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ON THE SPECTRUM OF CAYLEY GRAPHS

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Abstract. Let p and q are prime numbers and $q > p > 2$. In the current paper, we determine the spectra of Cayley graphs of groups of order p^2q in terms of their character table.

Keywords: Cayley graph, character table, spectrum of graph.

1. Introduction

Throughout this paper all groups are assumed to be finite. There are many connections between graph spectra and other aspects of mathematics such as diameter of graphs, automorphism group of graphs, hamiltonicity, etc. The aim of this paper is to investigate the spectrum of Cayley graphs of order p^2q $(p < q)$ via their character table, where $p, q > 2$ are distinct prime numbers, and the case $q < p$ will be solved in a subsequent paper. The most important works on the problem of computing the eigenvalues of Cayley graphs was done by Babai in 1979, see [2]. Babai used the methods based on the results of algebraic graph theory. In [7] authors proposed a formula for computing the spectrum of Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ with respect to the character table of G where S is a symmetric normal subset of G. The main results of this paper are related to this formula. In the next section, we give the necessary definitions and some preliminary results. In section three, we introduce all groups of order p^2q and finally in section four, we determine the spectrum of Cayley graphs of order p^2q, where $p < q$ in terms of their character tables. Here, our notation is standard and mainly taken from the standard books of algebraic graph theory and representation theory such as [3, 4].

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2. Definitions and Preliminaries

A representation of group G is a homomorphism $\alpha : G \to GL(n, p^m)$ and the degree of α is n, where p is a prime number. Let $\varphi : G \to GL(n, p^m)$ be a representation with $\varphi(g) = \varphi_g$, the character $\chi_\varphi : G \to \mathbb{C}$ afforded by φ is defined as $\chi_\varphi(g) = tr(\varphi_g)$. An irreducible character is the character of an irreducible representation and the character χ is linear, if $\chi(1) = 1$. We denote the set of all irreducible characters of G by $Irr(G)$. The number of irreducible characters of G is equal to $|G/G^{'}|$ where $G^{'}$ denotes the commutator subgroup of G. A character table is a matrix whose rows and columns are correspond to the irreducible characters and the conjugacy classes of G, respectively.

Let G be a group, for every element $g \in G$, we denote the conjugacy class of g by g^G. Assume that N be a normal subgroup of G and $\widetilde{\chi}$ is a character of G/N, then the character χ of G which is given by $\chi(g) = \widetilde{\chi}(Ng)$ for all $g \in G$ is called the lift of $\widetilde{\chi}$ to G.

Let G and H be two finite groups, then the direct product group $G \times H$ is one whose the set of elements of $G \times H$ is the Cartesian product of the sets G, H and for $(g_1, h_1), (g_2, h_2) \in G \times H$ the related binary operation is defined as $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$.

Theorem 2.1 ([5]). *Let G and H be two finite groups with irreducible characters $\varphi_1, \varphi_2, \cdots, \varphi_r$ and $\eta_1, \eta_2, \cdots, \eta_s$, respectively. Then the direct product group $G \times H$ has exactly rs irreducible characters $\varphi_i\eta_j$, where $1 \leq i \leq r$ and $1 \leq j \leq s$. In particular, let $\mathcal{M}(G)$ and $\mathcal{M}(H)$ be character tables of G and H, respectively then the character table $\mathcal{M}(G \times H)$ is the Kronecker product $\mathcal{M}(G) \otimes \mathcal{M}(H)$.*

3. Main Results

Let $p > q$ be prime numbers where $q|p-1$. A Frobenius group of order pq has the following presentation:

$$F_{p,q} = \langle a, b : \ a^p = b^q = 1, b^{-1}ab = a^u \rangle \tag{1}$$

where u is an element of order n in multiplicative group $\mathbb{Z}_p^*$. Hölder in [6] introduced the presentation of groups of order p^2q. By using the results of [6], we can prove that all groups of order p^2q $(2 < p < q)$ are isomorphic with exactly one of the following presentations:

- $G_1 = \mathbb{Z}_{p^2q}$,
- $G_2 = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $G_3 = \mathbb{Z}_p \times F_{q,p}(p|q-1)$,
- $G_4 = F_{q,p^2}$ $(p^2|q-1)$,
- $G_5 = \langle a, b : a^{p^2} = b^q = 1, a^{-1}ba = b^z,\ z^p \equiv 1 \pmod{q} \rangle$ $(p|q-1)$.

In this section, we compute the character table of groups of order p^2q. The character table of cyclic and Frobenius groups can be found in [5]. On the other hand, the character table of product groups can be computed directly by using Theorem 2.1. Hence, it remains to compute the character table of group $G = G_5$. It is not difficult to prove that $G^{'} = \langle b \rangle$ and $Z(G) = \langle a^p \rangle$, where $Z(G)$ denotes the center of G.

Lemma 3.1. *The conjugacy classes of G are*

$$\begin{aligned}&\{1\},\{a^p\},\cdots,\{a^{p(p-1)}\},(b^{v_i})^G=\{b^{v_iu}\mid u\in U\},\ 1\leq i\leq (q-1)/p,\\ &(a^n)^G=\{a^nb^i\mid 1\leq i\leq q-1\},\ 1\leq n\leq p^2-1,p\nmid n,\\ &(a^{kp}b^{v_i})^G=\{a^{kp}b^{v_iu}\mid u\in U\},\ 1\leq i\leq (q-1)/p,1\leq k\leq p-1,\end{aligned}$$

where $U=\langle z\rangle$ is a subgroup of order p in $\mathbb{Z}_q^*$ and v_i's are distinct coset representative of U in $\mathbb{Z}_q^*$.

Proof. It is clear that G has p singleton conjugacy classes $\{1\}$, $\{a^p\}$,$\cdots$,$\{a^{p(p-1)}\}$. One can see that $a^{-j}b^va^j=b^{z^jv}$ and so b^v and b^{vu} are conjugate, for all $u\in U$. Let v_i be coset representatives of U in $\mathbb{Z}_q^*$. Hence, there are $(q-1)/p$ conjugacy classes $(b^{v_i})^G$ of size p, where $1\leq i\leq (q-1)/p$. Assume now that $1\leq n\leq p^2-1$ and $p\nmid n$, then for $1\leq j\leq q$, we have $b^ja^nb^{-j}=a^nb^{jz^n-j}$ and thus $|(a^n)^G|\geq q$. On the other hand, $|C_G(a)|\geq p^2$ and so $|(a^n)^G|\leq q$. This implies that

$$(a^n)^G=\{a^n,a^nb,a^nb^2,\cdots,a^nb^{q-1}\},\quad 1\leq n\leq p^2-1\ \ and\ \ p\nmid n.$$

Finally, one can prove easily that there are $(p-1)(q-1)/p$ conjugacy classes of size p as $(a^{kp}b^i)^G=\{a^{kp}b^{v_i},\cdots,a^{kp}b^{v_iz^{p-1}}\}$, where $1\leq k\leq p-1$ and $1\leq i\leq (q-1)/p$. This completes the proof.

Theorem 3.2. *Let $p,q(p<q)$ be distinct prime numbers, $0\leq m,n\leq p^2-1,p\nmid n,1\leq i,j\leq (q-1)/p,1\leq r,k\leq p-1,1\leq t\leq q-1$, $\tau=e^{\frac{2\pi i}{p}}$ and $\gamma=e^{\frac{2\pi i}{q}}$. Then all irreducible characters of G are as reported in Table 3.*

Proof. It follows from Lemma 3.1 that G has $p^2+(q-1)/p+(p-1)(q-1)/p$ irreducible characters and among them p^2 characters are linear, since $|G/G^{'}|=p^2$. On the other hand, $|G/G^{'}|=p^2$ and then all its irreducible characters are linear as $\tilde{\chi}_i:G/G^{'}\to\mathbb{C}$ with $\tilde{\chi}_i((a)^jG^{'})=\epsilon^{ij}$ where $\epsilon=e^{\frac{2\pi i}{p^2}}$ and $1\leq i,j\leq p^2$. By lifting these characters, we get p^2 linear characters $\chi_m(0\leq m\leq p^2-1)$ such that

$$\begin{aligned}\chi_m(a^{kp})&=\tilde{\chi}_m(a^{kp}G^{'})=\epsilon^{kpm},\ 0\leq k\leq p-1,\\ \chi_m(b^{v_i})&=\tilde{\chi}_m(b^{v_i}G^{'})=\tilde{\chi}_m(G^{'})=1,\ 1\leq i\leq (q-1)/p,\\ \chi_m(a^n)&=\tilde{\chi}_m(a^nG^{'})=\epsilon^{nm},\ p\nmid n\ \ and\ \ 1\leq n\leq p^2-1,\\ \chi_m(a^{kp}b^{v_i})&=\tilde{\chi}_m(a^{kp}b^{v_i}G^{'})=\tilde{\chi}_m(a^{kp}G^{'})=\epsilon^{kpm},\ 0\leq k\leq p-1.\end{aligned}$$

Let $H=Z(G)=\langle a^p\rangle$, it is not difficult to see that $G/H\cong F_{q,p}$ and so by using [5, Theorem 25.10] it has p linear characters and $(q-1)/p$ non-linear characters of degree p. Let us to show all non-linear characters of G by $\tilde{\varphi}_j$, where $1\leq j\leq (q-1)/p$. We can check that $\tilde{\varphi}_j(H)=p,\tilde{\varphi}_j(b^{v_i}H)=\sum_{u\in U}e^{\frac{2\pi iv_iv_ju}{q}},\tilde{\varphi}_j(b^ya^xH)=0,1\leq x\leq p-1$. By lifting them, we achieve $(q-1)/p$ irreducible characters of degree p of G as follows:

$$\begin{aligned}\varphi_j(a^{kp})&=\tilde{\varphi}_j(a^{kp}H)=\tilde{\varphi}_j(H)=p,\ 0\leq k\leq p-1,\\ \varphi_j(b^{v_i})&=\tilde{\varphi}_j(b^{v_i}H)=\sum_{u\in U}\gamma^{v_iv_ju},\\ \varphi_j(a^n)&=\tilde{\varphi}_j(a^nH)=\tilde{\varphi}_j(a^{kp+r}H)=0,\ 1\leq r\leq p-1,\ p\nmid n,\\ \varphi_j(a^{kp}b^{v_i})&=\tilde{\varphi}_j(a^{kp}b^{v_i}H)=\tilde{\varphi}_j(b^{v_i}H)=\sum_{u\in U}\gamma^{v_iv_ju}.\end{aligned}$$

Consider now the subgroup $K \leq G$ by the following presentation:

$$K = \langle a^p, b | \ (a^p)^p = b^q = 1, \ a^p b = b a^p \rangle \cong \langle a^p b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_q.$$

Since K is abelian, their irreducible characters are linear and then K has pq linear characters $\psi_r \xi_t : K \to \mathbb{C}$ with $\psi_r \xi_t(a^{kp} b^i) = \tau^{rk} \gamma^{ti}$, where $\tau = e^{\frac{2\pi i}{p}}$, $\gamma = e^{\frac{2\pi i}{q}}$, $0 \leq r, k \leq p-1$ and $0 \leq i, t \leq q-1$. For given $(r,t) \in \mathbb{Z}_p^* \times \mathbb{Z}_q^*$, the degree of induced character $\psi_r \xi_t \uparrow G$ is p. On the other hand, for $1 \leq i \leq q-1$, we have

$$\begin{aligned}
\psi_r \xi_t \uparrow G(a^{kp}) &= p^2 q (\frac{\psi_r(a^{kp})}{pq}) = p\tau^{rk}, \\
\psi_r \xi_t \uparrow G(b^{v_i}) &= \sum_{u \in U} \xi_t(b^{v_i u}) = \sum_{u \in U} \gamma^{v_i u t} = \sum_{j=0}^{p-1} \gamma^{v_i t z^j} \\
\psi_r \xi_t \uparrow G(a^n) &= 0, \ p \nmid n \quad (a^n \notin K) \\
\psi_r \xi_t \uparrow G(a^{kp} b^{v_i}) &= \sum_{u \in U} (\psi_r \xi_t)(a^{kp} b^{v_i}) = \tau^{rk} \sum_{u \in U} \gamma^{v_i u t},
\end{aligned}$$

where $0 \leq k \leq p-1$, $1 \leq r \leq p-1$ and $1 \leq t \leq q-1$. It is clear that $\psi_r \xi_{tz^i} \uparrow G = \psi_r \xi_t \uparrow G$ and so we get $(p-1)(q-1)/p$ characters of G with degree p. It is not difficult to see that these are all irreducible characters of G and the the proof is completed.

g	e	a^{kp}	b^{v_i}	a^n	$a^{kp} b^{v_i}$
χ_m	1	ϵ^{kpm}	1	ϵ^{mn}	ϵ^{kpm}
φ_j	p	p	$\sum_{u \in U} \gamma^{v_i v_j u}$	0	$\sum_{u \in U} \gamma^{v_i v_j u}$
$\eta_{r,t}$	p	$p\tau^{rk}$	$\sum_{j=0}^{p-1} \gamma^{v_i t z^j}$	0	$\tau^{rk} \sum_{u \in U} \gamma^{v_i u t}$

Table 3. The character table of group G_5.

4. Spectrum of Cayley graphs

A symmetric subset of group G is a subset $S \subseteq G$, where $1 \notin S$ and $S = S^{-1}$. Let G be a finite group with symmetric subset S. We recall that S is a normal subset if and only if $g^{-1} S g = S$, for all $g \in G$.

The Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ with respect to S is a graph whose vertex set is $V(\Gamma) = G$ and the vertex x is adjacent with y if and only if $yx^{-1} \in S$. It is a well-known fact that $\mathrm{Cay}(G, S)$ is connected if and only if S generates the group G, see [3].

Let Γ be a simple graph with the adjacency matrix $A(\Gamma)$. The characteristic polynomial $\chi_\lambda(\Gamma)$ of $A(\Gamma)$ is defined as $\chi_\lambda(\Gamma) = |\lambda I - A|$ and the roots of this polynomial are called the spectrum of graph Γ, see [4]. The study of spectrum of Cayley graphs is closely related to irreducible characters of G. The following theorem is implicitly contained in [7, 8].

Theorem 4.1. *Let G be a finite group with a normal symmetric subset S. Let A be the adjacency matrix of* $\mathrm{Cay}(G, S)$*. Then the eigenvalues of A are given by* $\left\{ [\frac{1}{\chi(1)} \sum_{s \in S} \chi(s)]^{\chi(1)^2} \right\}$, *where $\chi \in Irr(G)$.*

By a circulant matrix, we mean a square $n \times n$ matrix whose rows are a cyclic permutation of the first row. A circulant matrix with the first row $[c_0, c_1, \cdots, c_{n-1}]$ is denoted by $[[c_0, c_1, c_2, \cdots, c_{n-1}]]$. By a circulant graph, we mean a graph whose adjacency matrix is circulant. The spectrum of a circulant matrix plays a significant role in the study of spectrum of Cayley graphs of order p^2q. Here, we recall that for $\omega = e^{\frac{2\pi}{n}i}$, the $n-$th root of unity, all eigenvalues of circulant matrix $[[c_0, c_1, c_2, \cdots, c_{n-1}]]$ are given by

$$\lambda_j = c_0 + c_{n-1}\omega^j + c_{n-2}\omega^{2j} + \cdots + c_1\omega^{(n-1)j}, \quad 0 \le j \le n-1. \tag{2}$$

Let A, B are two arbitrary sets. In what follows assume that

$$\delta_A(B) = \begin{cases} 1 & A \subseteq B \\ 0 & A \not\subseteq B \end{cases}.$$

Let $C_g = g^G \cup (g^{-1})^G$. It is clear that every normal subset of G is the union of its conjugacy classes. In other words, if S is a symmetric normal subset of G, then $S \subseteq \cup_{g \in G} C_g$ and for every $\chi \in Irr(G)$, the corresponding eigenvalue of $\Gamma = \mathrm{Cay}(G, S)$ is

$$\lambda_\chi = \frac{1}{2\chi(1)} \sum_{g \in G} \sum_{s \in C_g} \delta_{C_g}(S)|C_g|[\chi(s) + \chi(s^{-1})].$$

Example 4.2. Consider the cyclic group $\mathbb{Z}_n$ in two following cases:
Case 1. n is odd, thus $C_i = \{x^i, x^{-i}\}$ $(1 \le i \le \frac{n-1}{2})$ are normal symmetric subsets of $\mathbb{Z}_n$ and so $S \subseteq \cup_{i=1}^{(n-1)/2} C_i$. For $0 \le j \le n-1$, $\chi_j(x^i) = \omega^{ij}$ are all irreducible characters of $\mathbb{Z}_n = \langle x \rangle$ and $\omega = e^{\frac{2\pi}{n}i}$. Hence

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}).$$

Case 2. n is even, hence all normal symmetric subsets are

$$C_i = \{x^i, x^{-i}\} \ (1 \le i \le \frac{n}{2} - 1) \ \textit{and} \ C_{\frac{n}{2}} = \{x^{n/2}\}.$$

Therefore, $S \subseteq \cup_{i=1}^{\frac{n}{2}} C_i$. Similar to the last case, we have

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n}{2}-1} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}) + (-1)^{\frac{n}{2}j}\delta_{C_{\frac{n}{2}}}(S).$$

The Cartesian product $\Gamma_1 \boxtimes \Gamma_2$ of two graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices $(u, v), (x, y) \in V(\Gamma_1 \boxtimes \Gamma_2)$ are adjacent if and only if either $u = x$ and $(v, y) \in E(\Gamma_2)$ or $(u, x) \in E(\Gamma_1)$ and $v = y$.

Theorem 4.3 ([4])**.** *Let $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_m$ be eigenvalues of graphs Γ_1 and Γ_2, respectively. Then, for $1 \le i \le n$ and $1 \le j \le m$, all eigenvalues of $\Gamma_1 \boxtimes \Gamma_2$ are $\lambda_i + \mu_j$.*

4.1. **Spectrum of Cayley graph of Frobenius group.** Consider the presentation of Frobenius group introduced in the begining of this section. Here, we compute the spectrum of Cayley graph $\Gamma = \mathrm{Cay}(F_{q,p}, S)$ where S is a minimal normal symmetric subset of $F_{q,p}$.

Theorem 4.4 ([5])**.** *The Frobenius group $F_{q,p}$ has precisely*

(1) *p linear characters $\chi_n (0 \leq n \leq p-1)$, where $\chi_n(a^x b^y) = \omega^{ny}$ for $0 \leq x \leq q-1$ and $0 \leq y \leq p-1$;*
(2) *r characters of degree p given by $\varphi_j(a^x b^y) = 0$, $(1 \leq y \leq p-1)$ and $\varphi_j(a^x) = \sum_{s\in S} \xi^{v_j s x}$, $(1 \leq j \leq r)$ where $v_1 T, \cdots, v_r T$ are the cosets in $\mathbb{Z}_q^*$ of the subgroup $T = \langle u \rangle$, $\xi = e^{2\pi i/p}$, $\omega = e^{2\pi i/q}$ and $r = (q-1)/p$.*

Corollary 4.5. *By notation of Theorem 4.4, for $1 \leq j \leq p-1$, we have $Spec(\Gamma) = \left\{[2q]^1, [q(\omega^j + \omega^{-j})]^1, [0]^{rp^2}\right\}$, where $S = \langle b^G \cup (b^{-1})^G \rangle$.*

Proof. Since $b^G = \{a^m b : 0 \leq m \leq q-1\}$, by putting $m = 0$, it follows that $b \in \langle b^G \rangle$ and consequently $a \in \langle b^G \rangle$. Hence, $F_{q,p} = \langle b^G \rangle$. Since $S^{-1} = S$, then necessarily $S = \langle b^G \cup (b^{-1})^G \rangle$. According to Theorem 4.1, for all $\chi \in Irr(F_{q,p})$ we have

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{s\in S} \chi(s) = \frac{|b^G|\chi(b) + |(b^{-1})^G|\chi(b^{-1})}{\chi(1)} = \frac{q(\chi(b) + \chi(b^{-1}))}{\chi(1)}.$$

4.2. **Spectrum of Cayley graphs of order p^2q.** In this section, we compute the spectrum of Cayley graphs on groups of order p^2q $(p < q)$.

Theorem 4.6 ([1])**.** *If $\Gamma_1 = \mathrm{Cay}(G, S_1)$ and $\Gamma_2 = \mathrm{Cay}(H, S_2)$, then $\Gamma = \mathrm{Cay}(G \times H, S)$ is Cayley graph if and only if $S = \{(s_1, 1_G), (1_H, s_2); s_1 \in S_1, s_2 \in S_2)\}$.*

Theorem 4.7. *The minimal normal symmetric generating subset of groups G_5 is $S = (a^{kp} b^{v_i})^G \cup ((a^{kp} b^{v_i})^{-1})^G$.*

Proof. The proof is straightforward.

Corollary 4.8. *Let $\Gamma_i = \mathrm{Cay}(G_i, S_i)$ where $G_1, \cdots, G_5$ are groups introduced in Section 1 and S_i be a minimal normal symmetric subset of G_i. Then*

(1) *All eigenvalues of Γ_1 are $[\omega^j + \omega^{-j}]^1$, where $\omega = e^{\frac{2\pi}{p^2 q} i}$ and $0 \leq j \leq p^2 q - 1$.*
(2) *All eigenvalues of Γ_2 are $[\alpha^j + \alpha^{-j} + \zeta^i + \zeta^{-i}]^1$, where $\alpha = e^{\frac{2\pi i}{pq}}$, $\zeta = e^{\frac{2\pi i}{p}}$, $0 \leq j \leq p-1$, $0 \leq i \leq pq-1$ and S is as defined in Theorem 4.7.*
(3) *All eigenvalues of Γ_3 are $[\xi^i + 2q]^1, [\xi^i + q(\alpha^j + \alpha^{-j})]^1, [\xi^i]^t$, where $t = p(q-1), \alpha = e^{\frac{2\pi i}{q}}$, $\xi = e^{\frac{2\pi i}{p}}$, $1 \leq j \leq p-1$, $0 \leq i \leq p-1$ and S is as defined in Theorem 4.7.*
(4) *All eigenvalues of Γ_4 are $[2q]^1, [q(\alpha^j + \alpha^{-j})]^1, [0]^t$, where $t = p^2(q-1), \alpha = e^{\frac{2\pi i}{p^2}}$, $1 \leq j \leq p^2 - 1$ and S is as defined in Theorem 4.7.*
(5) *All eigenvalues of Γ_5 are $[p]^1, [A + \bar{A}]^{p^2-1}, [B + \bar{B}]^{p(q-1)}, [D + \bar{D}]^{p(p-1)(q-1)}$, where A, B and D are given in Theorem 4.7.*

Proof. One can easily prove that the adjacency matrix of Γ_1 is a circulant matrix with first row $[0, 1, 0, \cdots, 0, 1]$. Now all eigenvalues of Γ_1 can be computed directly from Eq. (2). By using Theorems 4.3, 4.6 and Example 4.2, we can compute eigenvalues of group Γ_2, Γ_3 and Γ_4. Finally, by using Theorem 4.1 and Corollary 4.8 all eigenvalues of Γ_5 are computed.

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