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MULTIPLICATIVE CONTROL PROBLEMS FOR NONLINEAR CONVECTION–DIFFUSION–REACTION EQUATION

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Abstract. Control problem for convection-diffusion-reaction equation, in which reaction coefficient depends nonlinearly on substance's concentration, is considered. Velocity vector, multiplicatively entered into the considered equation, is chosen as a control function. Extremum problem's solvability for reaction coefficient of common type is proved. Optimality system for quadratic reaction coefficient is obtained and on its basis local uniqueness of control problem's solutions for particular cost functionals is proved.

Keywords: convection-diffusion-reaction equation, multiplicative control problems, optimality system, local uniqueness.

1. Introduction. Boundary value problem

In a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ the following boundary value problem is considered

$$-\lambda\Delta\varphi + \mathbf{u}\cdot\nabla\varphi + k\varphi = f \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Gamma. \tag{1}$$

Here function φ means polluting substance's concentration, $\mathbf{u}$ is a given vector of velocity, f is a volume density of external sources of substance, λ − constant diffusion coefficient, function $k = k(\varphi)$ is a reaction coefficient. This problem (1) will be called problem 1 below.

The study of optimal control problems for model (1) is directed at searching efficient mechanisms to control chemical reactions' behavior. The decision to choose

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a vector of velocity $\mathbf{u}$ as a controlling function can signify the regularization of combustion process at the expense of fuel feed's intensity changing (see [1]). The efficiency criterion of such kind of control is the measured concentration of unburned fuel in a subdomain. It should be mentioned that some inverse problems can be reduced to optimal control ones. Particularly, the optimal control problem, which is considered in this paper, can mean the indentification problem of wind rose with the help of data about fire spread intensity. See also [3, 4, 5, 6, 7, 8] about similar methods and approaches.

The global solvability of problem 1, when reaction coefficients belong to rather wide class of functions, is proved in [9]. In this paper it is shown that power coefficients from [10, 11] are particular cases of the reaction coefficients considered in [9], with which nonlocal uniqueness of boundary value problem's solution takes place. The solvability of multiplicative control problem with common reaction coefficients is proved further. For quadratic reaction coefficient optimality system is obtained, on the analysis of which sufficient conditions for local uniqueness of multiplicative control problems' solutions for particular cost functionals are received.

While studing problem 1 and optimal control problems Sobolev spaces will be used: $H^s(D)$, $\mathbf{H}^s(D) \equiv H^s(D)^3$, $s \in \mathbb{R}$ and $L^r(D)$, $1 \le r \le \infty$, where D is either a domain Ω or its boudary Γ. Scalar products in $L^2(\Omega)$, $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ are denoted by $(\cdot,\cdot)$ and $(\cdot,\cdot)_1$, scalar products in $L^2(\Gamma)$ – by $(\cdot,\cdot)_\Gamma$, norm in $L^2(\Omega)$ – by $\|\cdot\|$, norm or semi-norm in $H^1(\Omega)$ – by $\|\cdot\|_1$ or $|\cdot|_1$.

It will be assumed that the domain Ω and its boundary Γ satisfy the following:

(i) Ω is a bounded domain in the space $\mathbb{R}^3$ with boundary $\Gamma \in C^{0,1}$.

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable functions with finite support in Ω, $L^p_+(\Omega) = \{k \in L^p(\Omega) : k \ge 0\}$, $p \ge 3/2$. Also let $\mathbf{Z} = \{\mathbf{v} \in \mathbf{L}^4(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}$, $\mathbf{V} \equiv \mathbf{Z} \cap \mathbf{H}^1(\Omega)$.

From Poincare-Friedrichs inequality and from continuity of embedding operator $H^1(\Omega) \subset L^4(\Omega)$ this lemma follows:

Lemma 1. *If conditions (i) hold, then there are such positive constants C_0, δ, C_4 and γ, depending on Ω, that for any functions $\varphi, S \in H^1(\Omega)$, $k \in L^p_+(\Omega)$, where $p \ge 3/2$, $\mathbf{u} \in \mathbf{Z}$ these relations are correct:*

(2)
$$|(\nabla\varphi, \nabla S)| \le \|\varphi\|_1 \|S\|_1,\ |(k\varphi, S)| \le C_0 \|k\|_{L^p(\Omega)} \|\varphi\|_1 \|S\|_1,\ \|\varphi\|_{L^4(\Omega)} \le C_4 \|\varphi\|_1,$$
$$|(\mathbf{u}\cdot\nabla\varphi, S)| \le \gamma \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)} \|\varphi\|_1 \|S\|_1 \le \gamma C_4 \|\mathbf{u}\|_1 \|\varphi\|_1 \|S\|_1,$$

$$(\mathbf{u}\cdot\nabla\varphi, \varphi) = 0\ \forall \varphi \in H^1_0(\Omega), \tag{3}$$

and for any function $S \in H^1_0(\Omega)$ the inequality takes place

$$(\nabla S, \nabla S) \ge \delta \|S\|_1^2. \tag{4}$$

From lemma 1 follows that while conditions (i) are satisfied with the constant $\lambda_* = \delta\lambda$ when $k \in L^{3/2}_+(\Omega)$, then the coercitive inequality is met

$$\lambda(\nabla S, \nabla S) + (kS, S) \ge \lambda_* \|S\|_1^2\ \forall S \in H^1_0(\Omega). \tag{5}$$

Let in addition to (i) the conditions hold:

(ii) $f \in L^2(\Omega)$, $\mathbf{u} \in \mathbf{Z}$.

(iii) $k \in L^p_+(\Omega)$, $p \geq 3/2$, wherein function $k = k(\varphi)$ is Lipschitz continuous of φ, i.e. if $\|\varphi_1\|_1 \leq c$ and $\|\varphi_2\|_1 \leq c$, then

$$\|k(\varphi_1) - k(\varphi_2)\|_{L^p(\Omega)} \leq L\|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \ \ \forall \varphi_1, \varphi_2 \in H^1_0(\Omega).$$

Let's multiply the equation in (1) by $S \in H^1_0(\Omega)$ and integrate over Ω. The following will be got

$$\lambda(\nabla\varphi, \nabla S) + (k(\varphi)\varphi, S) + (\mathbf{u} \cdot \nabla\varphi, S) = (f, S) \ \forall S \in H^1_0(\Omega). \tag{6}$$

As a result, the weak formulation of problem 1 is obtained. It consists in finding function $\varphi \in H^1_0(\Omega)$ from (6).

Definition 1. *A function $\varphi \in H^1_0(\Omega)$ which satisfies (6) will be called a weak solution of problem 1.*

The following theorem takes place [9].

Theorem 1. *If conditions (i)–(iii) hold, then a weak solution $\varphi \in H^1_0(\Omega)$ of problem 1 exists and the estimate takes place:*

$$\|\varphi\|_1 \leq M_\varphi = (1/\lambda_*)\|f\|. \tag{7}$$

If, besides, this condition is met

$$C_0 L\|f\| \leq \lambda_*^2, \tag{8}$$

then problem 1's weak solution is unique.

From [9, 10, 11] it ensues that the power dependence is interesting as an example of particular cases of function $k = k(\varphi)$, $k = \varphi^2$ and $k(\varphi) = \varphi^2|\varphi|$, for instance. As the case of quadratic reaction coefficient was analysed in detail in [9], in particular it was shown that a function $k = \varphi^2$ satisfies conditions (iii) and for this function there is nonlocal uniqueness of problem 1's weak solution , so let's consider the function $k = \varphi^2|\varphi|$.

For $k = \varphi^2|\varphi|$ the equality is true:

$$k(\varphi_1) - k(\varphi_2) = \varphi_1^2(|\varphi_1| - |\varphi_2|) + (\varphi_1 - \varphi_2)(\varphi_1 + \varphi_2)|\varphi_2| \text{ a.e. in } \Omega$$

and also an estimate takes place:

$$\left(\int_\Omega (\varphi_1 - \varphi_2)^{3/2}\varphi_1^3 \, d\Omega\right)^{2/3} \leq \|\varphi_1 - \varphi_2\|_{L^3(\Omega)}\|\varphi_1\|^2_{L^6(\Omega)}.$$

In such case function $k = \varphi^2|\varphi|$ satisfies the condition (iii).

When $k = \varphi^2|\varphi|$ nonlocal uniqueness of problem 1's solution takes place. Actually, let $k = \varphi^2|\varphi|$ and $\varphi_1, \varphi_2 \in H^1(\Omega)$ be two solutions of problem 1. Then their difference $\varphi = \varphi_1 - \varphi_2 \in H^1_0(\Omega)$ satisfies the ratio

$$\lambda(\nabla\varphi, \nabla h) + (\varphi_1^3|\varphi_1| - \varphi_2^3|\varphi_2|, h) + (\mathbf{u} \cdot \nabla\varphi, h) = 0 \ \forall h \in H^1_0(\Omega). \tag{9}$$

It's clear that

$$(\varphi_1^3|\varphi_1| - \varphi_2^3|\varphi_2|)(\varphi_1 - \varphi_2) = \varphi_1^4|\varphi_1| - \varphi_2^3|\varphi_2|\varphi_1 - \varphi_1^3|\varphi_1|\varphi_2 + \varphi_2^4|\varphi_2| \text{ a.e. in } \Omega$$

and on the strength of Young's inequality

$$\varphi_2^4\varphi_1 \leq (4/5)\varphi_2^5 + (1/5)\varphi_1^5, \ \ \varphi_1^4\varphi_2 \leq (4/5)\varphi_1^5 + (1/5)\varphi_2^5 \text{ a.e. in } \Omega.$$

In such case $(\varphi_1^3|\varphi_1| - \varphi_2^3|\varphi_2|, \varphi) \geq 0$ a.e. in Ω. Assuming $h = \varphi$ in (9), on the strength of lemma 1 it can be concluded that $\varphi = 0$ or $\varphi_1 = \varphi_2$ in Ω.

From aforesaid and [9] follows

Theorem 2. *Let conditions (i), (ii) hold. Then when $k = \varphi^2$ and $k = \varphi^2|\varphi|$, there is a unique weak solution $\varphi \in H^1(\Omega)$ of problem 1 and the estimate (7) is met.*

2. Statement of optimal control problem and its solvability

Let's formulate an optimal control problem for problem 1. For this purpose the whole set of initial data will be devided into two groups: the group of fixed functions, in which function f is included, and the group of controlling functions, in which $\mathbf{u}$ will be included, assuming that it can be changed in some subset K.

Let's introduce an operator $F : H_0^1(\Omega) \times K \to H^{-1}(\Omega)$ by formula

$$\langle F(\varphi, \mathbf{u}), S\rangle = \lambda(\nabla\varphi, \nabla S) + (\mathbf{u} \cdot \nabla\varphi, S) + (k(\varphi)\varphi, S) - (f, S).$$

Then (6) can be rewritten in the following form:

$$F(\varphi, \mathbf{u}) = 0. \tag{10}$$

Let's suppose that these conditions hold

(j) $K \subset \mathbf{V}$ is a nonempty convex closed set;

(jj) $\mu_i \geq 0, i = 1, 2$ and K is a bounded set $\mu_l > 0, l = 0, 1$ and functional I is bounded below.

Treating (10) as a conditional restriction on the state $\varphi \in H_0^1(\Omega)$ and on the control $\mathbf{u} \in K$, the problem of conditional minimization can be formulated as follows:

$$J(\varphi, k) \equiv \frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2}\|\mathbf{u}\|_1^2 \to \inf,\ F(\varphi, \mathbf{u}) = 0,\ (\varphi, \mathbf{u}) \in H_0^1(\Omega) \times K. \tag{11}$$

The following cost functionals can be used in the capacity of the possible ones:

$$I_1(\varphi) = \|\varphi - \varphi_d\|_Q^2 = \int_\Omega |\varphi - \varphi_d|^2 d\mathbf{x}, \quad I_2(\varphi) = \|\varphi - \varphi_d\|_{1,Q}^2.$$

Here $\varphi_d \in L^2(Q)$ is a given function in some subdomain - $Q \subset \Omega$. The set of possible pairs for problem (11) is denoted by $Z_{ad} = \{(\varphi, \mathbf{u}) \in H_0^1(\Omega) \times K : F(\varphi, \mathbf{u}){=}0, J(\varphi, \mathbf{u}){<}\infty\}$.

Theorem 3. *Let conditions (i)–(iii) and (j), (jj) hold. Then there is at least one solution of optimal control problem (11).*

Proof. Let $(\varphi_m, \mathbf{u}_m)$ be a minimizing sequence, for which the following is true:

$$\lim_{m\to\infty} J(\varphi_m, \mathbf{u}_m) = \inf_{(\varphi_m, \mathbf{u}_m)\in Z_{ad}} J(\varphi_m, \mathbf{u}_m) \equiv J^*.$$

That and the conditions of theorem for functional J from (11) imply the estimate $\|\mathbf{u}_m\|_1 \leq c_1$. From theorem 1 follows directly that $\|\varphi_m\|_1{\leq}c_2$, where constant c_2 doesn't depend on m.

Then the weak limits $\varphi^* \in H_0^1(\Omega)$ and $\mathbf{u}^* \in \mathbf{V}$ of some subsequences of sequences $\{\varphi_m\}$ and $\{\mathbf{u}_m\}$ exist. Corresponding sequences will be also denoted by $\{\varphi_m\}$ and $\{\mathbf{u}_m\}$. With this in mind it can be considered that

$$\varphi_m \to \varphi^* \in H^1(\Omega) \text{ weakly in } H^1(\Omega) \text{ ang strongly in } L^4(\Omega), \tag{12}$$

$$\mathbf{u}_m \to \mathbf{u}^* \in \mathbf{H}^1(\Omega) \text{ weakly in } \mathbf{H}^1(\Omega) \text{ and strongly in } \mathbf{L}^4(\Omega). \tag{13}$$

Let's show that $F(\varphi^*, \mathbf{u}^*) = 0$, i.e.

$$\lambda(\nabla\varphi^*, \nabla S) + (k(\varphi^*)\varphi^*, S) + (\mathbf{u}^* \cdot \nabla\varphi^*, S) = (f, S) \ \ \forall S \in H_0^1(\Omega). \tag{14}$$

And it should be taken into account that φ_m and $\mathbf{u}_m$ satisfy the relations

$$\lambda(\nabla\varphi_m,\nabla S)+(k(\varphi_m)\varphi_m,S)+(\mathbf{u}_m\cdot\nabla\varphi_m,S)=(f,S)\ \ \forall S\in H_0^1(\Omega). \tag{15}$$

Let's pass to the limit in (15) at $m\to\infty$. All linear summands in (15) turn into corresponding ones in (14). For nonlinear summand $(k(\varphi_m)\varphi_m,S)$ the inequality takes place

$$|(k(\varphi_m)\varphi_m,S)-(k(\varphi^*)\varphi^*,S)|\le|(k(\varphi_m)(\varphi_m-\varphi^*),S)|+|(k(\varphi_m)-k(\varphi^*),\varphi^*S)|.$$

On the strength of lemma 1 and condition (iii) for function $k=k(\varphi)$ it is obtained that

$$|(k(\varphi_m)-k(\varphi^*),\varphi^*S)|\le L\|\varphi_m-\varphi^*\|_{L^4(\Omega)}\|\varphi^*\|_{L^4(\Omega)}\|S\|_{L^4(\Omega)}\to 0 \text{ at } m\to\infty.$$

To apply the property (12) for summand $|(k(\varphi_m)(\varphi_m-\varphi^*),S)|$, embedding density by norm $\|\cdot\|_1$ will be used. Let $\{S_n\}\in\mathcal{D}(\Omega)$ be such a sequence of functions that $\|S_n-S\|_1\to 0$ at $n\to\infty$.

This inequality holds:

$$|(k(\varphi_m)(\varphi_m-\varphi^*),S_n)|\le\|k(\varphi_m)\|_{L^{3/2}(\Omega)}\|S_n\|_{L^{12}(\Omega)}\|\varphi_m-\varphi^*\|_{L^4(\Omega)}\to 0 \text{ at } m\to\infty.$$

As far as

$$\begin{aligned}||(k(\varphi_m)(\varphi_m-\varphi^*),S_n)|-|(k(\varphi_m)(\varphi_m-\varphi^*),S)||&\le|(k(\varphi_m)(\varphi_m-\varphi^*),S_n-S)|\\ \le\|k(\varphi_m)\|_{L^{3/2}(\Omega)}\|\varphi_m-\varphi^*\|_{L^6(\Omega)}\|S_n-S\|_{L^6(\Omega)}&\to 0 \text{ at } n\to\infty,\ m=1,2,...\end{aligned}$$

Then

$$\lim_{m\to\infty}(k(\varphi_m)\varphi_m,S)=(k^*(\varphi^*)\varphi^*,S).$$

For nonlinear summand $(\mathbf{u}_m\cdot\nabla\varphi_m,S)$ this relation is satisfied

$$(\mathbf{u}_m\cdot\nabla\varphi_m,S)-(\mathbf{u}^*\cdot\nabla\varphi^*,S)=(\mathbf{u}^*\cdot\nabla(\varphi_m-\varphi^*),S)+((\mathbf{u}_m-\mathbf{u}^*)\cdot\nabla\varphi_m,S)\ \forall S\in H_0^1(\Omega). \tag{16}$$

On the strength of (12) a weak convergence takes place: $\nabla\varphi_m\to\nabla\varphi^*$ in $\mathbf{L}^2(\Omega)$, according to which

$$(\mathbf{u}^*\cdot\nabla(\varphi_m-\varphi^*),S)=(\nabla(\varphi_m-\varphi^*),\mathbf{u}^*S)\to 0 \text{ at } m\to\infty\ \forall S\in H_0^1(\Omega),$$

and from (13) follows that

$$|((\mathbf{u}_m-\mathbf{u}^*)\cdot\nabla\varphi_m,S)|\le\|\nabla\varphi_m\|_{\mathbf{L}^2(\Omega)}\|\mathbf{u}_m-\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|S\|_{L^4(\Omega)}\to 0$$

as $m\to\infty\ \forall S\in H_0^1(\Omega)$.

Then, taking (16) into account, it is obtained

$$\lim_{m\to\infty}(\mathbf{u}_m\cdot\nabla\varphi_m,S)=(\mathbf{u}^*\cdot\nabla\varphi^*,S).$$

As the functional J is weakly semicontinuous below on $H_0^1(\Omega)\times\mathbf{V}$, then from aforesaid follows that

$$J^*=\lim_{m\to\infty}J(\varphi_m,\mathbf{u}_m)=\underline{\lim}_{m\to\infty}J(\varphi_m,\mathbf{u}_m)\ge J(\varphi^*,\mathbf{u}^*)\ge J^*.\ \blacksquare$$

Further the case of $k(\varphi)=\varphi^2$ will be considered and the principle of Lagrange multipliers for problem (11) will be justified.

Let's introduce a Lagrange multiplier $(\lambda_0,\theta)\in\mathbb{R}\times H_0^1(\Omega)$ and Lagrangian $L: H_0^1(\Omega)\times\mathbf{V}\times\mathbb{R}\times H_0^1(\Omega)\to\mathbb{R}$ by formula

$$\mathcal{L}(\varphi,\mathbf{u},\lambda_0,\theta)=\lambda_0 J(\varphi,\mathbf{u})+\langle\theta,F(\varphi,\mathbf{u})\rangle\equiv\lambda_0 J(\varphi,\mathbf{u})+\langle F(\varphi,\mathbf{u}),\theta\rangle. \tag{17}$$

A common analysis shows that Frechet derivative of operator F with respect to φ in (10) for $k=\varphi^2$ in the point $(\hat\varphi,\hat{\mathbf{u}})\in H_0^1(\Omega)\times\mathbf{V}$ is a linear continuous operator

$F'_\varphi(\hat{\varphi}, \hat{\mathbf{u}}) : H_0^1(\Omega) \to H^{-1}(\Omega)$, which assigns to each element $\tau \in H_0^1(\Omega)$ an element $\hat{l} \in H^{-1}(\Omega)$, where

$$\langle \hat{l}, S \rangle = \lambda(\nabla\tau, \nabla S) + 3(\hat{\varphi}^2\tau, S) + (\mathbf{u}\cdot\nabla\tau, S).$$

From lemma 1 follows that operator $F'_\varphi(\hat{\varphi}, \hat{\mathbf{u}})$ is an isomorphism. Then according to[13, 14] the theorem takes place:

Theorem 4. *While conditions (i), (ii) and (j), (jj) hold, let* $(\hat{\varphi}, \hat{\mathbf{u}}) \in H_0^1(\Omega) \times \mathbf{V}$ *be an element, on which the local minimum is achieved in problem (11) if* $k = \varphi^2$. *Then there is a unique nonzero Lagrange multiplier* $(1, \theta)$, *where* $\theta \in H_0^1(\Omega)$, *such as Euler–Lagrange equation is satisfied*

$$\langle J'_\varphi(\hat{\varphi}, \hat{\mathbf{u}}), \tau\rangle + \langle F'_\varphi(\hat{\varphi}, \hat{\mathbf{u}})\tau, \theta\rangle = 0 \ \ \forall \tau \in H_0^1(\Omega), \tag{18}$$

and is equivalent to

$$\lambda(\nabla\tau, \nabla\theta) + 3(\hat{\varphi}^2\tau, \theta) + (\mathbf{u}\cdot\nabla\tau, \theta) = -\mu_0(\hat{\varphi} - \varphi_d, \tau)_Q \ \ \forall \tau \in H_0^1(\Omega), \tag{19}$$

and also the minimum principle is true:

$$\langle \mathcal{L}'_{\mathbf{u}}(\hat{\varphi}, \hat{\mathbf{u}}, 1, \theta), \mathbf{u} - \hat{\mathbf{u}}\rangle \geq 0 \ \ \forall\, \mathbf{u} \in \mathbf{V},$$

which is equivalent to the inequality

$$\mu_1(\hat{\mathbf{u}}, \mathbf{u} - \hat{\mathbf{u}})_1 + ((\mathbf{u} - \hat{\mathbf{u}})\cdot\nabla\hat{\varphi}, \theta) \geq 0 \ \forall \mathbf{u} \in \mathbf{V}. \tag{20}$$

The relation (19) together with the variational inequality (20) and operational restriction (10), which is equivalent to the ratio (6), are the optimality system for problem (11) when $k = \varphi^2$.

3. Optimal control problem's solution's uniqueness

In this section sufficient conditions for uniqueness of problem (11)'s solutions (11) when $k = \varphi^2$ will be obtained. Let $(\varphi_i, \mathbf{u}_i) \in H_0^1(\Omega) \times \mathbf{V}$ be solutions of problem (11) and $(1, \theta_i)$, $i = 1, 2$ be nontrivial Lagrange multipliers, corresponding to the mentioned solutions.

Assuming $\varphi = \varphi_1 - \varphi_2$, $\theta = \theta_1 - \theta_2$, $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, let's subtract the equation (6), written for $(\varphi_2, \mathbf{u}_2)$, from the equation (6) for $(\varphi_1, \mathbf{u}_1)$. Taking into consideration that $(\mathbf{u}_1\cdot\nabla\varphi_1, S) - (\mathbf{u}_2\cdot\nabla\varphi_2, S) = (\mathbf{u}_1\cdot\nabla\varphi, S) + (\mathbf{u}\cdot\nabla\varphi_2, S)$, the relation can be got:

$$\lambda(\nabla\varphi, \nabla S) + (\varphi_1^3 - \varphi_2^3, S) + (\mathbf{u}_1\cdot\nabla\varphi, S) = -(\mathbf{u}\cdot\nabla\varphi_2, S) \ \ \forall S \in H_0^1(\Omega). \tag{21}$$

It's obvious that $\varphi_1^3 - \varphi_2^3 = (\varphi_1 - \varphi_2)(\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2)$ and $\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2 \geq 0$ a.e. in Ω. Supposing $S = \varphi$ in (21), the following estimate is deduced for φ subject to (5):

$$\|\varphi\|_1 \leq (1/\lambda_*)\gamma C_4 M_\varphi \|\mathbf{u}\|_1. \tag{22}$$

After writing (19) for $\theta = \theta_i$, $\hat{\varphi} = \varphi_i$, $\mathbf{u} = \mathbf{u}_i$, $i = 1, 2$, let's assume $\tau = \theta_i \in H_0^1(\Omega)$ there. This will be obtained:

$$\lambda(\nabla\theta_i, \nabla\theta_i) + 3(\hat{\varphi}_i^2\theta_i, \theta_i) + (\mathbf{u}_i\cdot\nabla\theta_i, \theta_i) = -\mu_0(\varphi_i - \varphi_d, \theta_i)_Q, \ i = 1, 2. \tag{23}$$

From (23) the following estimate for θ_i can be got:

$$\|\theta_i\|_1 \leq (\mu_0/\lambda_*)\|\varphi_i - \varphi_d\|_Q \leq \mu_0(1/\lambda_*)(M_\varphi + \|\varphi_d\|_Q), \ \ i = 1, 2. \tag{24}$$

Let's then write the equation (19) for $(\varphi_2, \mathbf{u}_2, \theta_2)$:

$$\lambda(\nabla\tau, \nabla\theta_2) + 3(\varphi_2^2\tau, \theta_2) + (\mathbf{u}_2\cdot\nabla\tau, \theta_2) = -\mu_0(\hat{\varphi}_2 - \varphi_d, \tau)_Q \ \ \forall \tau \in H_0^1(\Omega),$$

and subtract it from (19) for $(\varphi_1, \mathbf{u}_1, \theta_1)$:

$$\lambda(\nabla\tau, \nabla\theta_1) + 3(\varphi_1^2\tau, \theta_1) + (\mathbf{u}_1 \cdot \nabla\tau, \theta_1) = -\mu_0(\varphi_1 - \varphi_d, \tau)_Q \ \ \forall\tau \in H_0^1(\Omega).$$

Taking into account the following

$$(\varphi_1^2\tau, \theta_1) - (\varphi_2^2\tau, \theta_2) = (\varphi_1^2\tau, \theta_1) - (\varphi_2^2\tau, \theta_1) + (\varphi_2^2\tau, \theta_1) - (\varphi_2^2\tau, \theta_2) =$$
$$= (\varphi_1^2 - \varphi_2^2, \tau\theta_1) + (\varphi_2^2\tau, \theta),$$

$$(\mathbf{u}_1 \cdot \nabla\tau, \theta_1) - (\mathbf{u}_2 \cdot \nabla\tau, \theta_2) - (\mathbf{u}_2 \cdot \nabla\tau, \theta_1) + (\mathbf{u}_2 \cdot \nabla\tau, \theta_1) = (\mathbf{u} \cdot \nabla\tau, \theta_1) + (\mathbf{u}_2 \cdot \nabla\tau, \theta),$$

it can be obtained that

$$(25)\quad \lambda(\nabla\theta, \nabla\tau) + 3(\varphi_1^2 - \varphi_2^2, \tau\theta_1) + 3(\varphi_2^2\tau, \theta) + (\mathbf{u}_2 \cdot \nabla\tau, \theta) = -\mu_0(\varphi, \tau)_Q - (\mathbf{u} \cdot \nabla\tau, \theta_1).$$

Considering $\tau = \theta$ in (25) :

$$\lambda(\nabla\theta, \nabla\theta) + 3(\varphi_2^2\theta, \theta) = -\mu_0(\varphi, \theta)_Q - 3(\varphi_1^2 - \varphi_2^2, \theta\theta_1) - (\mathbf{u} \cdot \nabla\theta, \theta_1).$$

From it, with the help of (24) and (22), the estimate can be obtained

$$\|\theta\|_1 \le \mu_0(1/\lambda_*)\|\varphi\|_Q + 6M_\varphi(1/\lambda_*)\|\theta_i\|_1\|\varphi\|_1 + (1/\lambda_*)\gamma C_4\|\mathbf{u}\|_1\|\varphi\|_1\|\theta_i\|_1$$
$$\le \mu_0(1/\lambda_*^2)\gamma C_4 M_\varphi\|\mathbf{u}\|_1 + 6\mu_0(1/\lambda_*^3)\gamma C_4 M_\varphi^2(M_\varphi + \|\varphi_d\|_Q)\|\mathbf{u}\|_1$$
$$+\mu_0(1/\lambda_*^2)\gamma C_4(M_\varphi + \|\varphi_d\|_Q)\|\mathbf{u}\|_1 \le$$
$$(26)\qquad \le \mu_0(1/\lambda_*^3)\gamma C_4[\lambda_* M_\varphi + (6M_\varphi^2 + \lambda_*)(M_\varphi + \|\varphi_d\|_Q)]\|\mathbf{u}\|_1.$$

Supposing $\tau = \varphi$ in (25), the ratio is gained

$$(27)$$
$$\lambda(\nabla\theta, \nabla\varphi) + 3(\varphi_1^2 - \varphi_2^2, \varphi\theta_1) + 3(\varphi_2^2\varphi, \theta) + (\mathbf{u}_2 \cdot \nabla\varphi, \theta) = -\mu_0(\varphi, \varphi)_Q - (\mathbf{u} \cdot \nabla\varphi, \theta_1).$$

Further, taking $S = \theta$ in (21), the following is obtained

$$(28)\qquad \lambda(\nabla\varphi, \nabla\theta) + (\varphi_1^3 - \varphi_2^3, \theta) + (\mathbf{u}_1 \cdot \nabla\varphi, \theta) = -(\mathbf{u} \cdot \nabla\varphi_2, \theta).$$

Subtracting (28) from (27), it can be got

$$3(\varphi_1^2 - \varphi_2^2, \varphi\theta_1) + 3(\varphi_2^2\varphi, \theta) - (\varphi_1^3 - \varphi_2^3, \theta) + \mu_0(\varphi, \varphi)_Q + (\mathbf{u}_2 \cdot \nabla\varphi, \theta) - (\mathbf{u}_1 \cdot \nabla\varphi, \theta)$$
$$= -(\mathbf{u} \cdot \nabla\varphi, \theta_1) + (\mathbf{u} \cdot \nabla\varphi_2, \theta)$$

or

$$3((\varphi_1 + \varphi_2)\theta_1, \varphi^2) + 3(\varphi_2^2, \varphi\theta) - (\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2, \varphi\theta) + \mu_0\|\varphi\|_Q^2 - (\mathbf{u} \cdot \nabla\varphi, \theta)$$
$$= -(\mathbf{u} \cdot \nabla\varphi, \theta_1) + (\mathbf{u} \cdot \nabla\varphi_2, \theta).$$

The last relation can be rewritten in the following form:

$$3((\varphi_1 + \varphi_2)\theta_1, \varphi^2) - (\varphi_1^2 + \varphi_1\varphi_2 - 2\varphi_2^2, \varphi\theta) + \mu_0\|\varphi\|_Q^2 - (\mathbf{u} \cdot \nabla\varphi, \theta)$$
$$(29)\qquad = -(\mathbf{u} \cdot \nabla\varphi, \theta_1) + (\mathbf{u} \cdot \nabla\varphi_2, \theta).$$

Let's assume $\mathbf{u} = \mathbf{u}_1$ in the inequality (20), written for $\mathbf{u}_2$, and $\mathbf{u} = \mathbf{u}_2$ in (20), written for $\mathbf{u}_1$. These inequalities results:

$$\mu_1(\mathbf{u}_2, \mathbf{u})_1 + (\mathbf{u} \cdot \nabla\varphi_2, \theta_2) \ge 0, \ \ -\mu_1(\mathbf{u}_1, \mathbf{u})_1 - (\mathbf{u} \cdot \nabla\varphi_1, \theta_1) \ge 0.$$

Adding them, taking into account that $(\mathbf{u} \cdot \nabla\varphi_2, \theta_2) - (\mathbf{u} \cdot \nabla\varphi_1, \theta_1) + (\mathbf{u} \cdot \nabla\varphi_2, \theta_1) - (\mathbf{u} \cdot \nabla\varphi_2, \theta_1) = -(\mathbf{u} \cdot \nabla\varphi, \theta_1) - (\mathbf{u} \cdot \nabla\varphi_2, \theta)$, coming to the ratio

$$(30)\qquad -(\mathbf{u} \cdot \nabla\varphi, \theta_1) - (\mathbf{u} \cdot \nabla\varphi_2, \theta) \ge \mu_1\|\mathbf{u}\|_1^2.$$

Let's tranform the summands in (29) in the following way:

$$-(\mathbf{u} \cdot \nabla\varphi, \theta) + (\mathbf{u} \cdot \nabla\varphi, \theta_1) - (\mathbf{u} \cdot \nabla\varphi_2, \theta)$$

$$= -(\mathbf{u}\cdot\nabla\varphi,\theta) + (\mathbf{u}\cdot\nabla\varphi_1,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta)$$
$$= -(\mathbf{u}\cdot\nabla\varphi,\theta) + (\mathbf{u}\cdot\nabla\varphi_1,\theta) + (\mathbf{u}\cdot\nabla\varphi_1,\theta_2) - (\mathbf{u}\cdot\nabla\varphi_2,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta)$$
$$= -(\mathbf{u}\cdot\nabla\varphi,\theta_1) + (\mathbf{u}\cdot\nabla\varphi,\theta_2) + (\mathbf{u}\cdot\nabla\varphi_1,\theta) + (\mathbf{u}\cdot\nabla\varphi_1,\theta_2) - (\mathbf{u}\cdot\nabla\varphi_2,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta)$$
$$= -(\mathbf{u}\cdot\nabla\varphi,\theta_1) + (\mathbf{u}\cdot\nabla\varphi,\theta_2) + (\mathbf{u}\cdot\nabla\varphi_1,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta)$$
$$= -(\mathbf{u}\cdot\nabla\varphi,\theta_1) + (\mathbf{u}\cdot\nabla\varphi,\theta_2) + (\mathbf{u}\cdot\nabla\varphi,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta)$$
$$= -(\mathbf{u}\cdot\nabla\varphi,\theta_1) - (\mathbf{u}\cdot\nabla\varphi_2,\theta) + (\mathbf{u}\cdot\nabla\varphi,\theta_1+\theta_2).$$

There is the resulting relation:

$$3((\varphi_1+\varphi_2)\theta_1,\varphi^2) - (\varphi_1^2+\varphi_1\varphi_2-2\varphi_2^2,\varphi\theta) + \mu_0\|\varphi\|_Q^2 + (\mathbf{u}\cdot\nabla\varphi,\theta_1+\theta_2)$$
$$= (\mathbf{u}\cdot\nabla\varphi,\theta_1) + (\mathbf{u}\cdot\nabla\varphi_2,\theta). \tag{31}$$

Then from (31) with (30) the inequality follows

$$3((\varphi_1+\varphi_2)\theta_1,\varphi^2) - (\varphi_1^2+\varphi_1\varphi_2-2\varphi_2^2,\varphi\theta) + (\mathbf{u}\cdot\nabla\varphi,\theta_1+\theta_2)$$
$$+\mu_0\|\varphi\|_Q^2 + \mu_1\|\mathbf{u}\|_1^2 \le 0. \tag{32}$$

Let's estimate the first two summands in the left part of (32), using (22) and (24):

$$3|((\varphi_1+\varphi_2)\theta_1,\varphi^2)| \le 3(\|\varphi_1\|_{L^6(\Omega)} + \|\varphi_2\|_{L^6(\Omega)})\|\theta_1\|_{L^6(\Omega)}\|\varphi\|^2_{L^6(\Omega)}$$
$$\le 6M_\varphi(\mu_0/\lambda_*)(M_\varphi+\|\varphi_d\|_Q)\|\varphi\|_1^2 \le 6\mu_0(1/\lambda_*^3)\gamma^2C_4^2M_\varphi^3(M_\varphi+\|\varphi_d\|_Q)\|\mathbf{u}\|_1^2. \tag{33}$$

With the help of (22) and (26), let's estimate $(\varphi_1^2+\varphi_1\varphi_2-2\varphi_2^2,\varphi\theta)$ further:

$$|(\varphi_1^2+\varphi_1\varphi_2-2\varphi_2^2,\varphi\theta)| \le 4M_\varphi^2\|\varphi\|_1\|\theta\|_1$$
$$\le 4\mu_0(1/\lambda_*^4)\gamma^2C_4^2M_\varphi^3[\lambda_*M_\varphi + (6M_\varphi^2+\lambda_*)(M_\varphi+\|\varphi_d\|_Q)]\|\mathbf{u}\|_1^2. \tag{34}$$

According to (22) and (24) the estimate for $(\mathbf{u}\cdot\nabla\varphi,\theta_1+\theta_2)$ is obtained:

$$|(\mathbf{u}\cdot\nabla\varphi,\theta_1+\theta_2)| \le 2\gamma C_4\|\mathbf{u}\|_1\|\varphi\|_1\|\theta_i\|_1 \le 2\mu_0(1/\lambda_*^2)\gamma^2C_4^2M_\varphi(M_\varphi+\|\varphi_d\|_Q)\|\mathbf{u}\|_1^2. \tag{35}$$

Let the inequality hold

$$2(1/\lambda_*^4)\gamma^2C_4^2M_\varphi(2M_\varphi^2[\lambda_*M_\varphi+(6M_\varphi^2+\lambda_*)(M_\varphi+\|\varphi_d\|_Q)]+$$
$$+(3\lambda_*M_\varphi^2+\lambda^2)(M_\varphi+\|\varphi_d\|_Q)) \le \mu_1/\mu_0. \tag{36}$$

Here λ_* and M_φ are introduced in (5) and (7) correspondingly, μ_i, $i=0,1$ are positive parameters from (11).

If conditions (36) from (32) and (22) are satisfied, then $\mathbf{u}=\mathbf{0}$ and $\varphi=0$.

Let's formulate the recieved result as

Theorem 5. *Let's further conditions (i)–(iii), (j) and (36) hold. Then the optimal control problem (11) has a unique solution* $(\varphi,\mathbf{u}) \in H_0^1(\Omega)\times K$.

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