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EMBEDDING CENTRAL EXTENSIONS OF SIMPLE LINEAR GROUPS INTO WREATH PRODUCTS

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Abstract. We find a criterion for the embedding of a nonsplit central extension of $\mathrm{PSL}_n(q)$ with kernel of prime order into the permutation wreath product that corresponds to the action on the projective space.

Keywords: finite simple groups, permutation module, central cover, group cohomology.

1. Introduction

A group B included in a short exact sequence of groups

$$1 \to A \xrightarrow{\iota} B \to C \to 1$$

is called an extension of A by C and denoted by $A.C$ in general, or by $A \rtimes C$ if it is split. The extension B is *central* if $\iota(A) \leqslant \mathrm{Z}(B)$.

We write $\mathbb{Z}_n$ for a cyclic group of order n and (m,n) for $\gcd(m,n)$.

This note concerns the following

Problem 1. *Let $G = \mathrm{PSL}_n(q)$, where q is a prime power, and let r be a prime divisor of $(n, q-1)$. Does the permutation wreath product $\mathbb{Z}_r \wr_\rho G$ contain a subgroup isomorphic to the nonsplit central extension $\mathbb{Z}_r.G$, where ρ is the natural permutation representation of G on the points of the projective space $\mathbb{P}^{n-1}(q)$?*

We remark that the nonsplit extension $\mathbb{Z}_r.G$ mentioned in Problem 1 is unique up to isomorphism and is a quotient of $\mathrm{SL}_n(q)$. This problem is a generalization of the one raised in [1, p. 67], where the case $n = r$ is considered. The case $n = 2$ was

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studied in [2], where it was shown that the embedding holds if and only if $q \equiv -1 \mod 4$. We generalize this result by proving the following

Theorem 1. *In the notation of Problem 1, the nonsplit central extension $\mathbb{Z}_r.G$ is embedded into $\mathbb{Z}_r \wr_\rho G$ if and only if r does not divide $(q-1)/(n, q-1)$.*

The main method that we use is based on some cohomological considerations and is similar to that of [2].

2. Preliminaries

Let G be a group and let L, M be right G-modules. Suppose

$$0 \to L \to M \tag{1}$$

and

$$1 \to M \to E \xrightarrow{\pi} G \to 1$$

are exact sequences of modules and groups, where the conjugation action of E on M agrees with the G-module structure, i. e. $m^e = m \cdot \pi(e)$ for all $m \in M$ and $e \in E$, and we identify M with its image in E. A subgroup $S \leqslant E$ such that

$$S \cap M = L, \qquad SM = E, \tag{2}$$

where we also identify L with its image in M, which is itself an extension of L by G, will be called a *subextension* of E that corresponds to the embedding (1).

It is known [3] that the equivalence classes of extensions of L by G are in a one-to-one correspondence with (thus are *defined by*) the elements of the second cohomology group $H^2(G, L)$. Furthermore, the sequence (1) gives rise to a homomorphism

$$H^2(G, L) \xrightarrow{\varphi} H^2(G, M). \tag{3}$$

Lemma 2. [2, Lemma 2] *Let L, M be G-modules and E an extension as specified above. Let $\bar{\gamma} \in H^2(G, M)$ be the element that defines E. Then the set of elements of $H^2(G, L)$ that define the subextensions S of E corresponding to the embedding* (1) *coincides with $\varphi^{-1}(\bar{\gamma})$, where φ is the induced homomorphism* (3). *In particular, E has such a subextension S if and only if $\bar{\gamma} \in \operatorname{Im} \varphi$.*

We now present a slight generalization of the argument in [2, Section 7].

Denote by $\mathbb{F}_q$ a finite field of order q. The order of a group element g will be denoted by $|g|$.

Let G be a finite group, X a set, and let ρ be a permutation representation of G on X. For a prime p, we consider the permutation $\mathbb{F}_p G$-module V that corresponds to ρ with basis (identified with) X and its trivial submodule I spanned by $\sum_{x \in X} x$. Clearly, the wreath product $\mathbb{Z}_p \wr_\rho G$ is the natural split extension $V \rtimes G$.

Lemma 3. *In the above notation, if a central extension $S = \mathbb{Z}_p.G$ is a subextension of $V \rtimes G$ that corresponds to the embedding of $\mathbb{F}_p G$-modules $I \to V$ then S has no element s that satisfies the following three conditions*

(*i*) $|s| = p^2$,

(*ii*) $|g| = p$, *where $g \in G$ is the image of s under the natural epimorphism $S \to G$.*

(*iii*) $\rho(g)$ *has a fixed point on X.*

Proof. Assume to the contrary that s is such an element. Denote $t = \sum_{x\in X} x \in I$. Since $S = I.G$, we have $s^p = ct$ for a nonzero $c \in \mathbb{F}_p$, and since S is a subextension of $V \rtimes G$ with respect to $I \to V$, there exists $v \in V$ such that $s = gv$. Therefore, we have $ct = s^p = (gv)^p = vh$, where

$$h = 1 + g + \ldots + g^{p-1}.$$

Let $x \in X$ be a fixed point of $\rho(g)$. We can write $v = a_x x + w$, for some $a_x \in \mathbb{F}_p$, where $w = \sum_{y\in X\setminus\{x\}} a_y y$. Clearly, wh is a linear combination of elements of $X \setminus \{x\}$, and

$$(a_x x)h = a_x(x + \ldots + x) = a_x\, px = 0.$$

Hence, the coefficient of x in $ct = vh$ is zero, which contradicts $c \neq 0$. □

3. A permutation module for $\mathrm{PSL}_n(q)$

We henceforth denote $G = \mathrm{PSL}_n(q)$ and fix a prime divisor r of $(n, q-1)$. The natural permutation action ρ of G on the points of the projective space $\mathcal{P} = \mathbb{P}^{n-1}(q)$ gives rise to a permutation $\mathbb{F}_r G$-module V. As every permutation module, V has a trivial submodule I spanned by $\sum_{x\in\mathcal{P}} x$, and the augmentation submodule V_0 that consists of the elements $\sum_{x\in\mathcal{P}} a_x x$ with $\sum_x a_x = 0$. Since $\dim V = 1 + q + \ldots + q^{n-1} \equiv 0 \pmod r$, we have $I \leqslant V_0$, and the quotient $U = V_0/I$ is known [4] to be absolutely irreducible whenever $n \geqslant 3$. It was noticed by various authors [5, 6] that U is one of the few known examples of modules with 2-dimensional 1-cohomology, namely we have:

Lemma 4. *In the above notation, $H^1(G, U) \cong \mathbb{Z}_r^2$, whenever $n \geq 3$.*

We will also require the 1-cohomology of V.

Lemma 5. *Let V be the above-defined permutation module. Then we have*

$$H^1(G, V) \cong \begin{cases} \mathbb{Z}_r, & r \text{ divides } (q-1)/(n, q-1), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We will assume that $n \geqslant 3$, as the claim holds for $n = 2$ by [2, Lemma 12]. Since the action of G on $\mathcal{P}$ is transitive, we have $V \cong T^G$, where T is the principal $\mathbb{F}_r H$-module for a point stabilizer H. By Shapiro's lemma [7, §6.3], we have $H^1(G, V) \cong H^1(H, T) \cong \mathrm{Hom}(H/H', T)$. The structure of H is known [8, Section 2] and has the shape $\mathrm{ASL}_{n-1}(q).\mathbb{Z}_{(q-1)/(n,q-1)}$. Since $n \geq 3$ and $(n, q-1) > 1$, the group $\mathrm{ASL}_{n-1}(q)$ is perfect and so $H/H' \cong \mathbb{Z}_{(q-1)/(n,q-1)}$. As T is cyclic of order r, the claim follows. □

4. Proof

We can now prove Theorem 1. Due to [2], we may assume that $n \geqslant 3$. We denote by S the nonsplit central extension $\mathbb{Z}_r.G$. Since G is simple, the only possibility for S to be a subgroup of the extension $V \rtimes G$ is if S is its subextension, and since I is the unique trivial submodule of V, this subextension must be with respect to the embedding $I \to V$. Being split, the extension $V \rtimes G$ is defined by the zero element of $H^2(G, V)$. Hence, Lemma 2 implies that all subextensions of $V \rtimes G$ with respect to $I \to V$ are defined by the elements of $\mathrm{Ker}\,\varphi$, where φ is the induced homomorphism

$$H^2(G, I) \xrightarrow{\varphi} H^2(G, V). \tag{4}$$

The short exact sequence of modules

$$0 \to I \to V \to V^0 \to 0,$$

where $V^0 \cong V/I$, gives rise to the long exact sequence

$$H^1(G,I) \to H^1(G,V) \stackrel{\alpha}{\to} H^1(G,V^0) \stackrel{\delta}{\to} H^2(G,I) \stackrel{\varphi}{\to} H^2(G,V), \tag{5}$$

which implies that $\operatorname{Ker}\varphi = \operatorname{Im}\delta$. Observe that $H^1(G,I) \cong \operatorname{Hom}(G/G',I) = 0$, since G is simple. Therefore, the map α in (5) is an embedding, and $\operatorname{Ker}\varphi \cong H^1(G,V^0)/H^1(G,V)$.

The structure of V, see [4, Lemma 2], allows us to include V^0 in the nonsplit short exact sequence

$$0 \to U \to V^0 \to I \to 0,$$

which gives rise to the exact sequence

$$H^0(G,V^0) \to H^0(G,I) \to H^1(G,U) \to H^1(G,V^0) \to H^1(G,I). \tag{6}$$

Now, $H^0(G,V^0) = 0$, since V^0 has no trivial submodules, and $H^1(G,I) = 0$ as above. Therefore, $H^1(G,V^0) \cong H^1(G,U)/H^0(G,I)$. Since $H^0(G,I) \cong \mathbb{Z}_r$, Lemma 4 implies $H^1(G,V^0) \cong \mathbb{Z}_r$, and so $\operatorname{Ker}\varphi$ is 0 or $\mathbb{Z}_r$ according as r divides $(q-1)/(n,q-1)$ or otherwise, by Lemma 5. It follows that the nonzero element of $H^2(G,I)$ that defines the nonsplit extension S lies in $\operatorname{Ker}\varphi$ if and only if r does not divide $(q-1)/(n,q-1)$. By Lemma 2, this completes the proof of the theorem. □

Remark. In the case when r divides $(q-1)/(n,q-1)$, we can also prove the nonembedding of S into $V \rtimes G$ in a different way. Suppose this is the case. Then $\mathbb{F}_q$ has an element a of multiplicative order $r(n,q-1)$. Let s be the image in S of $\operatorname{diag}(a,a,\dots,a,a^{1-n})$ under the epimorphism $\mathrm{SL}_n(q) \to S$. We have $|s| = r^2$ and $|g| = r$, where g is the image of s under the epimorphism $S \to G$. Observe that $\rho(g)$ has a fixed point on $\mathcal{P}$, because every diagonal element of $\mathrm{SL}_n(q)$ fixes a point on $\mathcal{P}$, e. g. the projective image of the basis vector $(1,0,\dots,0)$. Therefore, S cannot be a subextension of $V \rtimes G$ by Lemma 3.

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