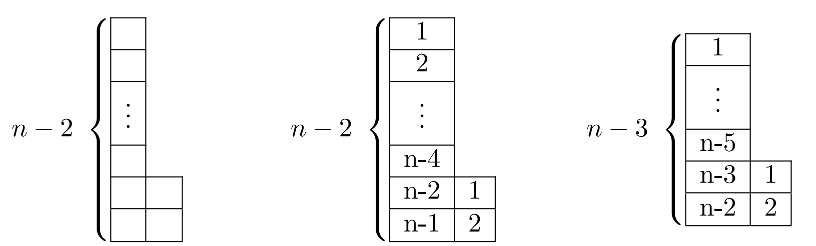
$$\prod_{(i,j)\in[\lambda_1]} h_{ij} = 1 \cdot 2 \cdot (n-1)(n-2)(n-4) \cdot \ldots \cdot 1 = 2(n-1)(n-2)(n-4)!,$$

$$\dim(V_{\lambda_1}) = \frac{n!}{2(n-1)(n-2)(n-4)!} = \frac{n(n-3)}{2},$$

and for $\lambda_2$ we have:

$$\prod_{(i,j)\in[\lambda_2]} h_{ij} = 1 \cdot 2 \cdot n(n-3)(n-4) \cdot \ldots \cdot 1 = 2n(n-3)!,$$

$$\dim(V_{\lambda_2}) = \frac{n!}{2n(n-3)!} = \frac{(n-1)(n-2)}{2}.$$



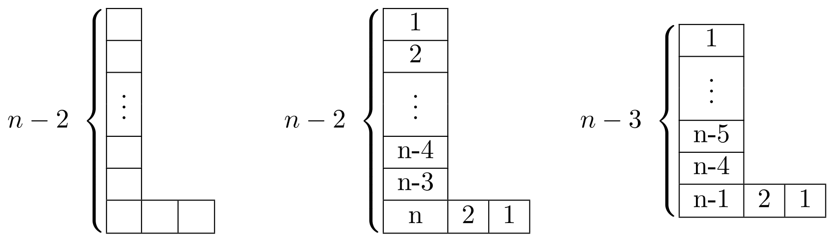

(a) $[\lambda_1]$ (b) $H_{\lambda_1}$ (c) $\bar{H}_{\lambda_1}$

Figure 2. $\lambda_1 = (2,2,1,\dots,1)$

Let us find $I_{\lambda_1}(n-3)$ and $I_{\lambda_2}(n-3)$ (see Figure 2(c) and Figure 3(c), correspondingly). By formula (9),

$$I_{\lambda_1}(n-3) = \frac{(n-1)!}{1 \cdot 2 \cdot (n-2)(n-3)(n-5)!} = \frac{(n-1)(n-4)}{2},$$

$$I_{\lambda_2}(n-3) = \frac{(n-1)!}{1 \cdot 2 \cdot (n-1)(n-4)!} = \frac{(n-2)(n-3)}{2}.$$




(a) $[\lambda_2]$ (b) $H_{\lambda_2}$ (c) $\bar{H}_{\lambda_2}$

Figure 3. $\lambda_2 = (3, 1, 1, \ldots, 1)$

Finally, from (10) we have:

$$\mathrm{mul}(n-3) = \frac{n(n-3)}{2} \cdot \frac{(n-1)(n-4)}{2} \cdot \frac{(n-1)(n-2)}{2} \cdot \frac{(n-2)(n-3)}{2} = \\ = \frac{(n-3)(n-1)}{2}(n^2 - 4n + 2)$$

for $n \geqslant 2k - 1$, which gives us (3) in Theorem 1.

Cases 3 and 4: $\mathrm{mul}(n-4)$ and $\mathrm{mul}(n-5)$

Formulas (4) and (5) are proved similarly. The Young diagrams of the corresponding shapes $\lambda$, the Hook tableaux $H_\lambda$ and $\bar{H}_\lambda$ are presented in Appendix.

In the case 3, when $c(n) = n - 4$, by Proposition 1 there are three Young diagrams of shapes $\lambda_1 = (4, 1, 1, \ldots, 1)$, $\lambda_2 = (3, 2, 1, \ldots, 1)$, $\lambda_3 = (2, 2, 2, 1, \ldots, 1)$ (see Figures 5–7 in Appendix). Using data presented in Appendix for the eigenvalue $(n - 4)$ of the Star graph, and by formulas (6), (8) and (9) we get the following results for the considered shapes:

| | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|
| $\dim(V_\lambda)$ | $\frac{x_1x_2x_3}{6}$ | $\frac{x_0x_2x_4}{3}$ | $\frac{x_0x_1x_5}{6}$ |
| $I_\lambda(n-4)$ | $\frac{x_2x_3x_4}{6}$ | $\frac{x_1x_3x_5}{3}$ | $\frac{x_1x_2x_6}{6}$ |

where $x_i = (n - i)$ for each $0 \leqslant i \leqslant 6$. Thus, by formula (10) we have:

$$\mathrm{mul}(n-4) = \frac{(n-2)(n-1)}{6}(n^4 - 12n^3 + 47n^2 - 62n + 12)$$

for $n \geqslant 2k - 1$, which gives us (4) in Theorem 1.

In the case 4, when $c(n) = n - 5$, by Proposition 1 there are five Young diagrams of shapes $\lambda_1 = (5, 1, 1, \ldots, 1)$, $\lambda_2 = (4, 2, 1, \ldots, 1)$, $\lambda_3 = (3, 3, 1, \ldots, 1)$, $\lambda_4 = (3, 2, 2, 1, \ldots, 1)$, $\lambda_5 = (2, 2, 2, 2, 1, \ldots, 1)$ (see Figures 8–12 in Appendix). Using data presented in Appendix for the eigenvalue $(n - 5)$ of the Star graph, and by formulas (6), (8) and (9) we get the following results for the considered shapes:

| | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|---|---|---|---|---|---|
| $\dim(V_\lambda)$ | $\frac{x_1x_2x_3x_4}{24}$ | $\frac{x_0x_2x_3x_5}{8}$ | $\frac{x_0x_1x_4x_5}{12}$ | $\frac{x_0x_1x_3x_6}{8}$ | $\frac{x_0x_1x_2x_7}{24}$ |
| $I_\lambda(n-5)$ | $\frac{x_2x_3x_4x_5}{24}$ | $\frac{x_1x_3x_4x_6}{8}$ | $\frac{x_1x_2x_5x_6}{12}$ | $\frac{x_1x_2x_4x_7}{8}$ | $\frac{x_1x_2x_3x_8}{24}$ |

where $x_i = (n-i)$ for each $0 \leqslant i \leqslant 8$. Hence, by formula (10) we have:

$$\mathrm{mul}(n-5) = \frac{(n-2)(n-1)}{24}(n^6 - 21n^5 + 169n^4 - 647n^3 + 1174n^2 - 820n + 60)$$

for $n \geqslant 2k-1$, which gives us (5) in Theorem 1. □

## 4. Lower bound on multiplicity of eigenvalues of the Star graph

In this section we improve (1) using standard Young tableaux (SYT). Let us put $t = n - k$.

**Theorem 4.** *In the Star graph $S_n$ for sufficiently large $n$ and for any fixed integer $t$ the multiplicity* $\mathrm{mul}(t)$ *of the eigenvalue $t$ is at least* $2^{\frac{1}{2}n\log n(1-o(1))}$.

*Proof.* Let us consider SYT of size $(k+t)\times k$. Let $i = k+t$ is the number of rows and $j = k$ is the number of columns. Then, $n$ appears in the rightmost and topmost box and $c(n) = i - j = t$. From (7) we have:

$$\mathrm{mul}(t) = \sum_{\lambda \vdash n} \dim(V_\lambda) I_\lambda(t). \tag{11}$$

Since $\dim(V_\lambda) \geqslant I_\lambda(t)$, then

$$\sum_{\lambda \vdash n} \dim(V_\lambda) I_\lambda(t) \geqslant I_\lambda^2(t). \tag{12}$$

Let us note that in (11) for any $t$ there are SYT of size $(k+t)\times k$ such that they have subtableaux $T$ of size $(k+1)\times k$, where $k \simeq n^{\frac{1}{2}}$. The number of tableaux $I_\lambda(t)$ is at least the number of a standard filling of the subtableaux $T$ with the elements $\{1, 2, \ldots, (k+1)k\}$ (see Figure 4).

| | | | | |
|---|---|---|---|---|
| | | $\ldots$ | $(k+1)k-1$ | $(k+1)k$ |
| | | | | $(k+1)k-2$ |
| | | | | $\vdots$ |
| $\vdots$ | | | | |
| 2 | | | | |
| 1 | 3 | $\ldots$ | | |

Figure 4. The subtableaux $T$

We consider the main diagonal and all diagonals above and below the main one in $T$. Then, $(k+1)k$ appears there in the rightmost and topmost box, and the element $(k+1)k-1$ can appear to the left or below of $(k+1)k$. The same we have for the element $(k+1)k-2$. So, permuting $(k+1)k-1$ with $(k+1)k-2$ along with the corresponding diagonal we get 2! different SYT. In general, the number of subtableaux $T$ is equal to $(1!2!\ldots(k-1)!k!k!(k-1)!\ldots 2!1!)$, when we permute elements along with corresponding diagonal. From (11) and (12), we get:

$$\mathrm{mul}(t) \geqslant (1!2!3!\ldots(k-1)!k!)^4 = \Big(\prod_{i=1}^{k} i!\Big)^4 \geqslant \Big(\prod_{i=\frac{k}{2}}^{k} i!\Big)^4 \geqslant \Big(\Big(\frac{k}{2}\Big)!\Big)^{2k}.$$

Representing Stirling's formula as:

$$n! \simeq n^{n(1-o(1))},$$

we have

$$\left(\left(\frac{k}{2}\right)!\right)^{2k} \geqslant \left(\left(\frac{k}{2}\right)^{\frac{k}{2}(1-o(1))}\right)^{2k} = \left(\frac{k}{2}\right)^{k^2(1-o(1))}.$$

Since $n \geqslant (k+1)k$, then $k \simeq n^{\frac{1}{2}}$ and we get:

$$\left(\frac{k}{2}\right)^{k^2(1-o(1))} = \left(\frac{n^{\frac{1}{2}}}{2}\right)^{n(1-o(1))} = \left(\frac{1}{2}\right)^{n(1-o(1))} n^{\frac{1}{2}n(1-o(1))} =$$

$$= 2^{-n(1-o(1))} 2^{\frac{1}{2}n\log n(1-o(1))} = 2^{\frac{1}{2}n\log n(1-\frac{2}{\log n})(1-o(1))} = 2^{\frac{1}{2}n\log n(1-o(1))}.$$

Finally, we have:

$$\mathrm{mul}(t) \geqslant 2^{\frac{1}{2}n\log n(1-o(1))},$$

which completes the proof of Theorem 4. Thus, for any fixed eigenvalue $t$ of the $S_n$ the order of logarithm of multiplicity $\mathrm{mul}(t)$ is the same that $n!$. □

## Acknowledgment

We thank the anonymous referee whose suggestions significantly improved the presentation of the paper.

## References


[1] A. Abdollahi, E. Vatandoost, *Which Cayley graphs are integral?* Electron. J. Comb., **16**:1 (2009), #122, 1–17. Zbl 1186.05064

[2] S.B. Akers, B. Krishnamurthy, *A group-theoretic model for symmetric interconnection networks*, IEEE Trans. Comput., **38**:4 (1989), 555–566. Zbl 0678.94026

[3] A.E. Brouwer, W.H. Haemers, *Spectra of Graphs*, Springer, New York, 2012. Zbl 1231.05001

[4] G. Chapuy, V. Féray, *A note on a Cayley graph of* $\mathfrak{S}_n$, arXiv:1202.4976v2 (2012).

[5] J.S. Frame, G. de B. Robinson, R.M. Thrall, *The hook graphs of the symmetric group*, Can. J. Math., **6** (1954), 316–324. Zbl 0055.25404

[6] J.S. Jwo, S.Lakshmivarahan, S.K. Dhall, *Embedding of cycles and grids in star graphs*, J. Circuits Syst. Comput., **1**:1 (1991), 43–74.

[7] E.N. Khomyakova, E.V. Konstantinova, *Note on exact values of multiplicities of eigenvalues of the Star graph*, Sib. Electron. Math. Reports, **12** (2015), 92–100. Zbl 1341.05152

[8] V.L. Kompel'makher, V.A. Liskovets, *Successive generation of permutations by means of a transposition basis*, Kibertetika, **1975**:3 (1975), 17–21 (in Russian). Zbl 0365.05003

[9] R. Krakovski, B. Mohar, *Spectrum of Cayley graphs on the symmetric group generated by transpositions*, Linear Algebra Appl., **437**:3 (2012), 1033–1039. Zbl 1243.05149

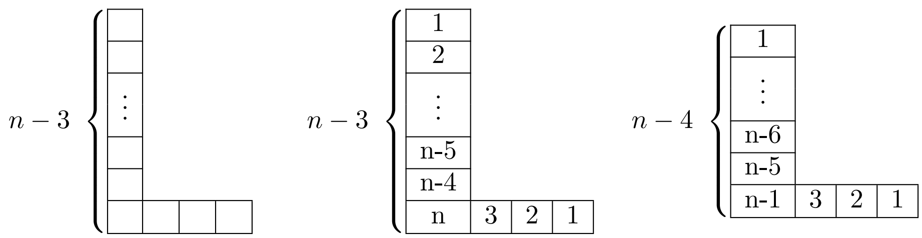
[10] B.E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed., Springer, New York, 2001. Zbl 0964.05070

[11] P.J. Slater, *Generating all permutations by graphical transpositions*, Ars Comb., **5** (1978), 219–225. Zbl 0405.05001

[12] The On-Line Encyclopedia of Integer Sequences. *Sequence A000041*. URL: https://oeis.org/A000041

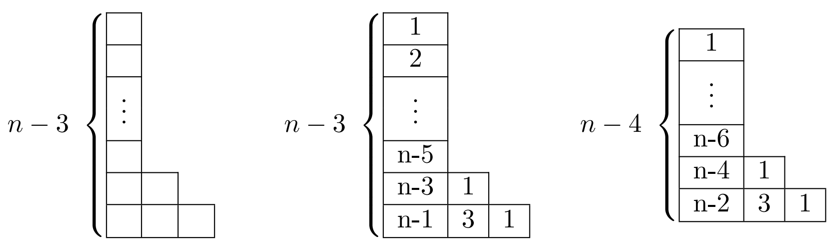
## Appendix

**Case 3.** $\mathrm{mul}(n-4)$



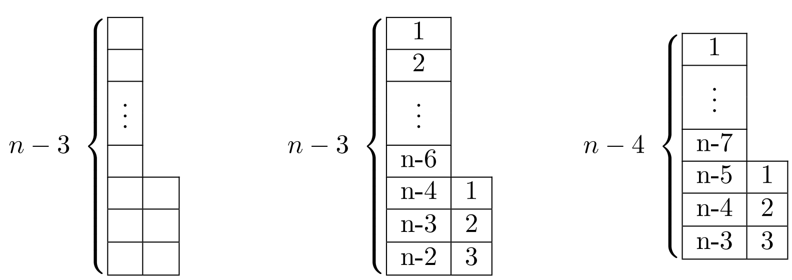

(a) $[\lambda_1]$ (b) $H_{\lambda_1}$ (c) $\bar{H}_{\lambda_1}$

Figure 5. $\lambda_1 = (4, 1, 1, \ldots, 1)$




(a) $[\lambda_2]$ (b) $H_{\lambda_2}$ (c) $\bar{H}_{\lambda_2}$

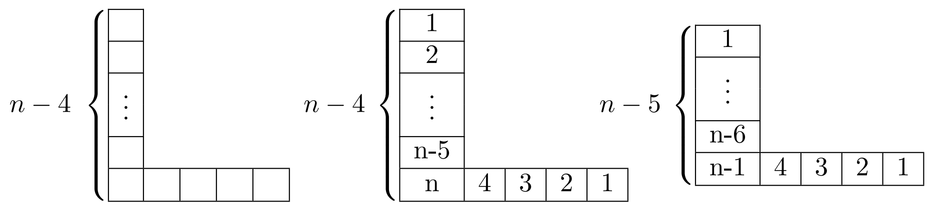
Figure 6. $\lambda_2 = (3, 2, 1, \ldots, 1)$




(a) $[\lambda_3]$ (b) $H_{\lambda_3}$ (c) $\bar{H}_{\lambda_3}$

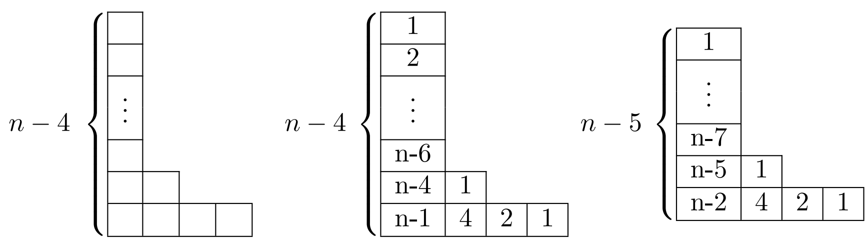
Figure 7. $\lambda_3 = (2, 2, 2, 1, \ldots, 1)$

**Case 4.** mul$(n-5)$



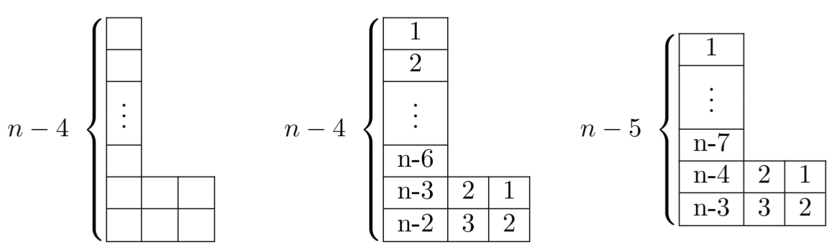

(a) $[\lambda_1]$ (b) $H_{\lambda_1}$ (c) $\bar{H}_{\lambda_1}$

Figure 8. $\lambda_1 = (5, 1, 1, \ldots, 1)$



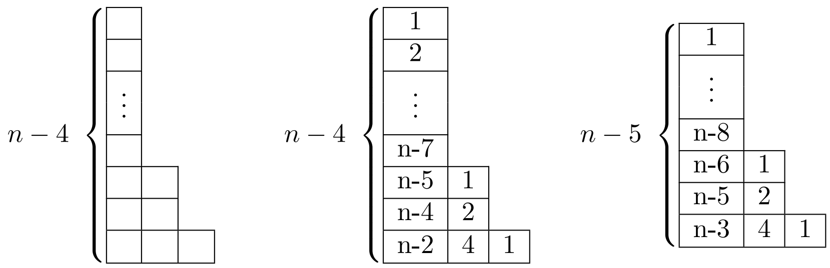

(a) $[\lambda_2]$ (b) $H_{\lambda_2}$ (c) $\bar{H}_{\lambda_2}$

Figure 9. $\lambda_2 = (4, 2, 1, \ldots, 1)$



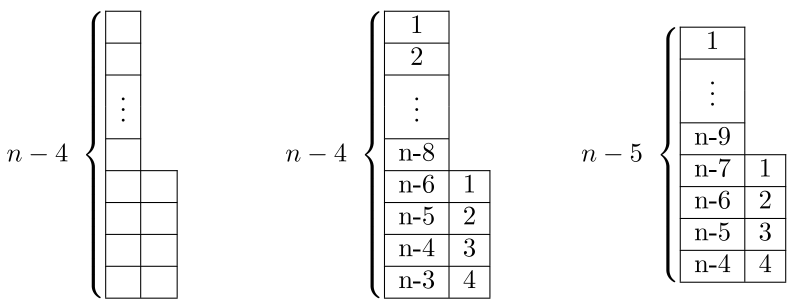

(a) $[\lambda_3]$ (b) $H_{\lambda_3}$ (c) $\bar{H}_{\lambda_3}$

Figure 10. $\lambda_3 = (3, 3, 1, \ldots, 1)$




(a) $[\lambda_4]$ (b) $H_{\lambda_4}$ (c) $\bar{H}_{\lambda_4}$

Figure 11. $\lambda_4 = (3, 2, 2, 1, \ldots, 1)$




(a) $[\lambda_5]$ (b) $H_{\lambda_5}$ (c) $\bar{H}_{\lambda_5}$

Figure 12. $\lambda_5 = (2, 2, 2, 2, 1, \ldots, 1)$

Sergey V. Avgustinovich
Sobolev Institute of Mathematics,
pr. Koptyuga 4,
630090, Novosibirsk, Russia
*E-mail address*: `avgust@math.nsc.ru`

Ekaterina N. Khomyakova
Novosibisk State University,
Pirogova, 2,
630090, Novosibirsk, Russia
*E-mail address*: `ekhomnsu@gmail.com`

Elena V. Konstantinova
Sobolev Institute of Mathematics,
pr. Koptyuga 4,
630090, Novosibirsk, Russia
Novosibisk State University,
Pirogova, 2,
630090, Novosibirsk, Russia
*E-mail address*: `e_konsta@math.nsc.ru`