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ARC-TRANSITIVE ANTIPODAL DISTANCE-REGULAR GRAPHS OF DIAMETER THREE RELATED TO $PSL_d(q)$

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Abstract. In this paper, we investigate antipodal distance-regular graphs of diameter three and valency $q(q^{d-1}-1)/(q-1)$ with arc-transitive automorphism group which induces an almost simple permutation group on the antipodal classes with the socle isomorphic to $PSL_d(q)$, where $d \geq 3$. We find that such a graph is necessarily bipartite.

Keywords: arc-transitive graph, distance-regular graph, antipodal cover.

1. Introduction

Antipodal covers of graphs have been studied since the 1970s, in particular, in the context of a general problem of description of imprimitivity systems in s-transitive extensions of graphs and in distance-transitive graphs. An important subclass of antipodal covers of graphs is the antipodal distance-regular graphs of diameter 3. Many natural questions on the structure of these graphs, even for those with large automorphism groups, are still open. This paper concerns the problem of the classification of arc-transitive antipodal distance-regular graphs of diameter 3, which emerged after classification of distance-transitive covers of complete graphs was obtained in [3].

Throughout the paper let Γ denote an antipodal distance-regular graph of diameter 3, let Σ denote the set of its antipodal classes, and $G = \mathrm{Aut}(\Gamma)$. Then Γ has intersection array $\{k, (r-1)\mu, 1; 1, \mu, k\}$ and Γ is an antipodal r-cover of a

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complete graph on $|\Sigma| = k + 1$ vertices (see [1] for definitions and further theory of distance-regular graphs). If Γ is arc-transitive, then G induces a 2-transitive permutation group $\bar{G}$ on Σ. So far arc-transitive antipodal distance-regular graphs of diameter 3 have been classified in the cases $r \in \{2, (k-1)/\mu, k\}$, in the case when G is distance-transitive or contains an arc-transitive subgroup isomorphic to $PSU_3(q)$ or $SU_3(q)$ and almost complete their description was obtained in case when $\bar{G}$ is an affine group (a brief overview can be found in a recent article [7]). A further study shows (see [5]) that the classification of graphs with an almost simple group $\bar{G}$ and $r \notin \{(k-1)/\mu, k\}$ is mostly reduced to investigation of the cases $(soc(\bar{G}), k+1) = (PSU_3(q), q^3+1)$ or $(PSL_d(q), (q^d-1)/(q-1))$, where $d \geq 3$. Our aim is to study the second possibility and show that in this case all such graphs are necessarily bipartite. Together with results of [5] and [7], this finalizes the classification of arc-transitive antipodal distance-regular graphs of diameter 3 in almost simple case.

In order to formulate the problem more precisely, we proceed with some notation and definitions. In this paper, we consider only undirected graphs without loops or multiple edges. By $V(\Gamma)$ and $A(\Gamma)$ we denote the vertex set of Γ and the set of ordered pairs of adjacent vertices of Γ, respectively. Further we suppose that G acts transitively on $A(\Gamma)$ and we use the following notation: K is the kernel of the action induced by G on Σ, $F \in \Sigma$, $a \in F$, G_a is the stabilizer of a in G, $E \in \Sigma - \{F\}$, $b \in E \cap \Gamma(a)$, $\bar{G} = G/K$, $\bar{T} = soc(\bar{G})$ and T is the (full) pre-image of $\bar{T}$ in G. For an arbitrary subgroup X of K, by Γ^X we denote the quotient graph of Γ on the set of X-orbits on $V(\Gamma)$, whose edges are the pairs of orbits $X(b_1)$ and $X(b_2)$ such that $b_1 \in \Gamma(b_2)$. We also suppose that $(\bar{T}, k+1) = (PSL_d(q), (q^d-1)/(q-1))$, where $d \geq 3$.

In [5], it was shown that under the conditions stated in the preceding paragraph, Γ is bipartite or one of the following assertions holds:

(1) $K = 1$, $G \leq P\Gamma L_d(q)$ and T acts transitively on $V(\Gamma)$;
(2) $1 < |K| < r$, $\bar{G} \leq P\Gamma L_d(q)$, $\bar{G}$ acts transitively on $A(\Gamma^K)$, and $\bar{T}$ acts transitively on $V(\Gamma^K)$;
(3) $|K| = r$, and for some proper subgroup X of K the group $\bar{T}$ acts transitively on $A(\Gamma^X)$.

We will show that if Γ is non-bipartite, then none of the cases (1)–(3) holds. Since the quotient graphs arising in (2) and (3) are again distance-regular covers of diameter 3 (which may be bipartite only when $\Gamma^X \cong \Gamma$), it suffices to consider the case when Γ satisfies the following property:

($*$) $|\Sigma| = (q^d-1)/(q-1)$, where $d \geq 3$, G contains a subgroup $\widehat{G}$ acting transitively on $A(\Gamma)$ such that $PSL_d(q) \cong N \trianglelefteq \widehat{G}$ and one of the following conditions is provided:
  (i) $\widehat{G} \leq P\Gamma L_d(q)$ and N acts transitively on $V(\Gamma)$;
  (ii) N acts transitively on $A(\Gamma)$.

Theorem. *If an antipodal distance-regular graph of diameter* 3 *satisfies the property* ($*$), *then it is bipartite.*

2. Proof of Theorem

Our notation and terminology are mostly standard. Let λ denote the number of common neighbours for any two adjacent vertices of Γ. Recall that Γ has spectrum

$k^1, n^f, (-1)^k, (-m)^g$, where $n, -m$ — are roots of the quadratic equation $x^2-(\lambda-\mu)x-k=0$, $f=m(r-1)(k+1)/(n+m)$ and $g=n(r-1)(k+1)/(n+m)$ (see [1]). By [4] and assumption $(*)$, we may assume that $\mu \neq \lambda$. Hence, eigenvalues of Γ are integral and parameters of Γ are expressed through r, n and m: $k=nm$, $\mu=(m-1)(n+1)/r$, $\lambda=\mu+n-m$.

Let $\mathbb{F}$ denote a finite field of $q=p^e$ elements, where p is the characteristic of $\mathbb{F}$, and let V denote a vector space of dimension d over $\mathbb{F}$ with $d \geq 3$. Put $\widetilde{N}=SL(V)=SL_d(q)$. Let $V=U\oplus W$, where $U=\langle u\rangle$ is a one-dimensional vector subspace of V and W is a hyperplane of V. Let ν and ρ denote the action of $\widetilde{N}$ on the one-dimensional subspaces of V and the action of $\widetilde{N}$ on the hyperplanes of V, respectively, which are induced by the natural action of $\widetilde{N}$ on V. Suppose that ψ is a homomorphism of $\widetilde{N}$ in G such that $N=\psi(\widetilde{N})$, and $\tilde{\psi}$ is a homomorphism of $\widetilde{N}$ in $\bar{G}$ induced by ψ.

Recall that ν and ρ are the only two non-equivalent 2-transitive permutation representations of $\widetilde{N}$ of degree $k+1$, and the stabilizer of a point in ν is conjugate in $\mathrm{Aut}(\widetilde{N})$ to the stabilizer of a point in ρ. Hence, the groups $\nu(\widetilde{N})$ and $\rho(\widetilde{N})$ are permutation isomorphic. Besides, if ν is equivalent to $\tilde{\psi}$, then that permutation isomorphism provides an isomorphism between Γ and a graph Γ' with a vertex-transitive group of automorphisms which is isomorphic to N and induces a faithful 2-transitive action on the antipodal classes of Γ' equivalent to ρ. Hence we may assume that $\tilde{\psi}$ is equivalent to ν.

Let $W=\langle w_1\rangle\oplus W_1$, where $W_1=\langle w_2, w_3\ldots, w_{d-1}\rangle$. Put $U_1=\langle u, w_1\rangle$. Without loss of generality, we may assume that U and $\langle w_1\rangle$ are the one-dimensional vector spaces that correspond to F and E, respectively. The following properties can be found in [6]. We have $\widetilde{N}_{\{F\}}=R\cdot S\cdot D$, where

$$R=\{\tau\in\widetilde{N}|\tau(u)=u,\ \tau(x)-x\in U \text{ for all } x\in V\}$$

is an elementary abelian group of order q^{d-1},

$$S=\{\sigma\in\widetilde{N}|\sigma(u)=u,\sigma(W)=W\}\cong SL_{d-1}(q)$$

and $D\cong Z_{q-1}$. Besides, R and RS are normal in $\widetilde{N}_{\{F\}}$, and $R\cap S=RS\cap D=1$. Moreover, $\widetilde{N}_{\{F\},\{E\}}=R_1\cdot S_1\cdot D$, where

$$R_1=\{\tau\in\widetilde{N}|\tau(u)=u,\tau(w_1)=w_1,\ \tau(x)-x\in U_1 \text{ for all } x\in V\}$$

is an elementary abelian group of order $q^{2(d-2)}$,

$$S_1=\{\sigma\in\widetilde{N}|\sigma(u)=u,\sigma(W_1)=W_1,\sigma(w_1)=\det(\sigma_{|W_1})w_1\}\cong GL_{d-2}(q),$$

$R_1\lhd\widetilde{N}_{\{F\},\{E\}}$, $R_1S_1\lhd\widetilde{N}_{\{F\},\{E\}}$ and $R_1\cap S_1=R_1S_1\cap D=1$.

Further, each transvection τ in R can be written uniquely in the form $\{\varphi,u\}$, where φ is a nonzero linear functional on V such that $u\in\ker\varphi$ and $\tau(v)=v+\varphi(v)u$. Hence R is isomorphic to the additive group of the space W^* of all linear functionals on W. The action of SD on R by conjugation is defined by the rule $\tau^x=\{\varphi,u\}^x=x^{-1}\{\varphi,u\}x=\{\varphi x, x^{-1}(u)\}=\{\varphi x,\alpha^{-1}u\}=\{\alpha^{-1}\varphi x,u\}$ for all $x\in SD$, where α is an element of $\mathbb{F}^*$ such that $\alpha u=x(u)$.

Note that if $(r,k)=1$, then N_a is transitive on $\Gamma(a)$ and $|N_{\{F\},\{E\}}:N_{a,b}|=r$.

Lemma 1. $S_{\{R(E)\}}=S_{\{E\}}$.

Proof. It is enough to note that S has exactly two orbits on $\Sigma - \{F\}$ (they have lengths $q^{d-1} - 1$ and $(q^{d-1} - 1)/(q - 1)$), each R-orbit on $\Sigma - \{F\}$ has length q, and S acts 2-transitively on R-orbits on $\Sigma - \{F\}$. □

Lemma 2. *$R_{\{E\}}$ has exactly $q(q^{d-2}-1)/(q-1)$ orbits of length q on $\Sigma - \{F\}$ and fixes exactly q points on $\Sigma - \{F\}$.*

Proof. We have $R_{\{E\}} = \{\{\varphi, u\} \in R |\ w_1 \in \ker\varphi\}$. Each nonzero linear functional φ on V with $u \in \ker\varphi$ is determined uniquely by its values on the basic vectors of W. Let $v = \alpha_0 u + \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_{d-1} w_{d-1}$ be an arbitrary vector in $V - \langle u, w_1\rangle$. Without loss of generality, we may assume that $\alpha_2 \neq 0$. Note that there are precisely $q - 1$ elements $\{\varphi_i, u\}$ in R (where $i \in \{1, ..., q-1\}$) with $\ker\varphi_i \cap W = \langle w_1, w_3, ..., w_{d-1}\rangle$. Suppose that $\{\varphi_i, u\}(v) = v + \alpha_2\varphi_i(w_2)u \in \langle v + \alpha_2\varphi_j(w_2)u\rangle$ for some $i, j \in \{1, ..., q-1\}, i \neq j$. Then for some $\alpha \in \mathbb{F}^*$ we have $v + \alpha_2\varphi_i(w_2)u = \alpha(v + \alpha_2\varphi_j(w_2)u)$, which implies $(1-\alpha)v = (\alpha\alpha_2\varphi_j(w_2) - \alpha_2\varphi_i(w_2))u$. But then $\alpha = 1$ and $\varphi_j(w_2) = \varphi_i(w_2)$, a contradiction. Since $R_{\{E\}} \lhd (RS)_{\{E\}}$, it follows that the length of each non-trivial $R_{\{E\}}$-orbit on $\Sigma - \{F\}$ equals q, while the trivial ones are precisely those which correspond to the one-dimensional subspaces $\langle v'\rangle$ with $v' \in \langle u, w_1\rangle - U$. □

Lemma 3. *If $r = 2$, then Γ is bipartite.*

Proof. Let $r = 2$. Then $\widehat{G}$ is a distance-transitive group. Since $d \geq 3$, $k = q(q^{d-1} - 1)/(q - 1)$ is not a prime power. Straightforward computation shows that $k \notin \{175, 275\}$. Suppose that Γ is not bipartite. By the classification of distance-transitive Taylor graphs (see, for example, [1] or [3]), we get $k = 2^{2s-1} \pm 2^{s-1} - 1$, $N \leq Sp_{2s}(2)$, where $s \geq 3$, and $\mu \in \{2^{2s-2}, 2^{2s-2} \pm 2^{s-1} - 2\}$. Hence q is odd and d is even. Then $R = R_a$. As S does not contain subgroups of index 2, we get $S = S_a$ and N is a distance-transitive group as well. By Lemma 1, if $b^* \in E - \{b\}$ is adjacent to a vertex in $R(b) - \{b\}$, then b^* is adjacent to each vertex in $R(b) - \{b\}$. Since μ is even, we obtain $\mu = q - 1$ or $\mu > q$. Suppose $\mu > q$. Then by Lemmas 1 and 2, $\mu = k - q = q^{d-1} + q^{d-2} + ... + q^2$. If $\mu = 2^{2s-2}$, then $k = 2^{2s-1} \pm 2^{s-1} - 1 = 2\mu \pm \sqrt{\mu} - 1 = k - q + q = \mu + q$ and $\mu \pm \sqrt{\mu} = q + 1$, a contradiction. If $\mu = 2^{2s-2} \pm 2^{s-1} - 2$, then $k = 2\mu \mp 2^{s-1} + 3 = \mu + q$ and $\mu \mp 2^{s-1} = 2^{2s-2} - 2 = q - 3$, a contradiction.

Suppose $\mu = q - 1$. Then $\lambda = k - q$. If $\mu = 2^{2s-2}$, then $k - \mu - 1 = \mu \pm \sqrt{\mu} - 2 = \lambda = k - q$ and $q - 1 \pm \sqrt{q-1} - 2 = k - q$, a contradiction. If $\mu = 2^{2s-2} \pm 2^{s-1} - 2$, then $k - \mu - 1 = \mu \mp 2^{s-1} + 2 = \lambda = k - q$ and $q - 1 \mp 2^{s-1} = 2^{2s-2} - 2 = k - q - 2$, a contradiction. □

Throughout the rest of the paper we will assume that $r > 2$.

We have $|\widehat{G}_a : N_a| = |\widehat{G}_{\{F\},\{E\}} : N_{\{F\},\{E\}}| = |\widehat{G} : N|$. By the assumption $(*)$, $|\widehat{G} : N|$ divides $e(d, q-1)$ or N_a is transitive on $\Gamma(a)$.

Lemma 4. *If r is divisible by q^{d-l} for some $l \in \mathbb{N}$, then $l > 1$ and, in particular, if $l = 2$, then $r/q^{d-2} + 1/(q^{d-2}\mu) < q/\mu + 3$.*

Proof. If r is divisible by $q^{d-l} > q$, then

$$q^{d-l}\mu - \mu \leq r\mu - \mu = (r-1)\mu = k - \lambda - 1 \leq q(q^{d-1} - 1)/(q - 1) - 1,$$
$$q^{d-l} - 1 \leq (q/\mu)(q^{d-1} - 1)/(q - 1) - 1/\mu.$$

Suppose that $\mu = 1$. Then Γ can be viewed as an induced subgraph of a biregular geodesic graph, say $\tilde{\Gamma}$, of diameter 2 which is defined as follows. Put $\Sigma = \{F_i\}_{i=1}^{k+1}$ and assume that $V(\tilde{\Gamma}) = V(\Gamma) \cup \{c_i\}_{i=0}^{k+1}$ so that $\tilde{\Gamma}(c_0) = \{c_i\}_{i=1}^{k+1}$, and, for all $i \in \{1, 2, ..., k+1\}$, c_i is adjacent to r vertices from F_i and Γ coincides with the subgraph induced by $\tilde{\Gamma}$ on $V(\Gamma)$. Then by [2, Proposition 4], k is a power of 2, a contradiction. Hence $l > 1$. If $r = q^{d-2}t$, then

$$q^{d-2}t - 1 + 1/\mu \leq (q/\mu)(qq^{d-2} - q + q - 1)/(q-1) = (q/\mu)q(q^{d-2} - 1)/(q-1) + q/\mu,$$

$$t - 1/q^{d-2} + 1/(q^{d-2}\mu) \leq (q/\mu)q(q^{d-2} - 1)/(q^{d-2}(q-1)) + 1/(\mu q^{d-3}) < (q/\mu)q/(q-1) + 1/(\mu q^{d-3}),$$

$$t + 1/(q^{d-2}\mu) < (q/\mu)q/(q-1) + 1/(\mu q^{d-3}) + 1/q^{d-2} < (q/\mu)q/(q-1) + 2 < q/\mu + 3.$$

□

Lemma 5. *R acts intransitively on F.*

Proof. On the contrary, suppose $R(a) = F$. Then R_a fixes F pointwise. Therefore, $\cup_{x \in \widetilde{N}_{\{F\}}} (R_a)^x \leq \widetilde{N}_a$. But S acts transitively by conjugation on $R - \{1\}$, hence $R_a = 1$ and $r = q^{d-1}$, which contradicts Lemma 4. □

Put $z = |R : R_a|$.

Lemma 6. *If $r \geq zk/q$ and z divides q, then $z\mu/2 < q$, and, in particular, if $z = 1$, then $q/\mu > 1 - (q-1)/(q^{d-1} - 1)$.*

Proof. By the equation $k = (r-1)\mu + \lambda + 1$, we get

$$k \geq (z(q^{d-1} - 1)/(q-1) - 1)\mu + \lambda + 1,$$

$$z(q^{d-1} - 1)\mu(q/(z\mu) - 1 + 1/(z(q^{d-1} - 1)/(q-1)))/(q-1) \geq \lambda + 1,$$

$$q/(z\mu) > 1 - 1/(z(q^{d-1} - 1)/(q-1)).$$

Hence $z\mu/2 < q$, and if $z = 1$, then $q/\mu > 1 - (q-1)/(q^{d-1} - 1)$. □

Lemma 7. *Each non-trivial S-orbit on F has length at least $(q^{d-1} - 1)/(q-1)$.*

Proof. By computations in GAP, we found that there are no antipodal distance-regular graphs of diameter three with $r > 2$ in cases $d = 3, q \in \{5, 7, 9, 11\}$ and $d = 5, q = 2$. By [6, Theorem 1], the degree of a minimal permutation representation of S equals $(q^{d-1} - 1)/(q-1)$ in the remaining cases. □

Lemma 8. *If $RS(a) = F$, then R has at least $(q^{d-1} - 1)/(q-1)$ orbits on F and z divides q.*

Proof. In this case S acts transitively on R-orbits on F. Hence the statement follows by the inequalities $|R : R_a|(q^{d-1} - 1)/(q-1) \leq r < q(q^{d-1} - 1)/(q-1)$, the former of which is true by Lemma 7. □

Lemma 9. *R fixes F pointwise.*

Proof. First, note that $\psi(R) \trianglelefteq \widehat{G}_{\{F\}}$. Indeed, R is the socle of $\widetilde{N}_{\{F\}}$, and hence $R \cong \psi(R)$ is the socle of $N_{\{F\}}$. Therefore $\psi(R) \trianglelefteq \widehat{G}_{\{F\}}$ and $\psi(R)_a \trianglelefteq \widehat{G}_a$. In particular, all R_a-orbits on $\Gamma(a)$ are of the same length.

Now we show that if Y is a non-trivial subgroup of R such that all Y-orbits on $\Sigma-\{F\}$ are of the same length, say $t+1$, then $Y = R$. The action of $\widetilde{N}_{\{F\}}$ on $\Sigma-\{F\}$ is permutation isomorphic to the action of $\widetilde{N}_{\{F\}}$ on the one-dimensional subspaces $\langle v'\rangle$ with $v' \in V - U$. Suppose that $Y_{w_1} = 1$. Then $Y = \{1\} \cup \{\{\varphi_i, u\}\}_{i=1}^t$, where $\{\varphi_i\}_{i=1}^t$ is a set of nonzero linear functionals on V such that for all $i \in \{1, ..., t\}$ we have $u \in \ker\varphi_i, w_1 \notin \ker\varphi_i$ and $|\{\varphi_i(w_1)\}_{i=1}^t| = t$. For all $\delta \in \mathbb{F}^*$ and $i \in \{1, ..., t\}$ we have $w_2, w_1 + \delta w_2 \notin \ker\varphi_i$ and $|\{\varphi_i(w_2)\}_{i=1}^t| = |\{\varphi_i(w_1 + \delta w_2)\}_{i=1}^t| = t$, a contradiction. Hence $Y_{w_1} \neq 1$. Put $X_j = \{\{\varphi, u\} \in R|\ w_i \in \ker\varphi \text{ for all } i \in \{1, 2, ..., d-1\} - \{j\}\}$, so that $R = X_1 \times \ldots \times X_{d-1}$. Consider the Y-orbits on the one-dimensional subspaces of V with representatives $\langle w\rangle$, where $w \in \langle w_1, w_2\rangle - \{0\}$. For all $w \in \langle w_1, w_2\rangle - \{0\}$ we have $|Y : Y_w||Y_w : Y_{w_1,w_2}| = (t+1)l$ and $l = |Y_{w_i}(w_j)| > 1$, where $\{i, j\} = \{1, 2\}$. For all $\delta \in \mathbb{F}^*$ and $\{\varphi, u\} \in Y$, we have $\{\varphi, u\}(\delta w_1 + w_2) = \delta w_1 + w_2 + (\delta\varphi_1(w_1) + \varphi_2(w_2))u$, where $\{\varphi_i, u\}$ is an element of X_i with $i \in \{1, 2\}$, and there are exactly l elements in $Y/Y_{w_1,w_2}$ which fix the point $\langle \delta w_1 + w_2\rangle$. It follows that $t + 1 = q = l$, and by similar argument we get $|Y| = q^{d-1}$, so $Y = R$. Hence, $R = R_a$. □

Lemma 10. *The case $RS(a) = F$ does not occur.*

Proof. Clearly, $R_{\{E\}}$ fixes F and E pointwise. By Lemma 8, $r - 1 \geq q$. Then by Lemma 2, there is a vertex in $E - \{b\}$ adjacent to q vertices in $\Gamma(a)$, and, by Lemma 6, $q \leq \mu < 2q$. If $\mu > q$, then there is at least one vertex in $E - \{b\}$ adjacent to μ vertices in $A = \Gamma(a) - R(b)$, and $|A| \geq q^2$, hence by Lemma 2, $\mu = 2q$, a contradiction. Thus, $\mu = q$ and $\lambda = q - 1 \pmod q$.

By the equation $k - \lambda - 1 = (r - 1)\mu$, we get

$$k/q - (\lambda + 1)/q = r - 1 \geq (q^{d-1} - 1)/(q - 1) - 1,$$

which implies $k = r\mu$, $\lambda = q - 1$, $m = n + 1$, and S acts 2-transitively on F. It follows that S_a is the stabilizer in S of a subspace of dimension 1 or $d - 2$ of W.

If S_a fixes a one-dimensional subspace $\langle h\rangle$ of W, then S_a acts transitively on the remaining $(q^{d-1} - 1)/(q - 1) - 1$ one-dimensional subspaces of W, and hence, acts transitively on $(q^{d-1} - 1)/(q - 1) - 1$ R-orbits on $\Gamma(a)$, which correspond to the one-dimensional subspaces $\langle w\rangle$ with $w \in W - \langle h\rangle$. Now suppose S_a fixes a hyperplane H of W. Then S_a acts transitively on $(q^{d-2} - 1)/(q - 1)$ R-orbits on $\Gamma(a)$, which correspond to the one-dimensional subspaces of H, and S_a acts transitively on q^{d-2} R-orbits on $\Gamma(a)$, which correspond to the one-dimensional subspaces $\langle w\rangle$ with $w \in W - H$.

In either case, N_a has at most two orbits on $\Gamma(a)$. If there are two such orbits, then they have lengths q and $k - q$, or q^{d-1} and $k - q^{d-1}$. But $N_a \trianglelefteq \widehat{G}_a$, hence $k/2 \in \{q, q^{d-1}\}$, a contradiction. Hence, N_a is transitive on $\Gamma(a)$. On the other hand, $RS_a \trianglelefteq \widetilde{N}_a$ and RS_a-orbits on $\Gamma(a)$ are of the same length, a contradiction. □

Lemma 11. *The case $RS(a) \neq F$ does not occur.*

Proof. We have $RS(a) = S(a)$. Suppose $S(a) \neq F$. Then $B = S(a)$ is an imprimitivity block of $\widetilde{N}$ on $V(\Gamma)$. Denote by π the imprimitivity system of $\widetilde{N}$ on $V(\Gamma)$ which contains B.

Put $P = \widetilde{N}_{\{B\}}(= RSD_{\{B\}}Z(\widetilde{N}))$. Denote by π_1 the imprimitivity system of $\widetilde{N}$ on $V - \{0\}$ which contains a P-orbit on $U - \{0\}$ as a block. The action of $\widetilde{N}$ on π is equivalent to the action of $\widetilde{N}$ on π_1. Suppose the subgraph induced by any two blocks of π is a coclique, or a perfect matching (this holds, in particular, provided that $S = S_a$). Then $\widetilde{N}_{\{B\}}$ acts transitively on the vectors in $V - U$. Hence B contains a vertex adjacent to at least two vertices in E, a contradiction. Hence, by Lemma 7, $|B| \geq k/q$. By Lemmas 2 and 6, $q \leq \mu < 2q$. Since $(r/|B|, q-1) > 1$ and $k - \lambda - 1 = (r-1)\mu$, it follows that

$$k/q - (\lambda+1)/q = (r-1)\mu/q \geq (2(q^{d-1}-1)/(q-1)-1)\mu/q \geq 2k/q - 1,$$

a contradiction. □

The theorem is proved.

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