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THE NUMBER OF SMALL CYCLES IN THE STAR GRAPH

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Abstract. The Star graph is the Cayley graph on the symmetric group Sym_n generated by the set of transpositions $\{(1\,i) \in Sym_n : 2 \leqslant i \leqslant n\}$. This graph is bipartite and does not contain odd cycles but contains all even cycles with a sole exception of 4-cycles. We denote as (π, id)-cycles the cycles constructed from two shortest paths between a given vertex π and the identity id. In this paper we derive the exact number of (π, id)-cycles for particular structures of the vertex π. We use these results to obtain the total number of 10-cycles passing through any given vertex in the Star graph.

Keywords: Cayley graphs; Star graph; cycle embedding; number of cycles.

1. Introduction

The Star graph $S_n = Cay(Sym_n, t)$, $n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \ldots \pi_n]$, where $\pi_i = \pi(i)$, $1 \leqslant i \leqslant n$, with the generating set $t = \{t_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all transpositions $t_i = (1\,i)$ swapping the 1st and ith elements of a permutation π when multiplied on the right, i.e. $[\pi_1\pi_2 \ldots \pi_{i-1}\pi_i\pi_{i+1} \ldots \pi_n]t_i = [\pi_i\pi_2 \ldots \pi_{i-1}\pi_1\pi_{i+1} \ldots \pi_n]$. It is a connected bipartite $(n-1)$–regular graph of order $n!$ with the diameter $D = \lfloor \frac{3(n-1)}{2} \rfloor$ and has the hierachical structure [4, 3]. Since this graph is bipartite it does not contain odd cycles but it contains all even ℓ–cycles where $\ell = 6, 8, \ldots, n!$ [5]. The hamiltonicity of this graph was also shown in papers on generating all permutations by transpositions [6, 7].

The shortest path between two arbitrary vertices π and τ in the graph S_n is denoted as (π, τ)-*path*. A cycle in the graph S_n constructed from two disjoint

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shortest (π, τ)-paths is called a (π, τ)-*cycle*. A cycle of length l is also called an l-*cycle*.

The characterization of 6- and 8-cycles has been obtained for the Star graph using the forms of cycles, which are indexed sequences of generating elements [3]. Along with the characterization, the number of these cycles passing through any given vertex is given for this graph. The results are presented in the following Theorems.

Theorem 1. [3] *Each of vertices of* $S_n, n \geqslant 3$, *belongs to* $\frac{1}{2}(n-1)(n-2)$ *distinct* 6*–cycles of the following form:*

$$C_6 = (t_k\, t_i)^3, \quad 2 \leqslant i < k \leqslant n. \tag{1}$$

Theorem 2. [3] *Each of vertices of* $S_n, n \geqslant 4$, *belongs to* $3(n-3)(n-2)(n-1)$ *distinct* 8*–cycles of the following forms:*

$$C_8^1 = t_k\, t_i\, t_j\, t_i\, t_k\, t_i\, t_j\, t_i, \quad 2 \leqslant i \neq j \leqslant k-1; \tag{2}$$

$$C_8^2 = t_k\, t_j\, t_i\, t_j\, t_k\, t_i\, t_j\, t_i, \quad 2 \leqslant i \neq j \leqslant k-1; \tag{3}$$

$$C_8^3 = t_k\, t_j\, t_i\, t_k\, t_j\, t_k\, t_i\, t_j, \quad 2 \leqslant i \neq j \leqslant k-1; \tag{4}$$

$$C_8^4 = t_k\, t_j\, t_k\, t_i\, t_k\, t_j\, t_k\, t_i, \quad 2 \leqslant i < j \leqslant k-1, \tag{5}$$

where $4 \leqslant k \leqslant n$.

The problem of cycle counting is a well-posed problem in graph theory and computer science [9, 8]. This problem has shown to have theoretical and practical use in case of vertex-transitive graphs. In [11, 12, 13] the number of cycles of small length is given for Pancake graph P_n, which is the Cayley graph on the symmetric group with the generating set of prefix–reversals. Comparing that numbers to the results on a Star graph we see that the Star graph has richer structure of small cycles. In the current paper we investigate the number of (π, τ)-cycles of length $2d$, where $3 \leqslant d \leqslant D$, passing through a given vertex τ in the Star graph S_n. Obtaining the number of such cycles is a part of a method of finding the critical value of the Oriented Percolation in a vertex-transitive graph, proposed in [10]. Also we prove the Lemma 10 that shows that the number of (π, τ)-cycles in the Star graph gives an asymptotic bound to the number of small cycles of length at most $2n$.

Since the Star graph is vertex-transitive, for the sake of convenience we can consider only cycles with vertex $\tau = id$, where id denotes the vertex, that corresponds to the identity permutation. Let us call such cycles as (π, id)-*cycles*. These cycles are characterized by the cyclic structure of permutation π and the distance d between vertices π and id.

The description of the cyclic structure of permutation is made using the definitions from [2]. Let a permutation $\pi \in Sym_n$ be represented as the product of disjoint cycles:

$$\pi = (\pi_1^0 \ldots \pi_{\ell_0}^0)(\pi_1^1 \ldots \pi_{\ell_1}^1) \ldots (\pi_1^k \ldots \pi_{\ell_k}^k) = (\pi^0)(\pi^1) \ldots (\pi^k),$$

where $\pi^i = \pi_1^i \pi_2^i \ldots \pi_{\ell_i}^i$ and ℓ_i is the number of elements in i'th cycle and $\ell_i \geqslant 1$, where $0 \leqslant i \leqslant k$. The cycle of length m that contains the element "1" is denoted as $m - CO$ and the cycle that does not contain "1" as $m - CN$ [2]. We also write simply CO-cycle or CN-cycle, if the length is not specified. We call a cycle to be *trivial* if it contains only one element. By the *cyclic structure* of the permutation we consider the number of elements in the CO-cycle and the number of non-trivial CN-cycles.

Example. Consider the permutation $\pi = [354126]$. The cyclic representation of the permutation is $\pi = (134)(25)(6)$ that contains a non-trivial $3 - CO$ (134) and one non-trivial $2 - CN$ (25) in its cyclic structure and one trivial CN (6).

The structure of (π, id)-paths is known from the shortest path routing algorithm described in [1]. We introduce the characterization of these paths which lets us calculate the number of (π, id)-cycles for particular cyclic structures of vertex π. Denote as $N_0(k)$ the number of distinct (π, id)-cycles, passing through all vertices π with trivial CO-cycle and k non-trivial CN-cycles and $N_1(k)$ the number of distinct (π, id)-cycles, passing through all vertices π with non-trivial CO-cycle and k non-trivial CN-cycles. We use the notation for the falling factorial $(n)_k := n(n-1)\dots(n-k+1)$. The results can be summarized as follows.

Theorem 3. *In the Star graph S_n, $n \geqslant 3$, the total number of distinct (π, id)-cycles of length $2d$, where $3 \leqslant d \leqslant n$, passing through all vertices π with trivial CO-cycle and one non-trivial CN-cycle in its cyclic structure is*

$$N_0(1) = \frac{d-2}{2}(n-1)_{d-1}.$$

Theorem 4. *In the Star graph S_n, $n \geqslant 3$, the total number of distinct (π, id)-cycles of length $2d$, where $3 \leqslant d \leqslant n$, passing through all vertices π with non-trivial CO-cycle and one non-trivial CN-cycle in its cyclic structure is*

$$N_1(1) = \frac{d(d-3)}{2}(n-1)_{d-1}.$$

Theorem 5. *In the Star graph S_n, $n \geqslant 3$, the number of distinct (π, id)-cycles of length $2d$, where $6 \leqslant d \leqslant n+1$, passing through all vertices π with trivial CO-cycle, and two non-trivial CN-cycles in its cyclic structure is*

$$N_0(2) = \frac{1}{240}(d-5)(7d^4 - 25d^3 + 40d^2 - 220d + 288)(n-1)_{d-2}.$$

We use these exact results to obtain the total number of 10-cycles passing through any given vertex without their explicit characterization, thus we state it as a Corollary.

Corollary 6. *In the Star graph S_n, $n \geqslant 4$, the total number of distinct 10-cycles passing through any given vertex is equal to*

$$N_{C_{10}} = \frac{41}{6}(n-1)_4 + \frac{5}{6}(n-2)_3.$$

The paper is structured as follows. In the second Section we present the basic notions required for presenting the main result, in the third Section we prove necessary technical Lemmas, in the fourth and fifth Sections we present the proof of main Theorems of the paper.

2. Basic concepts

Let a permutation $\pi \in Sym_n$ be represented as the product of disjoint cycles. Since any rearrangement of cycles does not change the permutation π itself, we write the CO-cycle at the beginning of the cyclic representation, then follow non-trivial CN-cycles and we omit trivial cycles since they do not take part in the construction of shortest paths. Then if the permutation π has an $m - CO$, where $m \geqslant 1$ and k non-trivial CN-cycles of lengths $\ell_1, \ldots, \ell_k \geqslant 2$ in its cyclic structure, then it is denoted as follows:

$$\pi = (1\, \pi_2^0 \ldots \pi_m^0)(\pi_1^1 \ldots \pi_{\ell_1}^1) \ldots (\pi_1^k \ldots \pi_{\ell_k}^k). \tag{6}$$

We label each non-trivial cycle with its length variable $m, \ell_1, \ldots, \ell_k$.

The shortest path routing algorithm *(SP-algorithm)* presented in [1] provides the way for constructing (π, id)-paths of length d for any given permutation π. Describe the algorithm.

SP**-algorithm.** Let us denote a (π, id)-path as $P = (\pi_0, \pi_1, \ldots, \pi_d)$, where $\pi_0 = \pi$ and $\pi_d = id$. Consider the permutation π_i, where $0 \leqslant i \leqslant d-1$. Suppose π_i has an $m - CO$, where $m \geqslant 1$ and $k \geqslant 0$ non-trivial CN-cycles of lengths $\ell_1, \ldots, \ell_k \geqslant 2$ in its cyclic structure. To obtain the vertex π_{i+1} along the shortest path we apply the transposition that performs either of two following operations:

(1) **Contraction.** If $m \geqslant 2$, apply the transposition $t_{\pi_2^0}$ and put element π_2^0 of $m - CO$ into its own trivial cycle of length 1, obtaining the permutation with cyclic structure

$$\pi_{i+1} = \pi_i\, t_{\pi_2^0} = (1\, \pi_3^0 \ldots \pi_m^0)(\pi^1)(\pi^2) \ldots (\pi^k)(\pi_2^0);$$

(2) **Merge.** If $k \geqslant 1$, apply one of transpositions $t_{\pi_1^i}, \ldots, t_{\pi_{\ell_i}^i}$ and merge $m - CO$ and $\ell_i - CN$, $i = 1, \ldots, k$, obtaining the permutation with cyclic structure

$$\pi_{i+1} = \pi_i\, t_{\pi_j^i} = (1\, \pi_j^i\, \pi_{j+1}^i \ldots \pi_{\ell_i}^i\, \pi_1^i \ldots \pi_{j-1}^i \pi_2^0 \ldots \pi_m^0)(\pi^2) \ldots \\ \ldots (\pi^{i-1})(\pi^{i+1}) \ldots (\pi^k)$$

where $1 \leqslant j \leqslant \ell_i$.

We characterize (π, id)-paths in terms of merge and contraction sequences. Let the vertex π in the Star graph be at the distance d from the identity id, where $3 \leqslant d \leqslant D$ and have an $m - CO$, where $m \geqslant 1$, and $k \geqslant 0$ non-trivial CN–cycles of lengths $\ell_1, \ldots, \ell_k \geqslant 2$ in its cyclic structure. According to the SP-algorithm in order to obtain a (π, id)-path we need to apply d transpositions that merge CN–cycles with a CO–cycle or contract a CO–cycle. Therefore, the (π, id)-paths in S_n can be represented by sequences of the following types of transpositions:

(1) τ_{CO} denotes transposition that contracts a CO–cycle;
(2) $\tau_{CN}^{\ell_i}$, where $1 \leqslant i \leqslant k$, denotes the transpositions that merge $\ell_i - CN$ and CO-cycle.

We call a *merge sequence* the ordered sequence $(\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_k})$ of labels of merged CN-cycles, where $\ell_{i_j} \in \{\ell_1, \ldots, \ell_k\} \backslash \{\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_{j-1}}\}$ and $1 \leqslant j \leqslant k$. Since there are ℓ_i ways to merge $\ell_i - CN$ with a CO-cycle then each application of $\tau_{CN}^{\ell_i}$ denotes ℓ_i transpositions, and since there is one way to contract CO-cycle, then τ_{CO} denotes a unique appropriate transposition. Once we fixed the order of merging of CN-cycles, we have to define the number of contractions between them. We call a *contraction sequence* the ordered sequence $[j_0, j_1, \ldots, j_k]$ of numbers of consecutive applications of τ_{CO}. The (π, id)-path with the merge sequence $(\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_k})$ and with the contraction sequence $[j_0, j_1, \ldots, j_k]$ can be represented graphically in the following way:

$$\underbrace{\tau_{CO}\cdots\tau_{CO}}_{j_0 \text{ times}}\tau_{CN}^{\ell_{i_1}}\underbrace{\tau_{CO}\cdots\tau_{CO}}_{j_1 \text{ times}}\tau_{CN}^{\ell_{i_2}}\underbrace{\tau_{CO}\cdots\tau_{CO}}_{j_2 \text{ times}}\tau_{CN}^{\ell_{i_3}}\cdots\tau_{CN}^{\ell_{i_k}}\underbrace{\tau_{CO}\cdots\tau_{CO}}_{j_k \text{ times}}, \tag{7}$$

where $0 \leqslant j_0 \leqslant \ell_{i_0}$, $0 \leqslant j_1 \leqslant \ell_{i_1} + \ell_{i_0} - j_0$, $0 \leqslant j_2 \leqslant \ell_{i_2} + \ell_{i_1} - j_1 + \ell_{i_0} - j_0$, $\ldots$, $0 \leqslant j_k = \ell_{i_k} + \sum\limits_{t=0}^{k-1}(\ell_{i_t} - i_t)$ and $\ell_{i_0} = m - 1$. We call the *intersecting* contraction sequence of a merge sequence $(\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_k})$ to be the contraction sequence $[j_0, j_1, \ldots, j_k]$ with at least one $j_s = \ell_{i_s} + \sum\limits_{t=0}^{s-1}(\ell_{i_t} - i_t)$, where $0 \leqslant s \leqslant k - 1$. It is easy to see that the paths described by an intersecting contraction sequence have at least one common vertex. The rest of contraction sequences we call *non-intersecting.*

The idea of counting (π, id)-cycles is based on the classification of (π, id)-paths by merge and contraction sequences and counting the number of disjoint pairs within the following three cases:

I. Within contraction sequence of a fixed merge sequence;
II. Between contraction sequences of a fixed merge sequence;
III. Between merge sequences.

In the proof of main Theorems we consider each situation independently and obtain the number of distinct cycles passing through a given vertex π with particular cyclic structure, then count through all vertices at the distance d with prescribed structure.

3. Technical lemmas

We start with the Lemma on cyclic structure of vertices at the distance d from the identity id in the Star graph S_n, $n \geqslant 3$.

Lemma 7. *In the graph S_n, $n \geqslant 3$, the vertices at the distance d from the identity id, where $1 \leqslant d \leqslant D$, have one of the following cyclic structures:*

a) *a non-trivial $(d+1)$-CO and trivial CN-cycles and thus having a unique shortest path to the identity;*
b) *an m-CO and k non-trivial CN-cycles of lengths $\ell_1, \ldots, \ell_k \geqslant 2$, where $k \geqslant 1$, such that*

$$d = k + (m-1) + \sum_{i=1}^{k} \ell_i. \tag{8}$$

Proof. Consider a vertex π in the Star graph at the distance d from the identity id, where $1 \leqslant d \leqslant D$ and use the SP-algorithm. Suppose vertex π has only non-trivial CO in its cyclic structure. Then on each step of the algorithm we can only contract one element from CO, thus π must have $(d+1) - CO$. Since the contraction step can be performed by unique transposition, we have a unique shortest path to the identity id.

To prove the second part, suppose the vertex π has an $m - CO$ and k non-trivial CN-cycles of lengths $\ell_1, \ldots, \ell_k \geqslant 2$. Then according to the SP-algorithm, one way to get a shortest path is to merge all CN-cycles together into one large CO-cycle and then contract it. These merges can be done in k steps, obtaining a permutation with one non-trivial $\left(m+\sum_{i=1}^{k}\ell_i\right)-CO$. Then we have to contract $\left(m+\sum_{i=1}^{k}\ell_i\right)-1$ elements from the CO-cycle and obtain the identity id. Since the total length of the shortest path is d, then we have the following equality:

$$d = k + (m-1) + \sum_{i=1}^{k} \ell_i,$$

which finishes the proof of the Lemma. □

Let $V(d;m,k)$ be the set of vertices with given cyclic structure of $m-CO$ and k non-trivial CN-cycles at the distance d from the identity id. The following Lemma provides the formula for counting through all vertices in $V(d;m,k)$.

Lemma 8. *Consider the set $V(d;m,k)$. Let $f(\pi) := f(m,\ell_1,\ldots,\ell_k)$ be the function of lengths of non-trivial CO- and CN-cycles. Then, the sum of values of this function through all vertices in $V(d;m,k)$ is given by:*

$$\sum_{\pi\in V(d;m,k)} f(\pi) = \frac{1}{k!}\sum_{(m-1)+\ell_1+\cdots+\ell_k=d-k} \frac{f(m,\ell_1,\ell_2,\ldots,\ell_k)}{\ell_1\ell_2\ldots\ell_k}(n-1)_{d-k-1}.$$

Proof. Consider the set of vertices $V(d;m,\ell_1,\ldots,\ell_k)$ at the distance d from the identity id that have an $m-CO$ and k non-trivial CN-cycles of lengths $\ell_1,\ldots,\ell_k$. Then the set $V(d;m,k)$ can be represented as the following union of sets:

$$V(d;m,k) = \bigcup_{(m-1)+\ell_1+\cdots+\ell_k=d-k} V(d;m,\ell_1,\ldots,\ell_k),$$

Any given permutation in the set $V(d;m,k)$ belongs to each set $V(d;m,\ell_{\omega(1)},\ldots,\ell_{\omega(k)})$, where $\omega\in Sym_k$. The number of such sets is $k!$, hence the total number of vertices in $V(d;m,k)$ is

$$|V(d;m,k)| = \frac{1}{k!}\sum_{(m-1)+\ell_1+\cdots+\ell_k=d-k} |V(d;m,\ell_1,\ldots,\ell_k)|. \tag{9}$$

The number of elements in $V(d;m,\ell_1,\ldots,\ell_k)$ equals to the number of ways to distribute $\sum_{i=1}^{k}\ell_i$ elements from $(n-1)$-element set through cyclic permutations of lengths $\ell_1,\ldots,\ell_k$ correspondingly and put the rest $m-1$ elements into a cyclic permutation where the first element equals to 1. This can be done in the following number of ways:

$$\begin{aligned} |V(d;m,\ell_1,\ldots,\ell_k)| &= \frac{(n-1)(n-2)\ldots(n-\sum_{i=1}^{k}\ell_i)(n-1-\sum_{i=1}^{k}\ell_i)!}{\ell_1\ell_2\ldots\ell_k(n-1-\sum_{i=1}^{k}\ell_i-(m-1))!} \\ &= \frac{1}{\ell_1\ell_2\ldots\ell_k}(n-1)\ldots(n-d+k) = \\ &= \frac{(n-1)_{d-k-1}}{\ell_1\ell_2\ldots\ell_k}, \end{aligned} \tag{10}$$

where the last equality is obtained using Lemma 7, b). Combining equations (9) and (10) provides us with the statement of Lemma:

$$\sum_{\pi\in V(d;m,k)} f(\pi) = \frac{1}{k!}\sum_{(m-1)+\ell_1+\cdots+\ell_k=d-k} f(m,\ell_1,\ell_2,\ldots,\ell_k)|V(d;m,\ell_1,\ldots,\ell_k)| =$$
$$= \frac{1}{k!}\sum_{(m-1)+\ell_1+\cdots+\ell_k=d-k} \frac{f(m,\ell_1,\ell_2,\ldots,\ell_k)}{\ell_1\ell_2\ldots\ell_k}(n-1)_{d-k-1},$$

which finishes the proof of the Lemma. □

The following Lemma provides us with the number of (π, id)-paths within a given contraction sequence of a given merge sequence.

Lemma 9. *In the graph S_n, $n \geqslant 3$, let a vertex π have an $m-CO$ and k non-trivial CN-cycles of lengths $\ell_1,\ldots,\ell_k$ in its cyclic structure, where $m,k \geqslant 1$. Then there are $\ell_1\ell_2\ldots\ell_k$ distinct (π, id)-paths defined by any given merge sequence $(\ell_{i_1},\ell_{i_2},\ldots,\ell_{i_k})$ with any given contraction sequence $[j_0,j_1,j_2,\ldots,j_k]$.*

Proof. The (π, id)-paths defined by the merge sequence $(\ell_{i_1},\ell_{i_2},\ldots,\ell_{i_k})$ with contraction sequence $[j_0,j_1,j_2,\ldots,j_k]$ can be graphically represented as:

$$\underbrace{\tau_{CO}\ldots\tau_{CO}}_{j_0 \text{ times}}\tau_{CN}^{\ell_{i_1}}\underbrace{\tau_{CO}\ldots\tau_{CO}}_{j_1 \text{ times}}\tau_{CN}^{\ell_{i_2}}\underbrace{\tau_{CO}\ldots\tau_{CO}}_{j_2 \text{ times}}\tau_{CN}^{\ell_{i_3}}\cdots\tau_{CN}^{\ell_{i_k}}\underbrace{\tau_{CO}\ldots\tau_{CO}}_{j_k \text{ times}}.$$

Transpositions of type τ_{CO} are unique, whereas $\tau_{CN}^{\ell_{i_k}}$ defines ℓ_{i_k} different transpositions. Therefore the considered number of paths equals to the number of combinations of different transpositions of type τ_{CN}. The number of such combinations is $\ell_{i_1}\ell_{i_2}\ldots\ell_{i_k} = \ell_1\ell_2\ldots\ell_k$, which finishes the proof of the Lemma. □

4. The number of small (π, id)-cycles in Star graph

We first prove the bound on the number of cycles of length $2d$, where $3 \leqslant d \leqslant n$.

Lemma 10. *In the Star graph S_n, $n \geqslant 3$, the number of cycles of length $2d$ passing through a fixed vertex, where $3 \leqslant d \leqslant n$, is at most $O(n^{d-1})$.*

Proof. Consider a form of a $2d$ cycle $C_{2d} = t_{i_1}\, t_{i_2} \ldots t_{i_{2d}}$. Each transposition in C_{2d} must appear twice, since once we applied it while traversing a cycle to move an element inside the permutation, we must apply it again to move the element back to obtain the initial permutation. Thus, the number of unique transpositions in the form C_{2d} is at most d.

Suppose the form C_{2d} has exactly d unique transpositions, then by the previous argument each transposition must be used exactly twice. Let k be the position of the last unique consecutive transposition in the form. Since the graph is vertex-transitive, apply C_{2d} to the permutation id. Then after applying first k transpositions from C_{2d} we arrive at the permutation with cyclic structure $(1 i_1 i_2 \ldots i_k)(.)\ldots(.)$. After applying the next transposition $t_{i_{k+1}}$ which is equal to t_{i_j} for some j, where $1 \leqslant j \leqslant k$, we split the CO into $1-CO$ and $j-CN$. Thus, to reach back the permutation id we have to merge the $j-CN$ back and contract it. Then the same transposition t_{k+1} will be used again thus arriving at contradiction. Therefore the number of unique transpositions in C_{2d} is at most $d-1$.

In order to compute the number of cycles described by C_{2d} passing through the fixed vertex we need to count the possible combinations of transpositions it involves. Since the number of transpositions in the generating set is $n-1$, then the number of cycles the form describes is of order $O(n^{d-1})$. Due to hierarchical structure of the graph, the number of forms of cycles is constant, thus finishing the proof of the Lemma. $\square$

4.1. **Proof of Theorem 3.** Consider a vertex π at the distance d from the identity id with one non-trivial $\ell_1 - CN$ in its cyclic structure. By Lemma 7, b), the length of the CN-cycle is $\ell_1 = d-1$. The only merge sequence for π is (ℓ_1) and contraction sequence is $[d-1]$ and it can be graphically represented in the following way:

$$\tau_{CN}^{d-1} \underbrace{\tau_{CO} \dots \tau_{CO}}_{d-1 \text{ times}}. \tag{11}$$

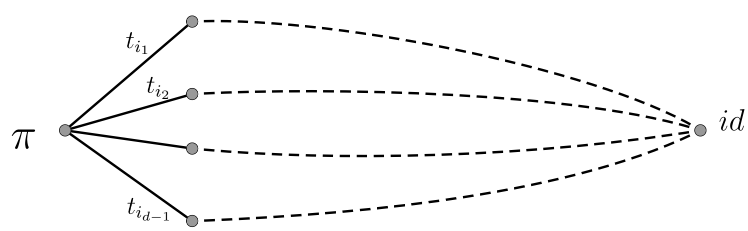

FIG. 1. (π, id)-paths with π having only non-trivial ℓ_1-CN.

Therefore (π, id)-cycles can only appear within the only contraction sequence of the only merge sequence. The contraction sequence is non-intersecting by definition and all (π, id)-paths are disjoint (see Figure 1). By Lemma 9 the number of distinct (π, id)-paths in this case is $(d-1)$, therefore the number of (π, id)-cycles for the given π is the following:

$$\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.$$

The total number of (π, id)-cycles is obtained by summing through all vertices π with given cyclic structure using Lemma 8:

$$N(1,1) = \sum_{\pi \in V(d;1,1)} \frac{(d-1)(d-2)}{2} = \frac{(d-1)(d-2)}{2} \frac{1}{d-1} (n-1)_{d-1} = \\ = \frac{d-2}{2}(n-1)_{d-1},$$

which finishes the proof of Theorem 3. $\square$

4.2. **Proof of Theorem 4.** Consider a vertex π at the distance d from the identity id with one non-trivial $m - CO$ and $\ell_1 - CN$ in its cyclic structure. By Lemma 7, b), the length of the CN-cycle is $\ell_1 = d - 1 - m$. The only merge sequence for π is (ℓ_1) and contraction sequences are $[j_0, j_1]$, where $0 \leqslant j_0 \leqslant m-1$ and $0 \leqslant j_1 = \ell_1 + m - 1 - j_0$, and they can be graphically represented as

$$\underbrace{\tau_{CO} \dots \tau_{CO}}_{j_0 \text{ times}} \tau_{CN}^{\ell_1} \underbrace{\tau_{CO} \dots \tau_{CO}}_{j_1 \text{ times}}. \tag{12}$$

It is easy to see that paths within any given contraction sequence are not disjoint. Then (π, id)-cycles can appear only within the only merge sequence and between contraction sequences. Divide the (π, id)-paths into Classes according to contraction sequences. Denote as **Class 1** the paths with the contraction sequence $[0, \ell_1+m-1]$, as **Class 2** the paths with the contraction sequence $[m-1, \ell_1]$ and let **Class 3** contain paths with the rest of contraction sequences. Paths inside Class 1 and Class 2 share same terminal vertices. Since each τ_{CO} is unique, then every path from Class 3 intersects with all paths from Class 1 and Class 2. Thus paths from Class 3 share same edges in the beginning with paths from Class 2 and in the end with paths from Class 1. Therefore, only disjoint pairs of paths are combined

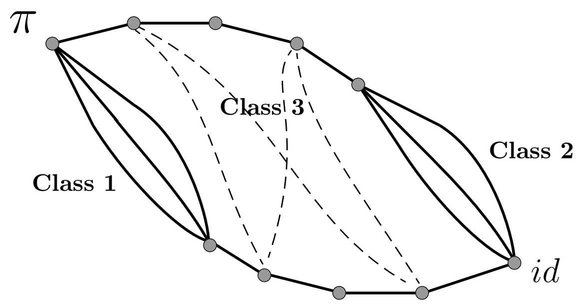

FIG. 2. Structure of Classes of (π, id)-paths.

from paths from Class 1 and Class 2 correspondingly. By Lemma 9, the number of distinct paths in each class is ℓ_1, therefore the number of (π, id)-cycles for a given π is $\ell_1^2 = (d-1-m)^2$.

The total number of (π, id)-cycles is obtained by summing through all vertices π with given cyclic structure using Lemma 8:

$$N(m, \ell_1) = \sum_{m=1}^{d-3} \sum_{\pi \in V(d;m,1)} \ell_1^2 = \sum_{m=1}^{d-3} \ell_1^2 \frac{1}{\ell_1}(n-1)_{d-1} = $$
$$= \sum_{m=1}^{d-3} (d-1-m)(n-1)_{d-1} = $$
$$= \sum_{m'=2}^{d-2} m'(n-1)_{d-1} = \frac{d(d-3)}{2}(n-1)_{d-1},$$

which finishes the proof of Theorem 4. □

4.3. **Proof of Theorem 5.** Consider a vertex π at the distance d from the identity with $1-CO$, and only non-trivial ℓ_1-CN and ℓ_2-CN. By Lemma 7, b), the lengths of CN-cycles are related as $d = \ell_1+\ell_2+2$. Since $\ell_1, \ell_2 \geqslant 2$, then $6 \leqslant d \leqslant n+1$. Let us denote two distinct (π, id)-paths as $P_1 = (\pi_0, \pi_1, \ldots, \pi_d)$ and $P_2 = (\pi'_0, \pi'_1, \ldots, \pi'_d)$, where $\pi_0 = \pi'_0 = \pi$ and $\pi_d = \pi'_d = id$. There are two different merge sequences:

1) (ℓ_1, ℓ_2) with contraction sequences $[j_1, j_2]$, which can be graphically represented as

$$\tau_{CN}^{\ell_1} \underbrace{\tau_{CO} \ldots \tau_{CO}}_{j_1 \text{ times}} \tau_{CN}^{\ell_2} \underbrace{\tau_{CO} \ldots \tau_{CO}}_{j_2 \text{ times}}, \tag{13}$$

where $0 \leqslant j_1 \leqslant \ell_1$ and $0 \leqslant j_2 = \ell_2 + \ell_1 - j_1$, and

2) (ℓ_2, ℓ_1) with contraction sequences $[j'_1, j'_2]$, which can be graphically represented as

$$\tau_{CN}^{\ell_2} \underbrace{\tau_{CO} \dots \tau_{CO}}_{j'_1 \text{ times}} \tau_{CN}^{\ell_1} \underbrace{\tau_{CO} \dots \tau_{CO}}_{j'_2 \text{ times}}, \tag{14}$$

where $0 \leqslant j'_1 \leqslant \ell_2$ and $0 \leqslant j'_2 = \ell_1 + \ell_2 - j'_1$.

We consider three cases of formation of (π, id)-cycles, mentioned in the end of Section 2. In cases I and II we consider the merge sequence (ℓ_1, ℓ_2) in details, while the result for the merge sequence (ℓ_2, ℓ_1) is obtained by analogy.

Case I. Consider a pair of paths P_1 and P_2 within the merge sequence (ℓ_1, ℓ_2) and contraction sequence $[j_1, j_2]$ (see Figure 3). The paths are disjoint if and only if the contraction sequence is non-intersecting. It is easy to see that the only intersecting contraction sequence in this case is $[\ell_1, \ell_2]$.

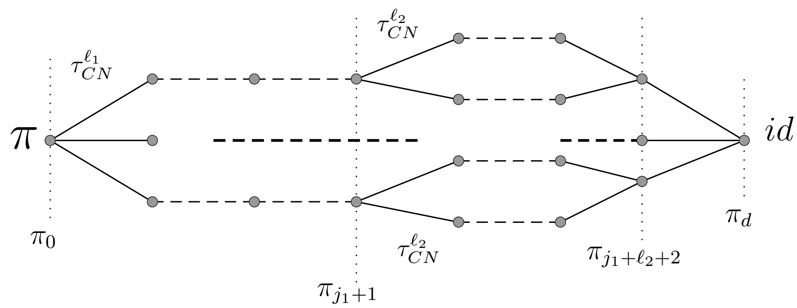

FIG. 3. Structure of (π, id)-paths within merge sequence and contraction sequence.

Since the paths start with a transposition of type $\tau_{CN}^{\ell_1}$, then to obtain a cycle we need to choose two distinct transpositions of this type and the rest of the paths will be disjoint. Since the transposition of type $\tau_{CN}^{\ell_2}$ describes ℓ_2 edges, then the number of cycles for a given contraction sequence is $\binom{\ell_1}{2}\ell_2^2$. Since we have ℓ_1 non-intersecting contraction sequences for a given merge sequence, then we have $\ell_1 \left(\binom{\ell_1}{2}\ell_2^2\right)$ cycles in this case. In the case of the merge sequence (ℓ_2, ℓ_1), by analogy, we have $\ell_2 \left(\binom{\ell_2}{2}\ell_1^2\right)$ cycles.

Case II. Consider a pair of paths P_1 and P_2 within merge sequence (ℓ_1, ℓ_2) with different contraction sequences $[j_1, j_2]$ and $[j'_1, j'_2]$ (see Figure 4).

In order to have P_1 and P_2 being disjoint, apply different transpositions of type $\tau_{CN}^{\ell_1}$ to the vertex π and obtain the following vertices:

$$\pi_1 = \pi\, t_{\pi_u^1} = (1\pi_u^1 \dots \pi_{\ell_1}^1 \pi_1^1 \dots \pi_{u-1}^1)(\pi_1^2 \dots \pi_{\ell_2}^2),$$

and

$$\pi'_1 = \pi\, t_{\pi_v^1} = (1\pi_v^1 \dots \pi_{\ell_1}^1 \pi_1^1 \dots \pi_{v-1}^1)(\pi_1^2 \dots \pi_{\ell_2}^2),$$

where $1 \leqslant u, v \leqslant \ell_1$. Since we require paths to be disjoint, we have $u \neq v$. Then after j_1 and j'_1 applications of τ_{CO} correspondingly we have the following permutations

$$\pi_{j_1+1} = (1\overline{\pi^1})(\pi_1^2 \dots \pi_{\ell_2}^2),$$

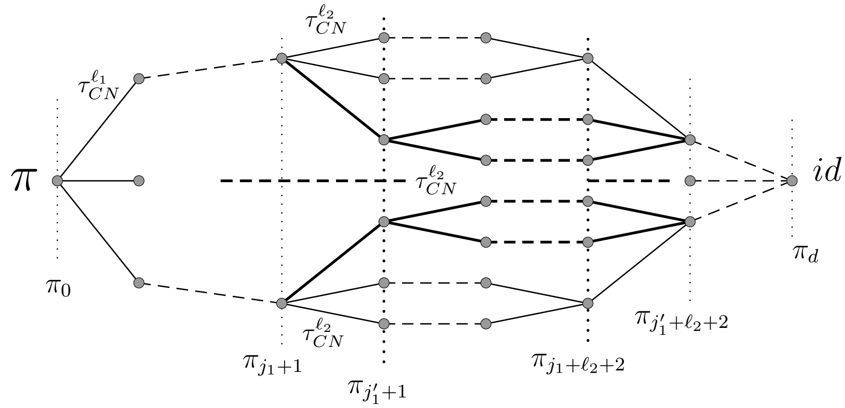

FIG. 4. Structure of (π, id)-paths between contraction sequences.

and

$$\pi_{j_1'+1} = (1\overline{\overline{\pi^1}})(\pi_1^2 \dots \pi_{\ell_2}^2),$$

where $\overline{\pi^1}$ and $\overline{\overline{\pi^1}}$ represent contracted segments of corresponding lengths $\ell_1 - j_1 \geqslant 0$ and $\ell_1 - j_1' \geqslant 0$. If $\overline{\pi^1}$ and $\overline{\overline{\pi^1}}$ are empty, then the paths P_1 and P_2 intersect in the same vertex $(1)(\pi_1^2 \dots \pi_{\ell_2}^2)$, thus we require at least one $j_1 \neq \ell_1$ or $j_1' \neq \ell_1$, i.e. one contraction sequence has to be non-intersecting.

Applying next the transposition of type $\tau_{CN}^{\ell_2}$ we have the following vertices:

$$\pi_{j_1+2} = \pi_{j_1+1}\, t_{\pi_u^2} = (1\pi_u^2 \dots \pi_{\ell_2}^2 \pi_1^2 \dots \pi_{u-1}^2 \overline{\pi^1}),$$

and

$$\pi_{j_1'+2} = \pi_{j_1'+1}\, t_{\pi_v^2} = (1\pi_v^2 \dots \pi_{\ell_2}^2 \pi_1^2 \dots \pi_{v-1}^2 \overline{\overline{\pi^1}}),$$

where $1 \leqslant u, v \leqslant \ell_2$. By Lemma 7, a), the rest of the paths $(\pi_{j_2+2}, \dots, \pi_d)$ and $(\pi'_{j_2+2}, \dots, \pi'_d)$ are unique.

Since contraction sequences are different, then $j_1 \neq j_1'$ and $\overline{\pi^1} \neq \overline{\overline{\pi^1}}$ and the paths are disjoint. Hence, for a given pair of contraction sequences we have $\binom{\ell_1}{2}\ell_2^2$ distinct (π, id)-cycles. We have $\binom{\ell_1}{2}$ distinct pairs of contraction sequences with $j_1 \neq \ell_1$ and $j_1' \neq \ell_1$ and ℓ_1 pairs of contraction sequences with one intersecting contraction sequence $[\ell_1, \ell_2]$. Therefore, we have total $\left(\binom{\ell_1}{2} + \ell_1\right)\binom{\ell_1}{2}\ell_2^2$ cycles in this case. In the case of merge sequence (ℓ_2, ℓ_1), by analogy, we have $\left(\binom{\ell_2}{2} + \ell_2\right)\binom{\ell_2}{2}\ell_1^2$ cycles.

Case III. Consider a pair of paths P_1 and P_2, such that P_1 has merge sequence (ℓ_1, ℓ_2) and contraction sequence $[j_1, j_2]$ and P_2 has merge sequence (ℓ_2, ℓ_1) and contraction sequence $[j_2', j_1']$.

It is easy to see that in the case when each contraction sequence is non-intersecting, paths P_1 and P_2 will be disjoint. By Lemma 9 each contraction sequence gives us $\ell_1\ell_2$ distinct paths, then for each pair of contraction sequences we have $(\ell_1\ell_2)^2$ distinct (π, id)-cycles. The number of pairs of contraction sequences is $\ell_1\ell_2$ in this case, then the total number of (π, id)-cycles in this case is $(\ell_1\ell_2)^3$.

Further let $[j_1, j_2]$ be intersecting and $[j_2', j_1']$ be non-intersecting. Then the path P_1 has the following vertex after the application of transposition of type $\tau_{CN}^{\ell_2}$:

$$\pi_{j_1+2} = (1\pi_u^2 \dots \pi_{\ell_2}^2 \pi_1^2 \dots \pi_{u-1}^2),$$

for some u, where $1 \leqslant u \leqslant \ell_2$.

The path P_2 has the following vertex after the application of transposition of type $\tau_{CN}^{\ell_1}$:

$$\pi'_{j_2'+2} = (1\pi_v^1 \dots \pi_{\ell_1}^1 \pi_1^1 \dots \pi_{v-1}^1 \overline{\overline{\pi^2}}),$$

for some v, where $1 \leqslant v \leqslant \ell_1$, and $\overline{\overline{\pi^2}}$ is a non-empty remainder of the merged cycle π^2. Notice that P_1 and P_2 may have the common vertex $\pi_{d-1} = \pi'_{d-1} = (1\pi_{u-1}^2)$. We exclude such pairs of paths by fixing P_1 and counting through P_2 with different vertex π'_{d-1}, which gives us $(\ell_2 - 1)\ell_1$ cycles for a given P_1. By Lemma 9 there are $\ell_1\ell_2$ choices of path P_1, hence the total number of (π, id)-cycles in this case equals to $\ell_1^2\ell_2(\ell_2 - 1)$. By analogy, in the case when $[j_2', j_1']$ is intersecting and $[j_1, j_2]$ is not, we have $\ell_2^2\ell_1(\ell_1 - 1)$.

Finally, when both $[j_1, j_2]$ and $[j_2', j_1']$ are intersecting, each pair of paths P_1 and P_2 is disjoint, which by Lemma 9 gives us $\ell_1^2\ell_2^2$ more cycles.

We have considered all cases of appearance of (π, id)-cycles. In order to obtain the total number of the number of (π, id)-cycles for a given π we have to sum up all the cases and we get $N_\pi(1, 2)$ cycles, where $N_\pi(1, 2)$ is equal to:

$$\ell_1\ell_2 \left(\frac{\ell_1^3\ell_2}{4} + \frac{\ell_1\ell_2^3}{4} + \frac{\ell_1^2\ell_2}{2} + \frac{\ell_1\ell_2^2}{2} + \ell_1^2\ell_2^2 + \frac{3\ell_1\ell_2}{2} - \ell_1 - \ell_2 \right).$$

The total number of (π, id)-cycles is obtained by summing through all vertices π with given cyclic structure using Lemma 8:

$$\begin{aligned} N(1,2) &= \sum_{\pi \in V(d;1,2)} N_\pi(1,2) = \\ &= \frac{1}{240}(d-5)(7d^4 - 25d^3 + 40d^2 - 220d + 288)(n-1)_{d-2}, \end{aligned}$$

which finishes the proof of the Theorem. □

5. The number of 10-cycles in the Star graph.

Consider a cycle C_{2d} of length $2d$, where $3 \leqslant d \leqslant D$, passing through a fixed vertex $\tau \in S_n$. Define the distance profile of a cycle C_{2d} as the sequence $(c_0, c_1, \dots, c_d)$, where c_i is the number of vertices of the cycle at the distance i from τ, where $0 \leqslant i \leqslant d$.

Proof of Corollary 6. Suppose we have a cycle C_{10} passing through a vertex $\tau \in S_n$. Since the graph S_n is vertex-transitive, we may assume $\tau = id$. There are no 4-cycles in this graph, the only possible distance profiles of C_{10} are the following: (a) $(1,2,2,2,2,1)$; (b) $(1,2,2,3,2,0)$; (c) $(1,3,4,2,0,0)$ and (d) $(1,2,4,3,0,0)$ and they are schematically represented in Fig 5. We consider each distance profile independently.

Consider the distance profile (a). By Lemma 7, b), vertices at the distance 5 from id have either only non-trivial $6-CO$, or $2-CO$ and one non-trivial $2-CN$ or trivial CO-cycle and one non-trivial $3-CN$. By Lemma 7, a), vertices with

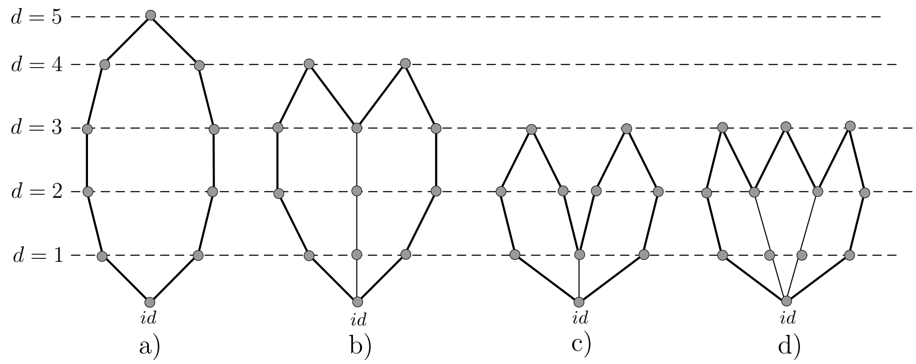

FIG. 5. Possible distance profiles of 10-cycles.

only non-trivial CO have unique path to the identity id, thus the number of cycles with distance profile (a) is the number of (π, id) cycles passing through vertices having $1 - CO$ and $3 - CN$ or $2 - CO$ and $2 - CN$ in its cyclic structure. Thus, by Theorem 3 and 4 the number of cycles in this case is given by:

$$\frac{13}{2}(n-1)_4.$$

Consider the distance profile (b). The 10-cycles with this distance profile can be considered as the union of two 8-cycles with a common 3-path. By Theorem 2, cycles of forms C_8^1 and C_8^3 have a common 3-path $t_i\, t_j\, t_k$ with same values of indices i, j and k, and forms C_8^1 and C_8^2 have a common 3-path $t_i\, t_j\, t_i$ with same values of indices i, j, but may have different values of k. In the first case, the forms can be applied in two ways: $id = id\, t_k\, t_j\, t_i \ldots$ or $id = id\, t_i\, t_j\, t_k \ldots$, thus for each pair of forms gives two 10-cycles. The number of pairs of such forms is given by the following formula:

$$\frac{1}{3}(n-2)_3.$$

In the second case, any pair of forms gives four 10-cycles, since we can combine cycles from different forms and cycles from the same form. The number of pairs of forms is given by:

$$\frac{1}{12}(n-1)_4.$$

Therefore, the total number of 10-cycles with distance profile (b) is equal to:

$$\frac{1}{3}(n-1)_4 + \frac{2}{3}(n-2)_3.$$

Consider the distance profile (c). The 10-cycles with this distance profile can be considered as the union of two 6-cycles with a common edge. By Theorem 1, the 6-cycles have the only form C_6 and there is the only way to make two cycles with a common edge is to take cycles of forms $C_6^1 = t_i\, t_j\, t_i\, t_j\, t_i\, t_j$ and $C_6^2 = t_i\, t_k\, t_i\, t_k\, t_i\, t_k$, where $2 \leqslant i < j < k \leqslant n$. The number of such pairs of cycles is equal to:

$$\frac{1}{6}(n-2)_3,$$

and equals to the number of 10-cycles.

Consider the distance profile (d). The 10-cycles with this distance profile can be considered as a gluing of two 6-cycles with a third one via different 2-paths. By Theorem 1 there are no such possible forms of 6-cycles, therefore there are no 10-cycles in this case. Summarizing the cases, we have the total number of 10-cycles is equal to:

$$\frac{41}{6}(n-1)_4 + \frac{5}{6}(n-1)_3,$$

which finishes the proof of the Theorem. $\square$

References

[1] S. B. Akers, B. Krishnamurthy, *The star graph: an attractive alternative to n-cube,* in: Proceedings of International Conference on Parallel Processing, ICPP-87, St. Charles, Illinois, August 1987, 393–400.

[2] L. Wang, S. Subramanian, S. Latifi, P. K. Srimani, *Distance distribution of nodes in star graphs*, Applied Mathematics Letters **19**:8 (2006), 780–784. Zbl 1119.05035

[3] E. V. Konstantinova, A.N. Medvedev, *Small cycles in the Star graph*, Siberian Electronic Mathematical Reports, **11** (2014), 906–914. (http://semr.math.nsc.ru/v11/p906-914.pdf) Zbl 1326.05062

[4] S. B. Akers, B. Krishnamurthy, *A group–theoretic model for symmetric interconnection networks*,IEEE Trans. Comput., **38**:4 (1989), 555–566. MR0984680

[5] J. S. Jwo, S. Lakshmivarahan, S. K. Dhall, *Embedding of cycles and grids in star graphs*, J. Circuits, Syst., Comput., **1**:1 (1991), 43–74.

[6] V. L. Kompel'makher, V. A. Liskovets, *Successive generation of permutations by means of a transposition basis*, Kibertetika, **3** (1975), 17–21 (in Russian). MR0498276

[7] P. J. Slater, *Generating all permutations by graphical transpositions*, Ars Combinatoria, **5** (1978), 219–225. MR0504868

[8] N. Alon, R. Yuster, U. Zwick, *Finding and counting given length cycles*, Algorithmica, **17**:3 (1997), 209–223. Zbl 0865.68093

[9] J. A. Bondy, M. Siminovits, *Cycles of even length in graphs*, Journal of Combinatorial Theory, **16**:B (1974), 97–105. Zbl 0283.05108

[10] J. A. Fill, R. Pemantle, *Percolation, First-passage Percolation and Covering Times for Richardson's Model on the N-cube*, The Annals of Applied Probability, **3**:2 (1993), 593–629. Zbl 0783.60102

[11] E. V. Konstantinova, A. N. Medvedev, *Cycles of length seven in the Pancake graph*, Diskretn. Anal. Issled. Oper., **17**:5 (2010), 46–55. (in Russian). Zbl 1249.05207

[12] E. V. Konstantinova, A. N. Medvedev, *Cycles of length nine in the Pancake graph*, Diskretn. Anal. Issled. Oper., **18**:6 (2011), 33–60. (in Russian) Zbl 1249.05208

[13] E. Konstantinova, A. Medvedev, *Small cycles in the Pancake graph*, Ars Mathematica Contemporanea, **7** (2014), 237–246. Zbl 1301.05126

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