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STRONGLY REGULAR GRAPHS WITH THE SAME PARAMETERS AS THE SYMPLECTIC GRAPH

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Abstract. We consider orbit partitions of groups of automorphisms for the symplectic graph and apply Godsil–McKay switching. As a result, we find four families of strongly regular graphs with the same parameters as the symplectic graphs, including the one discovered by Abiad and Haemers. Also, we prove that switched graphs are non-isomorphic to each other by considering the number of common neighbors of three vertices.

Keywords: cospectral graphs; switching; strongly regular graph; symplectic graphs.

1. Introduction

Godsil–McKay switching is often used to construct cospectral graphs. However, to apply that, a partition of the vertex set of a graph has to satisfy two very strong conditions. The orbit partition of a group of automorphisms satisfies one of them automatically, so if we can find an orbit partition that satisfies the other one, we can apply Godsil–McKay switching and we might be able to get cospectral graphs.

For the symplectic graph $Sp(2\nu, 2)$, Abiad and Haemers [2] considered a special 4-subset S and the partition $\{S, V(Sp(2\nu, 2)) \setminus S\}$. And then by applying Godsil–McKay swithcing, they obtained many graphs with the same parameters as the symplectic graph. We also aim to construct many graphs with the same parameters as the symplectic graph by applying Godsil–McKay switching, but partitions of the

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vertex set we consider are the orbit partitions of groups of automorphisms. In this paper, we consider the following groups:

- The automorphism group that fixes the standard basis.
- The automorphism group that fixes a special 4-subset by Abiad and Haemers.

As a result, we obtain four families of strongly regular graphs with the same parameters as the symplectic graphs. Also, we see one of them is isomorphic to the one discovered by Abiad and Haemers. More precisely, we see the edges involved with switching are the same.

Additionally, on the symplectic graph, we can regard the set of common neighbors of some vertices as the solution set of a system of linear equations. From this point of view, we investigate the number of common neighbors of three vertices as an invariant for isomorphism. As a result, we prove that the graphs in the five families, which are the four switched ones and the original one, are certainly pairwise non-isomorphic.

2. Preliminaries

Let $\mathbb{F}_2^{2\nu}$ be the 2ν-dimensional vector space over $\mathbb{F}_2$, and let

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The *symplectic graph* $Sp(2\nu, 2)$ over $\mathbb{F}_2$ is the graph defined by the following vertex set $V(Sp(2\nu, 2))$ and edge set $E(Sp(2\nu, 2))$:

$$V(Sp(2\nu, 2)) = \mathbb{F}_2^{2\nu} \setminus \{\mathbf{0}\},$$
$$E(Sp(2\nu, 2)) = \{xy \mid x^T K y = 1\},$$

where $K = I_\nu \otimes R$ (I_ν is the identity matrix of size ν). It is not difficult to see that $Sp(2\nu, 2)$ is a strongly regular graph with parameters $(2^{2\nu} - 1,\ 2^{2\nu-1},\ 2^{2\nu-2},\ 2^{2\nu-2})$.

In general, the spectrum of a strongly regular graph is determined by its parameters. Conversely, parameters are also characterized by the spectrum. Therefore if a graph X' has the same spectrum as a strongly regular graph X, then X' is also strongly regular with the same parameters as X.

Returning on the subject of the symplectic graph $Sp(2\nu, 2)$, when $\nu = 1$ we just have the complete graph K_3. Also, $Sp(4, 2)$ is a strongly regular graph with parameters $(15, 8, 4, 4)$, which is known to be determined by its parameters, so we suppose that $\nu \geq 3$ in the rest of this paper.

Let X be a graph and let $\pi = \{C_1, \ldots, C_t\}$ be a partition of $V(X)$. This partition π is called an *equitable partition* if for all i, j any two vertices in C_i have the same number of neighbors in C_j.

Godsil and McKay [6] introduced the following on constructing cospectral graphs.

Theorem 2.1. *Let X be a graph and let $\pi = \{C_1, \ldots, C_t, D\}$ be a partition of $V(X)$. Assume that π satisfies the following two conditions:*

(i) *$\{C_1, \ldots, C_t\}$ is an equitable partition of $V(X) \setminus D$.*

(ii) *For every $x \in D$ and every $i \in \{1, \ldots, t\}$ the vertex x has either 0, $\frac{1}{2}|C_i|$, or $|C_i|$ neighbors in C_i.*

Construct a new graph X' by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in C_i whenever x has $\frac{1}{2}|C_i|$ neighbors in C_i. Then X and X' have the same spectrum.

The operation that transforms X into X' is called *Godsil–McKay switching*. We will call a partion π of $V(X)$ a *Godsil–McKay partition* if we can apply the above theorem with respect to π. Also, we will call the special cell D a *Godsil–McKay cell* of π.

On the other hand, the orbit partition of a subgroup of automorphisms of a graph forms an equitable partition, so this automatically satisfies the condition (i) of Godsil–McKay switching no matter what orbit we choose as D.

Tang and Wan [10] determined the automorphism group of $Sp(2\nu, 2)$.

Proposition 2.2.

$$\mathrm{Aut}(Sp(2\nu, 2)) \simeq Sp_{2\nu}(\mathbb{F}_2),$$

where $Sp_{2\nu}(\mathbb{F}_2) = \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T K A = K\}$.

However, $Sp(2\nu, 2)$ is vertex-transitive. We aim to find Godsil–McKay cells in the orbit partition of a group of automorphisms, so we have to choose a proper subgroup of the automorphism group.

3. Automorphisms that fix the standard basis

In this section, we consider the subgroup of all automorphisms of $\mathrm{Aut}(X)$ that fix the set of the standard basis of $\mathbb{F}_2^{2\nu}$. To apply Godsil–McKay switching, we determine the orbit partition and confirm that it is a Godsil–McKay partition. After that, we prove that a switched graph is not isomorphic to the original symplectic graph.

Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν and let e_i be the vector in $\mathbb{F}_2^{2\nu}$ with a 1 in the ith coordinate and 0's elsewhere and put $\mathcal{E} = \{e_1, \dots, e_{2\nu}\}$. Also, let

$$\mathrm{Aut}(X)_{\mathcal{E}} = \{g \in \mathrm{Aut}(X) \mid \mathcal{E}^g = \mathcal{E}\}.$$

3.1. **Determination of the orbit partition of** $\mathrm{Aut}(X)_{\mathcal{E}}$**.** Let P be a permutation matrix of size ν and let $A_1, \dots, A_\nu$ be matrices of size 2. We define the matrix $P(A_1, \dots, A_\nu)$ of size 2ν as follows:

$$P_{ij} \mapsto \begin{cases} O & \text{if } P_{ij} = 0, \\ A_i & \text{if } P_{ij} = 1, \end{cases}$$

where O is the zero matrix. For example, when $\nu = 3$ and $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,

$$P(I_2, I_2, I_2) = \begin{bmatrix} O & I_2 & O \\ O & O & I_2 \\ I_2 & O & O \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $K = I_\nu(R, \dots, R)$. Let $[\nu] = \{1, \dots, \nu\}$.

Lemma 3.1. *Let $A_1, \dots, A_\nu$ be matrices of size 2 over $\mathbb{F}_2$, and set*

$$B = \begin{bmatrix} A_1 \\ \vdots \\ A_\nu \end{bmatrix}.$$

Suppose the two column vectors b_1, b_2 of B satisfy the following conditions:

- $b_1 \neq b_2$,
- $\mathrm{wt}(b_1) = \mathrm{wt}(b_2) = 1$, *where the weight* $\mathrm{wt}(x)$ *of a vector x is the number of non-zero coordinates equal to 1 of x.*

If $A_1^T RA_1 + \cdots + A_\nu^T RA_\nu = R$, then there exists a unique $i \in [\nu]$ such that $A_i \in \{I_2, R\}$ and $A_j = O$ for all $j \in [\nu] \setminus \{i\}$.

Proof. By the second condition on B, there exists $i \in [\nu]$ such that $A_i \neq O$ and the number of coordinates equal to 1 in A_i is 1 or 2.

Case 1: Suppose that the number of coordinates equal to 1 in A_i is 1. There exists another $j \in [\nu] \setminus \{i\}$ such that $A_j \neq O$. By the second condition on B, the number of coordinates equal to 1 in A_j has to be 1 and $A_k = O$ for all $k \in [\nu] \setminus \{i, j\}$. However $A_i^T RA_i = A_j^T RA_j = O$ and clearly $A_k^T RA_k = O$ for all $k \in [\nu] \setminus \{i, j\}$. Therefore $A_1^T RA_1 + \cdots + A_\nu^T RA_\nu = O$. This is a contradiction.

Case 2: Suppose that the number of coordinates equal to 1 in A_i is 2. By the two conditions on B, we have $A_i = I_2$ or R. Moreover by the second condition on B, it follows that $A_j = O$ for all $j \in [\nu] \setminus \{i\}$. $\square$

Lemma 3.2.

$$\mathrm{Aut}(X)_{\mathcal{E}} \simeq \left\{ P(A_1, \dots, A_\nu) \,\middle|\, P : \textit{a permutation matrix}, A_i \in \{I_2, R\} \right\}.$$

Proof. Put $\mathcal{P} = \left\{ P(A_1, \dots, A_\nu) \,\middle|\, P : \text{a permutation matrix}, A_i \in \{I_2, R\} \right\}$. By Proposition 2.2,

$$\mathrm{Aut}(X)_{\mathcal{E}} \simeq \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T KA = K, A\mathcal{E} = \mathcal{E}\},$$

so we set $Sp_{2\nu}(\mathbb{F}_2)_{\mathcal{E}} = \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T KA = K, A\mathcal{E} = \mathcal{E}\}$ and prove that $\mathcal{P} = Sp_{2\nu}(\mathbb{F}_2)_{\mathcal{E}}$.

First, let $P(A_1, \dots, A_\nu) \in \mathcal{P}$. Since $P(A_1, \dots, A_\nu)$ is a permutation matrix of size 2ν, $P(A_1, \dots, A_\nu)\mathcal{E} = \mathcal{E}$. Also,

$$\begin{aligned} P(A_1, \dots, A_\nu)^T KP(A_1, \dots, A_\nu) &= P^T(A_1^T, \dots, A_\nu^T)KP(A_1, \dots, A_\nu) \\ &= P^T(A_1, \dots, A_\nu)KP(A_1, \dots, A_\nu) \\ &= P^T(A_1, \dots, A_\nu)I_\nu(R, \dots, R)P(A_1, \dots, A_\nu) \\ &= (P^T I_\nu P)(A_1RA_1, \dots, A_\nu RA_\nu) \\ &= I_\nu(A_1RA_1, \dots, A_\nu RA_\nu) \\ &= I_\nu(R, \dots, R) \\ &= K. \end{aligned}$$

Consequently, we see that $P(A_1, \dots, A_\nu) \in Sp_{2\nu}(\mathbb{F}_2)_{\mathcal{E}}$.

Conversely, let $A \in Sp_{2\nu}(\mathbb{F}_2)_{\mathcal{E}}$. Since $A\mathcal{E} = \mathcal{E}$, A is a permutation matrix. We set A as a block matrix as follows:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1\nu} \\ \vdots & \ddots & \vdots \\ A_{\nu 1} & \cdots & A_{\nu\nu} \end{bmatrix} \qquad (A_{ij} \in M_2(\mathbb{F}_2)).$$

By $A^T K A = K$, we get

$$\begin{bmatrix} \sum_{i=1}^{\nu} A_{i1}^T R A_{i1} & & & \\ & \sum_{i=1}^{\nu} A_{i2}^T R A_{i2} & & \\ & & \ddots & \\ & & & \sum_{i=1}^{\nu} A_{i\nu}^T R A_{i\nu} \end{bmatrix} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}.$$

By comparing the $(1,1)$ blocks, we have $\sum_{i=1}^{\nu} A_{i1}^T R A_{i1} = R$. Since A is a permutation matrix, weights of the two column vectors of

$$\begin{bmatrix} A_{11} \\ \vdots \\ A_{\nu 1} \end{bmatrix}$$

are both 1 and these two column vectors are distinct. Therefore we can apply Lemma 3.1, that is, there exists a unique $i_1 \in [\nu]$ such that $A_{i_1,1} = I_2$ or R and $A_{j1} = O$ $(j \neq i_1)$. Moreover, A is a permutation matrix, so we get $A_{i_1,2} = A_{i_1,3} = \cdots = A_{i_1,\nu} = O$. Next, we compare the $(2,2)$ blocks. By a similar argument as above, we see that there exists a unique $i_2 \in [\nu] \setminus \{i_1\}$ such that $A_{i_2,2} = I_2$ or R and $A_{j2} = O\,(j \neq i_2)$, so we get $A_{i_2,3} = A_{i_2,4} = \cdots = A_{i_2,\nu} = O$. Continuing this argument repeatedly, we eventually have a permutation matrix P of size ν and ν matrices $B_i \in \{I_2, R\}$ $(i = 1, \ldots, \nu)$ such that $A = P(B_1, \ldots, B_\nu)$. Consequently, we see that $A \in \mathcal{P}$. $\square$

Hereafter, we often divide a vector $x \in \mathbb{F}_2^{2\nu}$ into ν blocks as follows:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_\nu \end{bmatrix} \qquad (x_i \in \mathbb{F}_2^2).$$

We define

$$O(i,j,k) = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_\nu \end{bmatrix} \in \mathbb{F}_2^{2\nu} \;\middle|\; \begin{array}{l} \#\{l \mid \mathrm{wt}(x_l) = 2\} = i, \\ \#\{l \mid \mathrm{wt}(x_l) = 1\} = j, \\ \#\{l \mid \mathrm{wt}(x_l) = 0\} = k \end{array} \right\}.$$

Note that $i + j + k = \nu$. A vector x is called the *initial vector* of $O(i,j,k)$ if $x_1 = \cdots = x_i = [11]^T$, $x_{i+1} = \cdots = x_{i+j} = [10]^T$, $x_{i+j+1} = \cdots = x_{i+j+k} = [00]^T$. For example, the initial vector of $O(1,2,1)$ is $[11101000]^T$.

Proposition 3.3. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν. The orbit partition of* $\mathrm{Aut}(X)_{\mathcal{E}}$ *on* $V(X)$ *is the following:*

$$V(X) = \bigsqcup_{\substack{i+j+k=\nu, \\ (i,j,k) \neq (0,0,\nu)}} O(i,j,k).$$

Proof. First, we prove that for all $x, y \in O(i,j,k)$ there exists $g \in \mathrm{Aut}(X)_{\mathcal{E}}$ such that $y = x^g$, but we can assume that y is the initial vector of $O(i,j,k)$ without loss of generality. Let $S = \{i \in [\nu] \mid x_i = [01]^T\}$ and consider the matrix A of size 2ν defined by the following:

$$A = I_\nu(A_1, \dots, A_\nu), \quad \text{where } A_i = \begin{cases} R & \text{if } i \in S, \\ I_2 & \text{otherwise.} \end{cases}$$

Then all weight-one blocks of Ax are $[10]^T$. After that, we can choose an appropriate permutation matrix P of size ν such that the weights of the ν blocks of the vector $P(I_2, \dots, I_2)Ax$ are in decreasing order. Then $P(I_2, \dots, I_2)Ax$ is nothing but the initial vector of $O(i,j,k)$, that is, $y = P(I_2, \dots, I_2)Ax$. By Lemma 3.2 and Proposition 2.2, the mapping

$$T_{P(I_2,\dots,I_2)A} : x \mapsto P(I_2, \dots, I_2)Ax$$

is certainly an automorphism that fixes the standard basis.

Next, we prove that for each $x \in O(i,j,k)$ and $y \in O(i',j',k')$ with $(i,j,k) \neq (i',j',k')$, $y \neq x^g$ for all $g \in \mathrm{Aut}(X)_{\mathcal{E}}$. By Lemma 3.2 and Proposition 2.2, for $g \in \mathrm{Aut}(X)_{\mathcal{E}}$ there exists $P(A_1, \dots, A_\nu) \in \mathcal{P}$ such that $g = T_{P(A_1,\dots,A_\nu)}$, where $T_{P(A_1,\dots,A_\nu)}$ is the mapping that maps $z \in V(X)$ to $P(A_1, \dots, A_\nu)z$. However, roles which $P(A_1, \dots, A_\nu)$ plays are only permuting blocks and exchanging the coordinates of a block, so if $x \in O(i,j,k)$ then $P(A_1, \dots, A_\nu)x \in O(i,j,k)$. Therefore, it follows that $y \neq x^g$ for all $g \in \mathrm{Aut}(X)_{\mathcal{E}}$. □

3.2. **Finding Godsil–McKay cells in orbit partitions.** We define $O(i,j,k)_{\mathrm{even}}$ and $O(i,j,k)_{\mathrm{odd}}$ as follows, to decompose $O(i,j,k)$ into two more sets:

$$O(i,j,k)_{\mathrm{even}} = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_\nu \end{bmatrix} \in O(i,j,k) \;\middle|\; \#\left\{ l \in [\nu] \;\middle|\; x_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \equiv 0 \pmod 2 \right\},$$

$$O(i,j,k)_{\mathrm{odd}} = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_\nu \end{bmatrix} \in O(i,j,k) \;\middle|\; \#\left\{ l \in [\nu] \;\middle|\; x_l = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \equiv 1 \pmod 2 \right\}.$$

Actually, we can see that there exists a bijection between $O(i,j,k)_{\mathrm{even}}$ and $O(i,j,k)_{\mathrm{odd}}$.

Lemma 3.4. $|O(i,j,k)_{\mathrm{even}}| = |O(i,j,k)_{\mathrm{odd}}|$.

Proof. If $j = 0$, $O(i,j,k)_{\mathrm{even}}$ and $O(i,j,k)_{\mathrm{odd}}$ are empty sets, so we have the above equality. Suppose that $j \geq 1$. For $x \in O(i,j,k)$, we can define $l_{\min} = \min\{l \in [\nu] \mid \mathrm{wt}(x_l) = 1\}$. Consider the following correspondence:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{l_{\min}} \\ \vdots \\ x_\nu \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_{l_{\min}} + \mathbf{1}_2 \\ \vdots \\ x_\nu \end{bmatrix},$$

where $\mathbf{1}_2 = [11]^T$. By this correspondence, parity of the number of blocks of $[10]^T$ change. Consequently, we get two mappings which are the one from $O(i,j,k)_{\mathrm{even}}$

to $O(i,j,k)_{\mathrm{odd}}$ and the other from $O(i,j,k)_{\mathrm{odd}}$ to $O(i,j,k)_{\mathrm{even}}$. Clearly, these are the inverse mappings each other, so we have the desired equality. □

Let $N(x)$ denote the set of all neighbors of a vertex x.

Proposition 3.5. *For all $x \in O(0,\nu,0)$ and an arbitrary orbit $O(i,j,k)$*

$$|N(x) \cap O(i,j,k)| = \begin{cases} \frac{1}{2}|O(i,j,k)| & \text{if } j \geq 1, \\ |O(i,j,k)| & \text{if } j = 0,\ i : \mathrm{odd}, \\ 0 & \text{if } j = 0,\ i : \mathrm{even}. \end{cases}$$

In particular, $O(0,\nu,0)$ is a Godsil–McKay cell in the orbit partition of $\mathrm{Aut}(X)_{\mathcal{E}}$.

Proof. For $x \in O(0,\nu,0)$ and $g \in \mathrm{Aut}(X)_{\mathcal{E}}$, since $O(i,j,k)$ is an orbit,

$$|N(x) \cap O(i,j,k)| = |N(x^g) \cap O(i,j,k)|,$$

so we can assume that $x = [0101\ldots01]^T$ as a special vertex in $O(0,\nu,0)$ without loss of generality. Then for $y \in N(x) \cap O(i,j,k)$,

$$\begin{aligned} 1 &= x^T K y \\ &= [1010\ldots10] \begin{bmatrix} y_1 \\ \vdots \\ y_\nu \end{bmatrix} \\ &= [10]y_1 + \cdots + [10]y_\nu, \end{aligned}$$

and $[10]y_l = 1$ if and only if $y_l = [11]^T$ or $[10]^T$; so, we get

$$\begin{aligned} 1 &\equiv \#\{l \in [\nu] \mid y_l = [11]^T\} + \#\{l \in [\nu] \mid y_l = [10]^T\} \pmod 2 \\ &= i + \#\{l \in [\nu] \mid y_l = [10]^T\}. \end{aligned}$$

We consider two cases:

Case 1 : Suppose $j = 0$. Then $\#\{l \in [\nu] \mid y_l = [10]^T\} = 0$, so $x^T K y = 1$ if and only if $i \equiv 1 \pmod 2$ by the above observation. Therefore,

$$\begin{aligned} |N(x) \cap O(i,j,k)| &= \#\{y \in O(i,j,k) \mid x^T K y = 1\} \\ &= \begin{cases} |O(i,j,k)| & \text{if } i : \mathrm{odd}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Case 2 : Suppose $j \geq 1$. Similarly,

$$
\begin{aligned}
&|N(x) \cap O(i,j,k)| \\
&\quad = \#\{y \in O(i,j,k) \mid x^T K y = 1\} \\
&\quad = \# \left\{ y \in O(i,j,k) \,\middle|\, i + \#\{l \in [\nu] \mid y_l = [10]^T\} \equiv 1 \pmod 2 \right\} \\
&\quad = \begin{cases} \# \left\{ y \in O(i,j,k) \,\middle|\, \#\{l \in [\nu] \mid y_l = [10]^T\} \equiv 0 \pmod 2 \right\} & \text{if } i : \text{odd}, \\ \# \left\{ y \in O(i,j,k) \,\middle|\, \#\{l \in [\nu] \mid y_l = [10]^T\} \equiv 1 \pmod 2 \right\} & \text{otherwise}, \end{cases} \\
&\quad = \begin{cases} |O(i,j,k)_{\text{even}}| & \text{if } i : \text{odd}, \\ |O(i,j,k)_{\text{odd}}| & \text{otherwise}, \end{cases} \\
&\quad = \frac{1}{2} |O(i,j,k)|
\end{aligned}
$$

by Lemma 3.4. □

By Proposition 3.5, we can apply Godsil–McKay switching to the symplectic graph with respect to the orbit partition of $\mathrm{Aut}(X)_{\mathcal{E}}$ with the Godsil–McKay cell $O(0,\nu,0)$. We denote this switched graph by $X^{O(0,\nu,0)}$. We will see that $X^{O(0,\nu,0)}$ is not isomorphic to the original graph $Sp(2\nu,2)$ in Section 5.

4. Automorphisms that fix a 4-subset

Let X be a graph and let $\{C_1, V(X) \backslash C_1\}$ be a partition of $V(X)$. If $|C_1| = 2$, then the partition $\{C_1, V(X) \setminus C_1\}$ is always a Godsil–McKay partition with the Godsil–McKay cell $V(X) \backslash C_1$, but the graph switched by this partition is always isomorphic to the original one. On the other hand, if $|C_1| \geq 4$, the partition $\{C_1, V(X) \setminus C_1\}$ is not always a Godsil–McKay partition, but if it is, then the switched graph can be non-isomorphic to the original one. In regard to this, Abiad, Brouwer and Haemers [1] studied some sufficient conditions for being non-isomorphic after Godsil–McKay switching, but nobody knows on necessary and sufficient conditions so far.

Even so, Abiad and Haemers [2] studied switched symplectic graphs with respect to partitions of the form $\{C_1, V(Sp(2\nu,2)) \setminus C_1\}$ with $|C_1| = 4$ and they obtained many graphs with the same parameters as $Sp(2\nu, 2)$.

In this section, we consider the subgroup of automorphisms that fix their 4-subset C_1. As a result, we find three Godsil–McKay cells including C_1.

Let $S = \{v_1, v_2, v_3, v_4\}$ be a 4-subset of $V(Sp(2\nu,2))$ satisfying the following two conditions:

- v_1, v_2, v_3 are linearly independent with $v_i^T K v_j = 0$ for all $i, j \in [3]$,
- $v_4 = v_1 + v_2 + v_3$.

Note that any three vectors $v_i, v_j, v_k \in S$ are linearly independent and for any $x \in V(Sp(2\nu,2))$, $x^T K v_1 + x^T K v_2 + x^T K v_3 + x^T K v_4 = x^T K \mathbf{0} = 0$, so $\#\{i \in [4] \mid x^T K v_i = 1\} = 0, 2, 4$. Therefore we can decompose $V(Sp(2\nu,2))$ into three subsets as follows:

$$V(Sp(2\nu,2)) = S_0 \sqcup S_2 \sqcup S_4,$$

where $S_i = \left\{ x \in V(Sp(2\nu,2)) \,\middle|\, \#\{j \in [4] \mid x^T K v_j = 1\} = i \right\}$.

4.1. **Determination of the orbit partition of** $\mathrm{Aut}(X)_S$**.** Let X be the symplectic graph $Sp(2\nu, 2)$ and S be the above 4-subset. We consider

$$\mathrm{Aut}(X)_S = \{g \in \mathrm{Aut}(X) \mid S^g = S\}.$$

Let $\langle S\rangle$ denote the subspace spanned by S. By Proposition 2.2, we get the following:

Lemma 4.1. *$\langle S\rangle^g = \langle S\rangle$ for all $g \in \mathrm{Aut}(X)_S$.*

Before determining the orbit partition, we recall the useful theorem known as Witt's theorem (see for example [3]).

Theorem 4.2. *Let V and V' be vector spaces equipped with a non-degenerate symplectic inner product and suppose that they are isometric. Let σ be an isometry from an arbitrary subspace U of V to V'. Then σ can be extended to a surjective isometry from V to V'.*

We can regard the value of $x^T K y$ as the value of an inner product (x, y), and preserving the value of the inner product is nothing but preserving the adjacency relation. Therefore Witt's theorem guarantees that an isometry constructed from a small subspace of $\mathbb{F}_2^{2\nu}$ can be extended to an automorphism of $Sp(2\nu, 2)$. This is a really strong tool to prove that any two vertices in a set, where we want to show it is an orbit, can be mapped to each other by an automorphism.

Let $T = \langle S\rangle \setminus (S \cup \{\mathbf{0}\}) = \{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$. Note that $S, T \subset S_0$.

Lemma 4.3. $\mathrm{Aut}(X)_S$ *acts on S as* $\mathrm{Sym}(S)$*, where* $\mathrm{Sym}(S)$ *is the symmetric group on S.*

Proof. Let $[4] = \{i_1, i_2, i_3, i_4\}$. We consider the subspace $U = \langle v_1, v_2, v_3\rangle$ and the linear mapping g from U to $\mathbb{F}_2^{2\nu}$ such that $v_j^g = v_{i_j}$ for $j \in [3]$. We see that g is an isometry, so there exists an automorphism g^* of $Sp(2\nu, 2)$ such that $g^*|_U = g$ by Witt's theorem. Also, $v_4^g = v_{i_4}$ since $v_{i_4} = v_{i_1} + v_{i_2} + v_{i_3}$, so we see that $\mathrm{Aut}(X)_S$ acts on S as $\mathrm{Sym}(S)$. □

Proposition 4.4. *Let X be the symplectic graph $Sp(2\nu, 2)$. The orbit partition of* $\mathrm{Aut}(X)_S$ *on $V(X)$ is $\{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$.*

Proof. First, we prove that any two vertices in different sets cannot be mapped to each other. For any $g \in \mathrm{Aut}(X)_S$ and for any $x \in V(X)$, we can define a mapping from $\{v_i \in S \mid x^T K v_i = 1\}$ to $\{v_i \in S \mid (x^g)^T K v_i = 1\}$ that maps v_i to v_i^g and it is clearly bijective, so the value of $\#\{v_i \in S \mid x^T K v_i = 1\}$ is invariant under $g \in \mathrm{Aut}(X)_S$. Therefore S_0, S_2, S_4 cannot be mapped to each other. By Lemma 4.1, $\langle S\rangle \setminus \{\mathbf{0}\}$ and $S_0 \setminus \langle S\rangle$ cannot be mapped to each other. Since $S^g = S$ for any $g \in \mathrm{Aut}(X)_S$, S and T cannot be mapped to each other. Consequently, we see that S, T, $S_0 \setminus (S \cup T)$, S_2, S_4 cannot be mapped to each other.

Next, we prove that for every $P \in \{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$, any two vertices in P can be mapped to each other by some $g \in \mathrm{Aut}(X)_S$. It is clear in the case $P \in \{S, T\}$ by Lemma 4.3. Thus, we consider $P \in \{S_0 \setminus (S \cup T), S_2, S_4\}$. Note that for three distinct vertices $v_i, v_j, v_k \in S$ and $x \in V(X) \setminus \langle S\rangle$, x, v_i, v_j, v_k are linearly independent. Assume that $P \in \{S_0 \setminus (S \cup T), S_4\}$. Let $x, y \in P$ and we consider the subspace $U = \langle x, v_1, v_2, v_3\rangle$ and the linear mapping g from U to $\mathbb{F}_2^{2\nu}$ such that $x^g = y$ and $v_i^g = v_i$ for $i \in [3]$. Then g preserves the value of the inner product and g is injective since x, v_1, v_2, v_3 are linearly independent, so g is an isometry. Therefore by Witt's theorem, there exists an automorphism g^* of X such that $g^*|_U = g$. This fixes S and maps x to y. The case $P = S_2$ is proved by a similar argument. $\square$

4.2. **Finding Godsil–McKay cells.** Let $x \in V(Sp(2\nu, 2))$. Since $v_4 = v_1+v_2+v_3$, $\#\{i \in [4] \mid v_i^T Kx = 1\} = 4$ if and only if $v_1^T Kx = v_2^T Kx = v_3^T Kx = 1$. Let

$$M = \begin{bmatrix} v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix}.$$

Then $S_4 = \{x \in V(Sp(2\nu, 2)) \mid Mx = \mathbf{1}_3\}$. Since v_1, v_2, v_3 are linearly independent, $\operatorname{rank} M = 3$, so the system of equations $Mx = \mathbf{1}_3$ has a solution. Thus, we have a bijection from S_4 to $\operatorname{Ker} T_M$, where T_M is the linear mapping $x \mapsto Mx$. So $|S_4| = 2^{2\nu-3}$. A similar argument gives us $|S_0| = 2^{2\nu-3} - 1$. Also, S_2 is the complement of $S_0 \cup S_4$, so we obtain $|S_2| = (2^{2\nu}-1)-(2^{2\nu-3}-1)-2^{2\nu-3} = 3\cdot 2^{2\nu-2}$. Summarizing above, we get the following:

Lemma 4.5.

$$|S_i| = \begin{cases} 2^{2\nu-3} - 1 & \text{if } i = 0, \\ 3 \cdot 2^{2\nu-2} & \text{if } i = 2, \\ 2^{2\nu-3} & \text{if } i = 4. \end{cases}$$

We decompose S_2 more. For distinct indices i, j with $i < j$, define

$$S_2(i, j) = \{x \in S_2 \mid x^T Kv_i = x^T Kv_j = 1\}.$$

By Lemma 4.3, we see that there is a bijection from $S_2(1, 2)$ to $S_2(i, j)$, so

$$|S_2| = \sum_{i,j} |S_2(i, j)| = 6|S_2(1, 2)|. \tag{1}$$

Let X be a graph and let $\{O_1, \dots, O_t\}$ be an orbit partition of a group of automorphisms of X. Then for all $x \in O_i$, $|N(x) \cap O_j|$ is a constant value. By counting the cardinality of $\{xy \in E(X) \mid x \in O_i, y \in O_j\}$ in two ways, we obtain the following useful formula:

Lemma 4.6. *For any $x \in O_i$ and $y \in O_j$,*

$$|O_i||N(x) \cap O_j| = |O_j||N(y) \cap O_i|.$$

Proposition 4.7. *Let X be the symplectic graph $Sp(2\nu, 2)$, and let $\{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$ be the orbit partition of $\mathrm{Aut}(X)_S$ on $V(X)$. For $P \in \{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$, the following statements hold:*

(i) *For any $x \in S$,*

$$|N(x) \cap P| = \begin{cases} 0 & \text{if } P = T \text{ or } S_0 \setminus (S \cup T), \\ |P| & \text{if } P = S_4, \\ \frac{1}{2}|P| & \text{if } P = S_2. \end{cases}$$

(ii) *For any $x \in S_4$,*

$$|N(x) \cap P| = \begin{cases} 0 & \text{if } P = T, \\ |P| & \text{if } P = S, \\ \frac{1}{2}|P| & \text{if } P = S_2 \text{ or } S_0 \setminus (S \cup T). \end{cases}$$

(iii) *For any* $x \in S_0 \setminus (S \cup T)$,

$$|N(x) \cap P| = \begin{cases} 0 & \textit{if } P = S \textit{ or } T, \\ \frac{1}{2}|P| & \textit{if } P = S_2 \textit{ or } S_4. \end{cases}$$

In particular, S*,* $S_0 \setminus (S \cup T)$*,* S_4 *are Godsil–McKay cells.*

Proof. First, we prove that S is a Godsil–McKay cell, but since $\{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$ is the orbit partition, it is sufficient to prove only that for all $P \in \{T, S_0 \setminus (S \cup T), S_2, S_4\}$ and a special vertex $x_0 \in S$, $|N(x_0) \cap P| = 0$, $\frac{1}{2}|P|$ or $|P|$. We take $v_1 \in S$ as a special vertex. It is easy to see that $N(v_1) \cap S_4 = S_4$ and $N(v_1) \cap S_0 = \emptyset$, so we get $|N(v_1) \cap S_4| = |S_4|$ and $|N(v_1) \cap T| = |N(v_1) \cap (S_0 \setminus (S \cup T))| = 0$. If $i < j$, then

$$N(v_1) \cap S_2(i,j) = \begin{cases} S_2(i,j) & \text{if } i = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Consequently, we have

$$N(v_1) \cap S_2 = \bigsqcup_{i,j} (N(v_1) \cap S_2(i,j)) = \bigsqcup_{j=2}^{4} S_2(1,j);$$

so $|N(v_1) \cap S_2| = 3|S_2(1,2)| = \frac{1}{2}|S_2|$ by the equality (1).

Next, we prove that S_4 is a Godsil–McKay cell. Let $x \in S_4$ be a special vertex. It is easy to see that $|N(x) \cap S| = |S|$ and $|N(x) \cap T| = 0$. To find the value of $|N(x) \cap (S_0 \setminus (S \cup T))|$, we calculate $|N(x) \cap S_0|$ first. Observe $N(x) \cap S_0 = \left\{ y \in V(Sp(2\nu, 2)) \,\middle|\, x^T K y = 1, \#\{i \in [4] \mid v_i^T K y = 1\} = 0 \right\}$, but $\#\{i \in [4] \mid v_i^T K y = 1\} = 0$ if and only if $v_1^T K y = v_2^T K y = v_3^T K y = 0$; so, $N(x) \cap S_0 = \{y \in \mathbb{F}_2^{2\nu} \mid My = [1000]^T\}$, where

$$M = \begin{bmatrix} x^T K \\ v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix}.$$

Since x, v_1, v_2, v_3 are linearly independent, there exists a bijection from $N(x) \cap S_0$ to $\operatorname{Ker} T_M$. Therefore we get $|N(x) \cap S_0| = 2^{2\nu-4}$. Consequently,

$$\begin{aligned} |N(x) \cap (S_0 \setminus (S \cup T))| &= |N(x) \cap S_0| - |N(x) \cap S_0 \cap (S \cup T)| \\ &= 2^{2\nu-4} - |N(x) \cap S_0 \cap S| - |N(x) \cap S_0 \cap T| \\ &= 2^{2\nu-4} - |N(x) \cap S| \\ &= 2^{2\nu-4} - 4. \end{aligned}$$

On the other hand, $|S_0 \setminus (S \cup T)| = 2^{2\nu-3} - 8$ by Lemma 4.5; so, we obtain $|N(x) \cap (S_0 \setminus (S \cup T))| = \frac{1}{2}|S_0 \setminus (S \cup T)|$. We can determine the value of $|N(x) \cap S_2|$ similarly as above. Observe

$$N(x) \cap S_2 = \left\{ y \in V(Sp(2\nu, 2)) \,\middle|\, x^T K y = 1, \#\{i \in [4] \mid v_i^T K y = 1\} = 2 \right\},$$

but $\#\{i \in [4] \mid v_i^T Ky = 1\} = 2$ if and only if $\#\{i \in [3] \mid v_i^T Ky = 1\} = 1$ or 2. Therefore

$$N(x) \cap S_2 = \bigsqcup_{\substack{\boldsymbol{b} \in \mathbb{F}_2^3, \\ \mathrm{wt}(\boldsymbol{b}) \in \{1,2\}}} \left\{ y \in \mathbb{F}_2^{2\nu} \,\middle|\, \begin{bmatrix} x^T K \\ v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix} y = \begin{bmatrix} 1 \\ \boldsymbol{b} \end{bmatrix} \right\}$$

and we get $\#\{y \in \mathbb{F}_2^{2\nu} \mid My = [1\boldsymbol{b}^T]^T\} = 2^{2\nu-4}$ for a fixed $\boldsymbol{b}$. Consequently, $|N(x) \cap S_2| = 6 \cdot 2^{2\nu-4} = \frac{1}{2}|S_2|$, so we can see that S_4 is a Godsil–McKay cell.

Finally, we prove that $S_0 \setminus (S \cup T)$ is a Godsi-McKay cell. Let $x \in S_0 \setminus (S \cup T)$ be a special vertex. It is easy to see that $|N(x) \cap S| = |N(x) \cap T| = 0$. Also,

$$\begin{aligned} |N(x) \cap S_2| &= \# \left\{ y \in V(Sp(2\nu, 2)) \,\middle|\, x^T Ky = 1, \#\{i \in [4] \mid v_i^T Ky = 1\} = 2 \right\} \\ &= \sum_{\substack{\boldsymbol{b} \in \mathbb{F}_2^3, \\ \mathrm{wt}(\boldsymbol{b}) \in \{1,2\}}} \# \left\{ y \in \mathbb{F}_2^{2\nu} \,\middle|\, \begin{bmatrix} x^T K \\ v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix} y = \begin{bmatrix} 1 \\ \boldsymbol{b} \end{bmatrix} \right\} \\ &= 6 \cdot 2^{2\nu-4} \\ &= \frac{1}{2}|S_2|. \end{aligned}$$

Furthermore, for $y \in S_4$,

$$\begin{aligned} |N(x) \cap S_4| &= \frac{1}{|S_0 \setminus (S \cup T)|}|S_4||N(y) \cap S_0 \setminus (S \cup T)| && \text{(by Lemma 4.6)} \\ &= \frac{1}{|S_0 \setminus (S \cup T)|}|S_4| \cdot \frac{1}{2}|S_0 \setminus (S \cup T)| && \text{(by part (ii))} \\ &= \frac{1}{2}|S_4|. \end{aligned}$$

Hence $S_0 \setminus (S \cup T)$ is a Godsil–McKay cell. □

Therefore on the orbit partition of $\mathrm{Aut}(X)_S$ on $V(X)$, we obtain three switched symplectic graphs with the Godsil–McKay cells $S, S_0 \setminus (S \cup T)$ and S_4. Let X^S, $X^{S_0 \setminus (S \cup T)}$ and X^{S_4} denote their switched graphs, respectively. In general, the set of edges deleted by Godsil–McKay switching with respect to the partition $\{C_1, \ldots .C_t, D\}$ is

$$\bigsqcup_{i=1}^{t} \bigsqcup_{x \in D} \left\{ \{x, y\} \,\middle|\, y \in C_i, x \sim y, |N(x) \cap C_i| = \frac{1}{2}|C_i| \right\}$$

and the set of added edges is similarly

$$\bigsqcup_{i=1}^{t} \bigsqcup_{x \in D} \left\{ \{x, y\} \,\middle|\, y \in C_i, x \not\sim y, |N(x) \cap C_i| = \frac{1}{2}|C_i| \right\}.$$

Abiad and Haemers [2] proved that the partition $\{S, V(X) \setminus S\}$ is a Godsil–McKay partition with a Godsil–McKay cell $D = V(X) \setminus S$, and constructed the switched symplectic graph that is not isomorphic to the original one. The set of deleted edges to construct this switched symplectic graph by Abiad and Haemers is

$$\bigsqcup_{x\in V(X)\setminus S}\left\{\{x,y\}\,\middle|\, y\in S, x\sim y, |N(x)\cap S|=\frac{1}{2}|S|\right\},$$

but it is easy to see that $|N(x)\cap S|=\frac{1}{2}|S|$ if and only if $x\in S_2$. Therefore this is equal to

$$\bigsqcup_{x\in S_2}\left\{\{x,y\}\,\middle|\, y\in S, x\sim y\right\}. \tag{2}$$

On the other hand, the set of deleted edges to construct X^S is

$$\bigsqcup_{P\in\{T,S_0\setminus(S\cup T),S_2,S_4\}}\ \bigsqcup_{x\in S}\left\{\{x,y\}\,\middle|\, y\in P, x\sim y, |N(x)\cap P|=\frac{1}{2}|P|\right\},$$

but we have already confirmed that for $x\in S$, $|N(x)\cap P|=\frac{1}{2}|P|$ if and only if $P=S_2$ by Proposition 4.7-(i). Therefore this is equal to (2) which is nothing but the one by Abiad and Haemers. Similarly, on the set of added edges,

$$\begin{aligned}&\bigsqcup_{x\in V(X)\setminus S}\left\{\{x,y\}\,\middle|\, y\in S, x\not\sim y, |N(x)\cap S|=\frac{1}{2}|S|\right\}\\ &\quad=\bigsqcup_{x\in S_2}\left\{\{x,y\}\,\middle|\, y\in S, x\not\sim y\right\}\\ &\quad=\bigsqcup_{P\in\{T,S_0\setminus(S\cup T),S_2,S_4\}}\ \bigsqcup_{x\in S}\left\{\{x,y\}\,\middle|\, y\in P, x\not\sim y, |N(x)\cap P|=\frac{1}{2}|P|\right\};\end{aligned}$$

so we can see the following:

Corollary 4.8. *X^S is isomorphic to the switched symplectic graph with respect to the Godsil–McKay partition $\{S, V(X)\setminus S\}$ with the Godsil–McKay cell $V(X)\setminus S$.*

We remark that for $x\in S_2(1,2)$ as a special vertex in S_2, $N(x)\cap T=\{v_2+v_3, v_3+v_1\}$, that is, $|N(x)\cap T|=\frac{2}{3}|T|$, so S_2 is not a Godsil–McKay cell. Therefore by Lemma 4.6, we get $|N(x)\cap S_2|=\frac{2}{3}|S_2|$ for $x\in T$, so T is not a Godsil–McKay cell either.

5. Not being isomorphic

In this section, we prove that the graphs in the five families X, $X^{O(0,\nu,0)}$, X^S, $X^{S_0\setminus(S\cup T)}$, X^{S_4} are not isomorphic to each other. To this end, we consider the number of common neighbors of three vertices as an invariant for isomorphism. First, we investigate how the value of the number of common neighbors of three vertices changes after switching. Next, for each family, by inspecting the non-zero minimum number of common neighbors of three vertices, we prove that the graphs in different families are not isomorphic.

5.1. **Formulas that give the number of common neighbors of three vertices in the switched graph.** Let X be a graph and let A, B be subsets of the vertex set $V(X)$ which are disjoint. We define

$$\mathcal{N}_X[A:B]=\left\{w\in V(X)\setminus(A\cup B)\,\middle|\, \begin{array}{l}w\sim a\,(\forall a\in A),\\ w\not\sim b\,(\forall b\in B)\end{array}\right\}.$$

Practically, we consider the case $|A \cup B| = 3$. For example, for three distinct vertices x, y, z in $V(X)$,

$$\mathcal{N}_X[\{x,y\} : \{z\}] = \{w \in V(X) \setminus \{x,y,z\} \mid w \sim x,\, w \sim y,\, w \not\sim z\},$$

but we will write $\mathcal{N}_X[xy : z]$ instead of $\mathcal{N}_X[\{x,y\} : \{z\}]$ for simplicity.

Let $\pi = \{C_1, \ldots, C_t, C_{t+1}\}$ be the orbit partition of a group of automorphisms of X. Assume that π is a Godsil–McKay partition with the Godsil–McKay cell $D = C_{t+1}$. Then for any $i \in [t]$,

$$\{|N(x) \cap C_i| \mid x \in D\} = \{0\}, \left\{\frac{1}{2}|C_i|\right\} \text{ or } \{|C_i|\},$$

so we can decompose the index set $[t]$ depending on these values. We define

$$\begin{aligned}
\mathcal{C}_0 &= \left\{ i \in [t] \,\middle|\, \{|N(x) \cap C_i| \mid x \in D\} = \{0\} \right\}, \\
\mathcal{C}_{\frac{1}{2}} &= \left\{ i \in [t] \,\middle|\, \{|N(x) \cap C_i| \mid x \in D\} = \left\{\frac{1}{2}|C_i|\right\} \right\}, \\
\mathcal{C}_1 &= \left\{ i \in [t] \,\middle|\, \{|N(x) \cap C_i| \mid x \in D\} = \{|C_i|\} \right\}.
\end{aligned}$$

Then $[t] = \mathcal{C}_0 \sqcup \mathcal{C}_{\frac{1}{2}} \sqcup \mathcal{C}_1$. Let X' be the switched graph with respect to π with $D = C_{t+1}$. To investigate the number of common neighbors of three vertices in X', we consider, for example, the case $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$. The set of pairs of vertices involved with switching is

$$\bigsqcup_{i \in \mathcal{C}_{\frac{1}{2}}} \left\{ \{v, w\} \,\middle|\, v \in D, w \in C_i \right\},$$

so the vertices in

$$\bigsqcup_{i \in \mathcal{C}_1} (C_i \cap \mathcal{N}_X[xyz :\,])$$

are also common neighbors of x, y, z in X'. On the other hand, in this case, the vertices in

$$\bigsqcup_{i \in \mathcal{C}_{\frac{1}{2}} \sqcup \{t+1\}} (C_i \cap \mathcal{N}_X[xyz :\,])$$

are no longer common neighbors of x, y, z after switching. However, the vertices in

$$\left(\bigsqcup_{i \in \mathcal{C}_{\frac{1}{2}}} (C_i \cap \mathcal{N}[yz : x]) \right) \cup (D \cap \mathcal{N}_X[xz : y])$$

become new common neighbors after switching. Consequently, we get

$$|\mathcal{N}_{X'}[xyz :\,]| = \sum_{i \in \mathcal{C}_1} |C_i \cap \mathcal{N}_X[xyz :\,]| + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} |C_i \cap \mathcal{N}_X[yz : x]| + |D \cap \mathcal{N}_X[xz : y]|.$$

For other cases, we can investigate $\mathcal{N}_{X'}[xyz :\,]$ by a similar argument as above, so we get the following formulas on the number of common neighbors of three vertices in X'.

Theorem 5.1. *Let X be a graph and $\pi = \{C_1, \ldots, C_t, C_{t+1}\}$ be the orbit partition of a group of automorphisms. Assume that π is a Godsil–McKay partition with a Godsil–McKay cell $D = C_{t+1}$. Let X' be the switched graph with respect to π. Let x, y, z be three distinct vertices in $V(X)$ and i_x, i_y, i_z be the indices of C_{i_x}, C_{i_y}, C_{i_z}, to which x, y, z belong, respectively. Then for each of the following ten cases, the values of $|\mathcal{N}_{X'}[xyz:]|$ are given in Table 5.1.*

(1) $x, y, z \notin D$
- (i) $i_x, i_y, i_z \notin \mathcal{C}_{\frac{1}{2}}$,
- (ii) $i_x \in \mathcal{C}_{\frac{1}{2}}$ *and* $i_y, i_z \notin \mathcal{C}_{\frac{1}{2}}$,
- (iii) $i_x, i_y \in \mathcal{C}_{\frac{1}{2}}$ *and* $i_z \notin \mathcal{C}_{\frac{1}{2}}$,
- (iv) $i_x, i_y, i_z \in \mathcal{C}_{\frac{1}{2}}$.

(2) $x \in D$ *and* $y, z \notin D$
- (i) $i_y, i_z \notin \mathcal{C}_{\frac{1}{2}}$,
- (ii) $i_y \in \mathcal{C}_{\frac{1}{2}}$ *and* $i_z \notin \mathcal{C}_{\frac{1}{2}}$,
- (iii) $i_y, i_z \in \mathcal{C}_{\frac{1}{2}}$.

(3) $x, y \in D$ *and* $z \notin D$
- (i) $i_z \notin \mathcal{C}_{\frac{1}{2}}$,
- (ii) $i_z \in \mathcal{C}_{\frac{1}{2}}$.

(4) $x, y, z \in D$.

(1)-(i)	$\lvert\mathcal{N}_X[xyz:]\rvert$
(1)-(ii)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_{\frac{1}{2}} \sqcup \mathcal{C}_1} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \lvert D \cap \mathcal{N}_X[yz:x]\rvert$
(1)-(iii)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_{\frac{1}{2}} \sqcup \mathcal{C}_1} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \lvert D \cap \mathcal{N}_X[z:xy]\rvert$
(1)-(iv)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_{\frac{1}{2}} \sqcup \mathcal{C}_1} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \lvert D \cap \mathcal{N}_X[:xyz]\rvert$
(2)-(i)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1 \sqcup \{t+1\}} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} \lvert C_i \cap \mathcal{N}_X[yz:x]\rvert$
(2)-(ii)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} \lvert C_i \cap \mathcal{N}_X[yz:x]\rvert + \lvert D \cap \mathcal{N}_X[xz:y]\rvert$
(2)-(iii)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} \lvert C_i \cap \mathcal{N}_X[yz:x]\rvert + \lvert D \cap \mathcal{N}_X[x:yz]\rvert$
(3)-(i)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1 \sqcup \{t+1\}} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} \lvert C_i \cap \mathcal{N}_X[z:xy]\rvert$
(3)-(ii)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} \lvert C_i \cap \mathcal{N}_X[z:xy]\rvert + \lvert D \cap \mathcal{N}_X[xy:z]\rvert$
(4)	$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1 \sqcup \{t+1\}} \lvert C_i \cap \mathcal{N}_X[xyz:]\rvert + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} \lvert C_i \cap \mathcal{N}_X[:xyz]\rvert$

Table 5.1. The number of common neighbors of three vertices in X'

5.2. **Investigating the non-zero minimum number of common neighbors of three vertices.** We use Table 5.1 to investigate the number of common neighbors of three vertices for each family. It is certainly difficult to determine all the possible values, but our goal is to prove that the graphs in the five families are not isomorphic to each other, so it is sufficient to find an easier invariant for isomorphism. From this point of view, we calculate the non-zero minimum number of common neighbors of three vertices. Similarly to the previous section, T_M denotes the linear mapping $x \mapsto Mx$ for a matrix M.

Proposition 5.2. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν and let x, y, z be three distinct vertices of X. Then,*

$$|\mathcal{N}_X[xyz:]| = \begin{cases} 0 & \textit{if } x+y+z = \mathbf{0}, \\ 2^{2\nu-3} & \textit{otherwise.} \end{cases}$$

In particular, the non-zero minimum number of common neighbors of three vertices in $Sp(2\nu, 2)$ is $2^{2\nu-3}$.

Proof. First, we assume $x+y+z = \mathbf{0}$. Suppose that there exists $w \in \mathcal{N}_X[xyz:]$. Then $x^TKw = y^TKw = z^TKw = 1$, but $1 = 1+1+1 = x^TKw + y^TKw + z^TKw = (x+y+z)^TKw = \mathbf{0}^TKw = 0$. This is a contradiction; so, $|\mathcal{N}_X[xyz:]| = 0$.

Next, we assume $x+y+z \neq \mathbf{0}$. Let

$$M = \begin{bmatrix} x^TK \\ y^TK \\ z^TK \end{bmatrix}.$$

Then $\mathcal{N}_X[xyz:] = \{w \in \mathbb{F}_2^{2\nu} \mid Mw = [111]^T\}$. Since $x+y+z \neq \mathbf{0}$, x, y, z are linearly independent; so, $\operatorname{rank} M = 3$. Therefore the system of equations $Mw = \mathbf{1}_3$ has a solution. This implies that there is a bijection from $\mathcal{N}_X[xyz:]$ to $\operatorname{Ker} T_M$. The dimension of $\operatorname{Ker} T_M$ is $2\nu - 3$, so we get $|\mathcal{N}_X[xyz:]| = |\operatorname{Ker} T_M| = 2^{2\nu-3}$. $\square$

Proposition 5.3. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν, and let $X' = X^S$. Take $x \in S_2(1,2)$, $y \in S_2(1,3)$ and set $z = x+y$. Then $z \in S_2(2,3)$ and $|\mathcal{N}_{X'}[xyz:]| = 1$. Therefore, the non-zero minimum number of common neighbors of three vertices in X^S is 1.*

Proof. Since

$$z^TKv_i = x^TKv_i + y^TKv_i = \begin{cases} 1 & \text{if } i = 2,3, \\ 0 & \text{if } i = 1,4, \end{cases}$$

we have $z \in S_2(2,3)$. We recall that for $C \in \{T, S_0 \setminus (S \cup T), S_2, S_4\}$ and for $v \in S$,

$$|N(v) \cap C| = \begin{cases} 0 & \text{if } C = T \text{ or } S_0 \setminus (S \cup T), \\ |C| & \text{if } C = S_4, \\ \frac{1}{2}|C| & \text{if } C = S_2, \end{cases}$$

by Proposition 4.7-(i). Thus,

$$\begin{aligned}
|\mathcal{N}_{X'}[xyz:]| &= |(T \cup (S_0 \setminus (S \cup T)) \cup S_4 \cup S_2) \cap \mathcal{N}_X[xyz:]| \\
&\quad + |S \cap \mathcal{N}_X[:xyz]| && \text{(by (1)-(iv) in Table 5.1)} \\
&= |S \cap \mathcal{N}_X[:xyz]| && \text{(by Proposition 5.2)} \\
&= |\{v_4\}| \\
&= 1.
\end{aligned}$$

□

Next, we consider the non-zero minimum number of common neighbors of three vertices in $X^{O(0,\nu,0)}$. If we decompose a vector $x \in \mathbb{F}_2^{2\nu}$ into ν blocks as follows:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_\nu \end{bmatrix} \qquad (x_i \in \mathbb{F}_2^2),$$

we can see

$$\begin{bmatrix} 1100 & \cdots & 00 \\ 0011 & \cdots & 00 \\ \vdots & \ddots & \vdots \\ 0000 & \cdots & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_\nu \end{bmatrix} \equiv \begin{bmatrix} \mathrm{wt}(x_1) \\ \mathrm{wt}(x_2) \\ \vdots \\ \mathrm{wt}(x_\nu) \end{bmatrix} \pmod 2.$$

Thus, for a vector $x \in \mathbb{F}_2^{2\nu}$, there exists j such that $x \in O(i,j,k)$ for some i, k if and only if there exists a vector $\boldsymbol{b} \in \mathbb{F}_2^{\nu}$ whose weight is j such that $(I_\nu \otimes [11])x = \boldsymbol{b}$, so we can regard also an orbit of $\mathrm{Aut}(X)_{\mathcal{E}}$ as the solution set of a system of linear equations.

Recall that for an orbit $O(i,j,k)$ of $\mathrm{Aut}(X)_{\mathcal{E}}$ and for a vertex $v \in O(0,\nu,0)$,

$$|N(v) \cap O(i,j,k)| = \begin{cases} \frac{1}{2}|O(i,j,k)| & \text{if } j \geq 1, \\ |O(i,j,k)| & \text{if } j = 0,\ i: \text{odd}, \\ 0 & \text{if } j = 0,\ i: \text{even}, \end{cases}$$

by Proposition 3.5. For three vertices x, y, z, define the $(\nu+3) \times 2\nu$ matrix M as follows:

$$M = \begin{bmatrix} x^T K \\ y^T K \\ z^T K \\ I_\nu \otimes [11] \end{bmatrix}.$$

Lemma 5.4. *Let X be the symplectic graph $Sp(2\nu,2)$ of order 2ν, and let $X' = X^{O(0,\nu,0)}$. For three distinct vertices x, y, z, $|\mathcal{N}_{X'}[xyz:]|$ is a multiple of $2^{\nu-2}$.*

Proof. For three distinct vertices x, y, z, we consider two cases.

Case 1: Suppose M has full rank. We only consider the case (1)-(ii) of Theorem 5.1, but on other cases, we can consider similarly. Assume that $x \in O(i,j,k)$

with $1 \le j \le \nu - 1$, $y \in O(l, 0, m)$ and $z \in O(l', 0, m')$. According to Table 5.1,

$$|\mathcal{N}_{X'}[xyz:]| = \sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1} |C_i \cap \mathcal{N}_X[xyz:]| + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} |C_i \cap \mathcal{N}_X[xyz:]| + |D \cap \mathcal{N}_X[yz:x]|.$$

The first term is equal to $\#\{w \in \mathbb{F}_2^{2\nu} \mid Mw = [111\mathbf{0}_\nu^T]^T\}$. Since M has full rank, it is $2^{2\nu-(\nu+3)} = 2^{\nu-3}$. The second term is equal to

$$\sum_{\substack{\boldsymbol{b} \in \mathbb{F}_2^\nu, \\ 1 \le \mathrm{wt}(\boldsymbol{b}) \le \nu-1}} \# \left\{ w \in \mathbb{F}_2^{2\nu} \;\middle|\; Mw = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boldsymbol{b} \end{bmatrix} \right\},$$

but M has full rank, so it is $(2^\nu - 2) \cdot 2^{\nu-3}$. The third term is equal to $\#\{w \in \mathbb{F}_2^{2\nu} \setminus \{x\} \mid Mw = [011\mathbf{1}_\nu^T]^T\}$, but since $x \in O(i, j, k)$ with $1 \le j \le \nu - 1$, $(I_\nu \otimes [11]^T)x \ne \mathbf{1}_\nu$. Thus, x is not a solution of $Mw = [011\mathbf{1}_\nu^T]^T$, so we get $|D \cap \mathcal{N}_X[yz:x]| = 2^{\nu-3}$. Consequently,

$$|\mathcal{N}_{X'}[xyz:]| = 2^{\nu-3} + (2^\nu - 2) \cdot 2^{\nu-3} + 2^{\nu-3} = 2^{2\nu-3}.$$

In particular, $|\mathcal{N}_{X'}[xyz:]|$ is a multiple of $2^{\nu-2}$. (Note that on other cases, if M has full rank, then $|\mathcal{N}_{X'}[xyz:]| = 2^{2\nu-3}$.)

Case 2: Suppose M does not have full rank, that is, $\operatorname{rank} M = \nu$, $\nu + 1$ or $\nu + 2$. We argue similarly to the case 1. For each case, there exist proper subsets A, B, A', B' of $V(X)$ such that

$$|\mathcal{N}_{X'}[xyz:]| = \sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1} |C_i \cap \mathcal{N}_X[xyz:]| + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} |C_i \cap \mathcal{N}_X[A:B]| + |D \cap \mathcal{N}_X[A':B']|$$

and we can confirm that x, y, z, $\mathbf{0}$ are not a solution of the system of linear equations determined by each term. Thus, each term is a multiple of $2^{2\nu - \operatorname{rank} M}$, but $\operatorname{rank} M \le \nu + 2$ in this case. Therefore, $|\mathcal{N}_{X'}[xyz:]|$ is a multiple of $2^{\nu-2}$. $\square$

Fortunately, we can take three vertices that give $|\mathcal{N}_{X'}[xyz:]| = 2^{\nu-2}$.

Proposition 5.5. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν, and let $X' = X^{O(0,\nu,0)}$. Then the non-zero minimum number of common neighbors of three vertices in $X^{O(0,\nu,0)}$ is $2^{\nu-2}$.*

Proof. Pick $x = [101000\mathbf{0}_{2\nu-6}]^T$, $y = [100010\mathbf{0}_{2\nu-6}] \in O(0, 2, \nu-2)$, and set $z = x + y$. Then $z = [001010\mathbf{0}_{2\nu-6}] \in O(0, 2, \nu-2)$. Thus, by the case (1)-(iv) of Theorem 5.1, we get $|\mathcal{N}_{X'}[xyz:]| = |D \cap \mathcal{N}_X[:xyz]|$. Also,

$$\begin{aligned} D \cap \mathcal{N}_X[:xyz] &= \{w \in \mathbb{F}_2^{2\nu} \setminus \{x, y, z, \mathbf{0}\} \mid Mw = [000\mathbf{1}_\nu^T]^T\} \\ &= \{w \in \mathbb{F}_2^{2\nu} \mid Mw = [000\mathbf{1}_\nu^T]^T\} \\ &= \left\{ w \in \mathbb{F}_2^{2\nu} \;\middle|\; \begin{bmatrix} x^T K \\ y^T K \\ I_\nu \otimes [11] \end{bmatrix} w = \begin{bmatrix} 0 \\ 0 \\ \mathbf{1}_\nu \end{bmatrix} \right\}, \end{aligned}$$

and the matrix $\begin{bmatrix} x^T K \\ y^T K \\ I_\nu \otimes [11] \end{bmatrix}$ has rank $\nu + 2$, which has full rank. Thus, we see that $|\mathcal{N}_{X'}[xyz :]| = |D \cap \mathcal{N}_X[: xyz]| = 2^{\nu-2}$. $\square$

Next, we consider the family X^{S_4}. Recall that for an orbit $C \in \{S, T, S_0 \setminus \{S \cup T\}, S_2\}$ of $\mathrm{Aut}(X)_S$ and for a vertex $v \in S_4$,

$$|N(v) \cap C| = \begin{cases} 0 & \text{if } C = T, \\ |C| & \text{if } C = S, \\ \frac{1}{2}|C| & \text{if } C = S_0 \setminus \{S \cup T\} \text{ or } S_2, \end{cases}$$

by Proposition 4.7-(ii). Also, for three vertices $x, y, z \in V(X)$, we redefine the matrix M as follows:

$$M = \begin{bmatrix} x^T K \\ y^T K \\ z^T K \\ v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix}.$$

Lemma 5.6. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν and let $X' = X^{S_4}$. Assume that x, y, z are three distinct vertices in X'. If $\mathcal{N}_{X'}[xyz :]$ is nonempty, then $|\mathcal{N}_{X'}[xyz :]| \geq 2^{2\nu-5}$.*

Proof. For three distinct vertices x, y, z, we consider two cases.

Case 1: Suppose M has full rank. We consider the case (2)-(i) of Theorem 5.1 for example, but we can consider similarly on other cases too. Assume that $x \in D = S_4$ and $y, z \in S \cup T$. By the case (2)-(i) of Theorem 5.1,

$$\begin{aligned} |\mathcal{N}_{X'}[xyz :]| &\geq |S_2 \cap \mathcal{N}_X[yz : x]| \\ &= \sum_{\substack{\boldsymbol{b} \in \mathbb{F}_2^3, \\ \mathrm{wt}(\boldsymbol{b}) \in \{1,2\}}} \# \left\{ w \in \mathbb{F}_2^{2\nu} \setminus \{x, y, z, \mathbf{0}\} \;\middle|\; Mw = \begin{bmatrix} 0 \\ 1 \\ 1 \\ \boldsymbol{b} \end{bmatrix} \right\}. \end{aligned}$$

Clearly, $y, z, \mathbf{0}$ are not a solution of $Mw = [011\boldsymbol{b}^T]^T$. Also, since $x \in S_4$,

$$\begin{bmatrix} v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, x is not a solution of $Mw = [011\boldsymbol{b}^T]^T$, either. Therefore, since M has full rank, we see that

$$|\mathcal{N}_{X'}[xyz :]| \geq 6 \cdot 2^{2\nu-6} \geq 2^{2\nu-5}.$$

Accordingly, $\mathcal{N}_{X'}[xyz :]$ is always nonempty in this case.

Case 2: Suppose M does not have full rank, that is, $\operatorname{rank} M = 3$, 4 or 5. We consider the case (1)-(ii) of Theorem 5.1 for example and we can argue similarly on other cases except three cases (2)-(iii), (3)-(ii) and (4). Assume that $x \in (S_0 \setminus (S \cup T)) \cup S_2$ and $y, z \in S \cup T$. By the case (1)-(ii) of Theorem 5.1,

$$\begin{aligned}\mathcal{N}_{X'}[xyz:] = ((S \cup T) \cap \mathcal{N}_X[xyz:]) \sqcup ((S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[xyz:]) \\ \sqcup (S_2 \cap \mathcal{N}_X[xyz:]) \sqcup (S_4 \cap \mathcal{N}_X[yz:x]).\end{aligned}$$

Since $y, z \in S_0$, $(S \cup T) \cap \mathcal{N}_X[xyz:] = \emptyset$. By $\mathcal{N}_{X'}[xyz:] \neq \emptyset$, one of $(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[xyz:]$, $S_2 \cap \mathcal{N}_X[xyz:]$ or $S_4 \cap \mathcal{N}_X[yz:x]$ is nonempty. We suppose $(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[xyz:] \neq \emptyset$ first. We see that

$$(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[xyz:] = \{w \in \mathbb{F}_2^{2\nu} \setminus (\{x, y, z, \mathbf{0}\} \cup S \cup T) \mid Mw = [111000]^T\},$$

but any vector in $\{x, y, z, \mathbf{0}\} \cup S \cup T$ is not a solution of $Mw = [111000]^T$ since $y, z \in S_0$. On the other hand, $Mw = [111000]^T$ has a solution, so

$$\begin{aligned}|(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[xyz:]| &= \#\{w \in \mathbb{F}_2^{2\nu} \mid Mw = [111000]^T\} \\ &= 2^{2\nu - \operatorname{rank} M} \\ &\geq 2^{2\nu-5}.\end{aligned}$$

Next, we suppose $S_2 \cap \mathcal{N}_X[xyz:] \neq \emptyset$. Then there exists a vector $\boldsymbol{b}$ whose weight is 1 or 2 such that

$$\{w \in \mathbb{F}_2^{2\nu} \setminus \{x, y, z, \mathbf{0}\} \mid Mw = [111\boldsymbol{b}^T]^T\} \neq \emptyset.$$

Since x, y, z, $\mathbf{0}$ are not a solution of $Mw = [111\boldsymbol{b}^T]^T$,

$$\begin{aligned}|S_2 \cap \mathcal{N}_X[xyz:]| &\geq \#\{w \in \mathbb{F}_2^{2\nu} \mid Mw = [111\boldsymbol{b}^T]^T\} \\ &= 2^{2\nu - \operatorname{rank} M} \\ &\geq 2^{2\nu-5}.\end{aligned}$$

Finally, we suppose $S_4 \cap \mathcal{N}_X[yz:x] \neq \emptyset$. Then

$$\{w \in \mathbb{F}_2^{2\nu} \setminus \{x, y, z, \mathbf{0}\} \mid Mw = [011111]^T\} \neq \emptyset,$$

but x, y, z, $\mathbf{0}$ are not a solution of $Mw = [011111]^T$ since $x \notin S_4$. Thus,

$$|S_4 \cap \mathcal{N}_X[yz:x]| = 2^{2\nu - \operatorname{rank} M} \geq 2^{2\nu-5}.$$

In this way, we can basically prove that for appropriate subsets A, A', A'', B, B', $B'' \subset V(X)$ determined by each case of Table 5.1,

- $(S \cup T) \cap \mathcal{N}_X[xyz:] = \emptyset$,
- If $(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[A:B] \neq \emptyset$, then $|(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[A:B]| \geq 2^{2\nu-5}$,
- If $S_2 \cap \mathcal{N}_X[A':B'] \neq \emptyset$, then $|S_2 \cap \mathcal{N}_X[A':B']| \geq 2^{2\nu-5}$,
- If $S_4 \cap \mathcal{N}_X[A'':B''] \neq \emptyset$, then $|S_4 \cap \mathcal{N}_X[A'':B'']| \geq 2^{2\nu-5}$,

so we can see that $|\mathcal{N}_{X'}[xyz:]| \geq 2^{2\nu-5}$ as a result. However, if $x, y, z \in S_2 \cup S_4$, then $(S \cup T) \cap \mathcal{N}_X[xyz:] \neq \emptyset$ can occur. Thus, we need other arguments on the cases (2)-(iii), (3)-(ii) and (4).

(I) The case (2)-(iii), especially, $x \in D = S_4$ and $y, z \in S_2$.

- If $y, z \in S_2(i,j)$, then $(S \cup T) \cap \mathcal{N}_X[xyz:] = \{v_i, v_j\}$, but $S_0 \cap \mathcal{N}_X[yz:x] \cap \langle S \rangle = \{v_i + v_k, v_j + v_k\}$ for $k \in [4] \setminus \{i,j\}$, so
$$|\mathcal{N}_{X'}[xyz:]| \geq 2 + (2^{2\nu - \operatorname{rank} M} - 2) \geq 2^{2\nu-5}.$$
- If $y \in S_2(i,j)$ and $z \in S_2(i,k)$ for distinct indices i, j, k, then $(S \cup T) \cap \mathcal{N}_X[xyz:] = \{v_i\}$, but $S_0 \cap \mathcal{N}_X[yz:x] \cap \langle S \rangle = \{v_j + v_k\}$, so
$$|\mathcal{N}_{X'}[xyz:]| \geq 1 + (2^{2\nu - \operatorname{rank} M} - 1) \geq 2^{2\nu-5}.$$
- If $y \in S_2(i,j)$ and $z \in S_2(k,l)$ for distinct indices i, j, k, l, then $(S \cup T) \cap \mathcal{N}_X[xyz:] = \emptyset$. Thus, this case is no problem because we can use a "basis" argument.

(II) The case (3)-(ii), especially, $x, y \in D = S_4$ and $z \in S_2$. Assume that $z \in S_2(i,j)$. Then $(S \cup T) \cap \mathcal{N}_X[xyz:] = \{v_i, v_j\}$, but $S_0 \cap \mathcal{N}_X[z:xy] \cap \langle S \rangle = \{v_i + v_k, v_j + v_k\}$ for $k \in [4] \setminus \{i,j\}$, so
$$|\mathcal{N}_{X'}[xyz:]| \geq 2 + (2^{2\nu - \operatorname{rank} M} - 2) \geq 2^{2\nu-5}.$$

(III) The case (4). Then $(S \cup T) \cap \mathcal{N}_X[xyz:] = S$, but $S_0 \cap \mathcal{N}_X[:xyz] \cap \langle S \rangle = T \cup \{\mathbf{0}\}$, so
$$|\mathcal{N}_{X'}[xyz:]| \geq 4 + (2^{2\nu - \operatorname{rank} M} - 4) \geq 2^{2\nu-5}.$$

Consequently, we can get the desired inequality for all cases. □

Proposition 5.7. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν and let $X' = X^{S_4}$. Then the non-zero minimum number of common neighbors of three vertices in X^{S_4} is $2^{2\nu-5}$.*

Proof. We take $x \in S_2(1,2)$ and $y \in S_2(2,3)$ and set $z = x + y$. Then $z \in S_2(1,3)$, so by the case (1)-(iv) of Theorem 5.1, we get
$$|\mathcal{N}_{X'}[xyz:]| = |(S_0 \cup S_2) \cap \mathcal{N}_X[xyz:]| + |S_4 \cap \mathcal{N}_X[:xyz]|.$$
Since $x + y + z = \mathbf{0}$, the first term of the right hand side is zero by Proposition 5.2. Thus,
$$\begin{aligned} |\mathcal{N}_{X'}[xyz:]| &= |S_4 \cap \mathcal{N}_X[:xyz]| \\ &= \#\{w \in \mathbb{F}_2^{2\nu} \mid Mw = [000111]^T\} \\ &= \#\left\{ w \in \mathbb{F}_2^{2\nu} \;\middle|\; \begin{bmatrix} x^T K \\ y^T K \\ v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix} w = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \end{aligned}$$
but x, y, v_1, v_2, v_3 are linearly independent. Therefore, $\begin{bmatrix} x^T K \\ y^T K \\ v_1^T K \\ v_2^T K \\ v_3^T K \end{bmatrix}$ has full rank, so we see that $|\mathcal{N}_{X'}[xyz:]| = 2^{2\nu-5}$. □

Finally, we consider the family $X^{S_0\setminus(S\cup T)}$. Recall that for an orbit $C \in \{S, T, S_2, S_4\}$ of $\mathrm{Aut}(X)_S$ and for a vertex $v \in S_4$,

$$|N(v) \cap C| = \begin{cases} 0 & \text{if } C = S \text{ or } T, \\ \frac{1}{2}|C| & \text{if } C = S_2 \text{ or } S_4, \end{cases}$$

by Proposition 4.7-(iii). We prove that the non-zero minimum number of common neighbors of three vertices in $X^{S_0\setminus(S\cup T)}$ is $2^{2\nu-5} - 2$, but its proof is similar to the one in X^{S_4} basically, that is, we can see the following for appropriate subsets A, A', A'', B, B', $B'' \subset V(X)$ determined by each case of Table 5.1.

(I) When M has full rank, we see that

$$|\mathcal{N}_{X'}[xyz :]| \geq |S_2 \cap \mathcal{N}_X[A : B]| \geq 2^{2\nu-5} \geq 2^{2\nu-5} - 2.$$

(II) When M does not have full rank, we can prove the following except the case (1)-(iv).

- $(S \cup T) \cap \mathcal{N}_X[xyz :] = \emptyset$,
- If $S_2 \cap \mathcal{N}_X[A : B] \neq \emptyset$, then $|S_2 \cap \mathcal{N}_X[A : B]| \geq 2^{2\nu-5}$,
- If $S_4 \cap \mathcal{N}_X[A' : B'] \neq \emptyset$, then $|S_4 \cap \mathcal{N}_X[A' : B']| \geq 2^{2\nu-5}$,
- If $(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[A'' : B''] \neq \emptyset$, then $|(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[A'' : B'']| \geq 2^{2\nu-5}$.

The exceptional case (1)-(iv) is proved as follows.

Lemma 5.8. *Let X be the symplectic graph $Sp(2\nu, 2)$ of order 2ν and let $X' = X^{S_0\setminus(S\cup T)}$. Suppose that $x, y, z \in S_2 \cup S_4$ and M does not have full rank. Then $|\mathcal{N}_{X'}[xyz :]| \geq 2^{2\nu-5} - 2$.*

Proof. By the case (1)-(iv) of Theorem 5.1,

$$|\mathcal{N}_{X'}[xyz :]| \geq |(S \cup T) \cap \mathcal{N}_X[xyz :]| + |(S_0 \setminus (S \cup T) \cap \mathcal{N}_X[: xyz]|.$$

Since x, y, z are not a solution of the system of equations $Mw = \mathbf{0}_6$,

$$(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[: xyz] = \{w \in F_2^{2\nu} \mid Mw = \mathbf{0}_6\} \setminus \langle S \rangle.$$

Thus,

$$\begin{aligned} |(S_0 \setminus (S \cup T)) \cap \mathcal{N}_X[: xyz]| &= |\mathrm{Ker}\, T_M| - |\mathrm{Ker}\, T_M \cap \langle S \rangle| \\ &\geq 2^{2\nu-5} - |\mathrm{Ker}\, T_M \cap \langle S \rangle|, \end{aligned}$$

but since $x, y, z \in S_2 \cup S_4$, $\dim \mathrm{Ker}\, T_M \cap \langle S \rangle \leq 2$. If $\dim \mathrm{Ker}\, T_M \cap \langle S \rangle = 0$ or 1, then we can get the desired inequality, so we assume $\dim \mathrm{Ker}\, T_M \cap \langle S \rangle = 2$. We aim to prove that $|(S \cup T) \cap \mathcal{N}_X[xyz :]| \geq 2$ in this case. (Actually, we can prove $|(S \cup T) \cap \mathcal{N}_X[xyz :]| = 4$.) Observe that there exist two distinct indices $i, j \in [4]$ such that $Mv_i \neq \mathbf{0}_6$ and $Mv_j \neq \mathbf{0}_6$. Since $|\mathrm{Ker}\, T_M \cap \langle S \rangle| = 4$, there exist two distinct indices $k, l \in [4]$ such that $v_k + v_l \in \mathrm{Ker}\, T_M \cap \langle S \rangle$. We can assume $k = 1$, $l = 2$ without loss of generality. It is sufficient to check the following two cases.

Case 1: Suppose that $\mathrm{Ker}\, T_M \cap \langle S \rangle = \langle v_1 + v_2, v_1 \rangle$. Observe $x, y, z \in S_2(3, 4)$, and we see that

$$|(S \cup T) \cap \mathcal{N}_X[xyz :]| = \#\{v_3, v_4, v_1 + v_3, v_2 + v_3\} = 4.$$

Case 2: Suppose that $\operatorname{Ker} T_M \cap \langle S \rangle = \langle v_1+v_2, v_2+v_3 \rangle$. If we suppose $z \in S_2(i,j)$, then $z^T K(v_i+v_k) = 1$ for $k \in [4] \setminus \{i,j\}$, but this is a contradiction. Thus, x, y, z have to be in S_4. Consequently,

$$|(S \cup T) \cap \mathcal{N}_X[xyz:]| = \#\{v_1, v_2, v_3, v_4\} = 4.$$

□

Proposition 5.9. *Let X be the symplectic graph $Sp(2\nu,2)$ of order 2ν and let $X' = X^{S_0 \setminus (S \cup T)}$. Then the non-zero minimum number of common neighbors of three vertices in $X^{S_0 \setminus (S \cup T)}$ is $2^{2\nu-5} - 2$.*

Proof. Take $y \in S_2(1,2)$ and $z \in S_2(3,4)$ and set $x = y + z$. Then $x \in S_4$ and by using the case (1)-(iv) of Theorem 5.1, we can check $|\mathcal{N}_{X'}[xyz:]| = 2^{2\nu-5} - 2$. □

Summarizing this subsection, we get the following:

Theorem 5.10. *Let X be the symplectic graph $Sp(2\nu,2)$ of order 2ν. The non-zero minimum numbers of common neighbors of three distinct vertices for each graphs are given in the Table 5.2.*

X	X^S	$X^{O(0,\nu,0)}$	X^{S_4}	$X^{S_0 \setminus (S \cup T)}$
$2^{2\nu-3}$	1	$2^{\nu-2}$	$2^{2\nu-5}$	$2^{2\nu-5} - 2$

Table 5.2. The non-zero minimum numbers of common neighbors of three distinct vertices

In particular, the five graphs X, $X^{O(0,\nu,0)}$, X^S, X^{S_4}, $X^{S_0 \setminus (S \cup T)}$ are not isomorphic to each other for a fixed ν.

Remark that $2^{\nu-2} = 2^{2\nu-5} = 2$ in the case $\nu = 3$, so we do not know whether $X^{O(0,\nu,0)}$ is isomorphic to X^{S_4} or not in this case. Actually, we see that the two graphs are isomorphic to each other by computer.

6. Recent studies and comparison for the case $\nu = 4$

Recently, other researchers also constructed strongly regular graphs with the same parameters as (the complement of) the symplectic graph by each methods. We refer to [4, 8, 9]. In this section, we compare with the graphs constructed by other researchers and confirm whether their and our graphs are isomorphic to each other or not for $\nu = 4$ by computer. As for notation, we use the same ones in the original papers written by each authors.

Barwick, Jackson and Penttila [4] constructed $\nu - 1$ strongly regular graphs Γ_0, $\dots$, $\Gamma_{\nu-2}$ with the same parameters as the complement of the symplectic graph for each ν. Moreover, they proved that their $\nu - 1$ graphs are not isomorphic to each other by counting the number of maximal cliques. (Of these, Γ_0 is isomorphic to $\overline{Sp(2\nu,2)}$.) Note that when we specify α_s as the subspace spanned by e_1, e_3, $\dots$,

e_{2s+1}, then their switching set $\mathcal{X}_s$ is equal to

$$\left\{ x \in \mathbb{F}_2^{2\nu+1} \;\middle|\; \begin{array}{r} x_2 = x_4 = \cdots = x_{2s+2} = 0, \\ [x_{2s+3}, x_{2s+4}, \dots, x_{2\nu+1}] \neq \mathbf{0}, \\ x_{2\nu+1}^2 = \displaystyle\sum_{i=1}^{\nu} x_{2i-1}x_{2i} \end{array} \right\}.$$

Thus, this corresponds to

$$\left\{ x \in \mathbb{F}_2^{2\nu} \;\middle|\; \begin{array}{r} x_2 = x_4 = \cdots = x_{2s+2} = 0, \\ [x_{2s+3}, x_{2s+4}, \dots, x_{2\nu+1}] \neq \mathbf{0} \end{array} \right\}$$

in the vertex set of the symplectic graph.

Ihringer [8] also constructed many strongly regular graphs with the same parameters as the complement of the symplectic graph. His graphs are defined by a $(\nu-1)$-dimensional isotropic subspace L and three permutations σ_1, σ_2, σ_3. By his construction, we can get many graphs, but because of its quantity, "which two graphs are non-isomorphic?" and "how many pairwise non-isomoephic graphs can we get?" are still open. On the other hand, as an example, he provides the permutation σ_i defined by $\sigma_i(M) = \tilde{P}_i \setminus M$ for a hyperplane M of $\tilde{P}_i$. Thus, we can consider three graphs $\Gamma(L, \sigma_1, \mathrm{id}_2, \mathrm{id}_3)$, $\Gamma(L, \sigma_1, \sigma_2, \mathrm{id}_3)$, $\Gamma(L, \sigma_1, \sigma_2, \sigma_3)$. In the case $\nu = 4$, we can check these graphs are not isomorphic to each other by computer.

Munemasa and Vanhove [9] constructed strongly regular graphs with parameters $(2^{2mn}-1, 2^{2mn-1}-1, 2^{2mn-2}-3, 2^{2mn-2}-1)$. Let $f : \mathbb{F}_{2^n}^{2m} \times \mathbb{F}_{2^n}^{2m} \to \mathbb{F}_{2^n}$ be an alternating form for positive integers m, n. The vertex set of their graph is $\mathbb{F}_{2^n}^{2m} \setminus \{\mathbf{0}\}$ and two vertices u, v are adjacent if and only if $\mathrm{Tr}(f(u,v)^i) = 0$, where i is a positive integer that satisfies $(i, 2^n - 1) = 1$. Let $G(m,n,i)$ denote this graph. The case $(m,n) = (4,1), (2,2), (1,4)$ give strongly regular graphs with the same parameters as $\overline{Sp(8,2)}$. By this construction, we have four graphs $G(4,1,1)$, $G(2,2,1)$, $G(1,4,1)$, $G(1,4,7)$, but the graph which is not isomorphic to $\overline{Sp(8,2)}$ is only $G(1,4,7)$.

Now, we have ten graphs Γ_1, Γ_2, $\Gamma(L, \sigma_1, \mathrm{id}_2, \mathrm{id}_3)$, $\Gamma(L, \sigma_1, \sigma_2, \mathrm{id}_3)$, $\Gamma(L, \sigma_1, \sigma_2, \sigma_3)$, $G(1,4,7)$, $\overline{X^S}$, $\overline{X^{O(0,4,0)}}$, $\overline{X^{S_4}}$, $\overline{X^{S_0 \setminus (S \cup T)}}$ which has the same parameters as $\overline{Sp(8,2)}$ but not isomorphic to it. As a result of computation, we can confirm the isomorphism classes of the ten graphs we have are $\{\Gamma_1\}$, $\{\Gamma_2, \Gamma(L, \sigma_1, \sigma_2, \sigma_3), \overline{X^{S_0 \setminus (S \cup T)}}\}$, $\{G(1,4,7)\}$, $\{\overline{X^{O(0,4,0)}}, \Gamma(L, \sigma_1, \sigma_2, \mathrm{id}_3)\}$, $\{\overline{X^S}\}$, $\{\overline{X^{S_4}}\}$, $\{\Gamma(L, \sigma_1, \mathrm{id}_2, \mathrm{id}_3)\}$.

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