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DOMAIN DECOMPOSITION METHOD FOR A MEMBRANE WITH A DELAMINATED THIN RIGID INCLUSION

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Abstract. The paper deals with the numerical solution of an equilibrium problem for an elastic membrane with a thin rigid inclusion. The thin inclusion is supposed to delaminate, therefore a crack between the inclusion and the membrane is considered. The boundary conditions for nonpenetration of the crack faces are fulfilled. We provide the relaxation of the problem and propose an iterative method for the numerical solution of the approximated problem. The method is based on a domain decomposition and the Uzawa algorithm for finding a saddle point of the Lagrangian. Examples of the numerical solution of the initial problem are presented.

Keywords: crack, thin rigid inclusion, nonpenetration condition, variational inequality, domain decomposition method, Uzawa algorithm.

1. Introduction

Rigid inclusions (also called stiffeners or anticracks) are used in solid mechanics to describe fibres embedded in a matrix material. Under compression, rigid inclusions may delaminate from the matrix material, thereby introducing cracks. There are different approaches to model cracks in solids. The classical formulation of the problems with cracks implies linear boundary conditions on the crack faces [7, 15, 33]. It is well known that such models have shortcoming because there can be situations when the crack faces penetrate each other. It is natural to impose boundary conditions that exclude mutual penetration of the crack faces. The book [21]

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and papers [22, 32, 36, 23, 28, 29] contain results for crack models with the nonpenetration conditions for a wide class of constituve laws.

Existence theorems and qualitative properties of solutions of equilibrium problems for elastic bodies with thin and volume rigid inclusions can be found in [24, 25, 40, 26, 27, 42, 43, 44, 48, 46, 47], where some results concerning asymptotics behaviour near the crack tips, shape sensitivity analysis, propagation of cracks are investigated. Practical aspects of using of rigid inclusions in composite materials are analyzed in [8, 5, 6].

Equilibrium problems for membranes with rigid inclusions are considered in [51, 19, 39, 50].

In the present paper, we propose an iterative method of the numerical solution an equilibrium problem for an elastic membrane with a delaminated thin rigid inclusion. We assume that the nonpenetration conditions are imposed on the crack faces. To construct an effective numerical algorithm, we provide the relaxation of the initial problem. Then for the approximated problem, we apply the domain decomposition method [41] and the Uzawa algorithm for finding a saddle point of the Lagrangian [12, 18]. To this end, the initial domain is divided into two subdomains. Two linear problems are solved in each subdomain at every iteration. One of problems describes the equilibrium of a membrane having a rigid inclusion on the external boundary. The presence of the rigid inclusion imposes constraints on a solution. In order to find the solution of such problem, we introduce three auxiliary Dirichlet problems without constraints and solve its. For "gluing" the solutions of both linear problems and providing the nonpenetration conditions, we use the Lagrange multipliers. Numerical experiments illustrate the performance of our algorithm.

The domain decomposition technique are widely used for finding approximate solutions to boundary value problems (see, e.g., [34, 2, 35, 30, 38, 13]). We refer the reader to the publications [3, 9, 17, 31, 4, 10, 11, 16] for details concerning different versions of the domain decomposition method for solving of contact problems. Finally, we mention the paper [45], where the domain decomposition technique apply to an equilibrium problem for a membrane having a crack under the nonpenetration conditions.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume that Ω is divided into two subdomains Ω_1 and Ω_2 with Lipschitz boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively, such that $\mathrm{meas}(\partial\Omega \cap \partial\Omega_i) > 0$, $i = 1, 2$; and denote by Σ the interface of Ω_i. Let γ_c be a curve lying on the interface Σ such that $\overline{\gamma}_c \cap \partial\Omega = \varnothing$, and let $\Omega_c = \Omega \setminus \overline{\gamma}_c$. Denote by Σ the rest part of γ_g, by ν_i the outward unit normals to Ω_i, $i = 1, 2$. In particular, we have $\nu_2 = -\nu_1 = \nu$ on Σ, see Figure 1 for an illustration of the geometry.

In what follows, an elastic membrane occupies the domain Ω_c, and γ_c corresponds to a crack in the membrane. In the domain Ω_c, we intend to consider a mixed boundary value problem for the displacement field u. Namely, denote by u_i the restriction of u to Ω_i, $i = 1, 2$. We suppose that the traces u_1 and u_2 on γ_c are, in general, different and satisfying nonlinear boundary conditions that prevent mutual penetration of the crack faces. Furthermore, due to the presence of a thin rigid inclusion the function u_2 is a constant function on γ_c. The equilibrium problem for the elastic membrane having the crack and the thin rigid inclusion reads as follows. For a given external force $f \in L^2(\Omega)$ acting on the membrane, we have to find a displacement u and a real number A such that

$$-\Delta u = f \quad \text{in} \quad \Omega_c, \tag{1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2}$$

$$[u] \geq 0 \quad \text{on} \quad \gamma_c, \tag{3}$$

$$\frac{\partial u_1}{\partial \nu} \geq 0, \quad \frac{\partial u_1}{\partial \nu}[u] = 0 \quad \text{on} \quad \gamma_c, \tag{4}$$

$$u_2 = A \quad \text{on} \quad \gamma_c, \tag{5}$$

$$\int_{\gamma_c} \left[\frac{\partial u}{\partial \nu}\right] ds = 0, \tag{6}$$

where $[u] = u_2 - u_1$ is a jump of the function u on γ_c.

To give a variational or weak formulation of the problem (1)–(6), we introduce the functional space

$$V = \{\, v \in H^1(\Omega_c) \mid v = 0 \text{ on } \partial\Omega \,\}$$

and the set of admissible displacements

$$K_c = \{\, v \in V \mid [v] \geq 0 \text{ on } \gamma_c,\ v_2 \text{ is a constant on } \gamma_c \,\},$$

where $v_2 = v|_{\Omega_2}$. The potential energy of the system is represented by the functional

$$\Pi(v) = \frac{1}{2}\int_{\Omega_c} |\nabla v|^2\, dx - \int_{\Omega_c} f v\, dx.$$

Then the weak formulation of the problem (1)–(6) is the following minimization problem:

$$\text{Find} \quad u \in K_c \quad \text{such that} \quad \Pi(u) = \inf_{v \in K_c} \Pi(v). \tag{7}$$

Note that the functional Π is coercive and weakly lower semicontinuous on the space V. Moreover, the set K_c is weakly closed. Hence, the minimization problem (7) has a unique solution $u \in K_c$ (see, e.g., [12, 21]), which satisfies the variational inequality

$$\int_{\Omega_c} \nabla u(\nabla v - \nabla u)\, dx \geq \int_{\Omega_c} f(v - u)\, dx \quad \forall v \in K_c. \tag{8}$$

The differential formulation of the problem we are interested in is equivalent to the variational formulation (see, e.g., [20, 24, 37, 49]).

3. Domain decomposition

Define the functional spaces

$$V_i = \{\, v_i \in H^1(\Omega_i) \mid \ v_i = 0 \text{ on } \partial\Omega \cap \partial\Omega_i \,\}$$

that we endow with the norms

$$\|v_i\|^2_{V_i} = \int_{\Omega_i} |\nabla v_i|^2\, dx, \quad i = 1, 2,$$

which, due to the Poincaré inequality are equivalent to the usual H^1-norms. Moreover, we introduce the one-dimensional subspace V_2^g of V_2 given by

$$V_2^g = \{\, v_2 \in V_2 \mid v_2 \text{ is a constant on } \gamma_c \,\}$$

and the set $K_{gc} \subset V^1 \times V_g^2$ given by

$$K_{gc} = \{\, (v_1, v_2) \in V_1 \times V_2^g \mid v_2 - v_1 \geq 0 \text{ on } \gamma_c,\ v_2 - v_1 = 0 \text{ on } \gamma_g \,\}.$$

Let us consider now the constrained minimization problem:

(9)
$$\text{Find} \quad (u_1, u_2) \in K_{gc} \quad \text{such that} \quad \Pi_1(u_1) + \Pi_2(u_2) = \inf_{(v_1,v_2)\in K_{gc}} (\Pi_1(v_1) + \Pi_2(v_2)),$$

with the energy functionals Π_i are defined on V_i by the relations

$$\Pi_i(v_i) = \frac{1}{2} \int\limits_{\Omega_i} |\nabla v_i|^2 \, dx - \int\limits_{\Omega_i} f v_i \, dx, \quad i = 1, 2.$$

Theorem 1. *Let u be a solution of the minimization problem* (7)*. Then* (9) *has a unique solution $(u_1, u_2) \in K_{gc}$, which satisfies the conditions*

$$u_i = u|_{\Omega_i}.$$

We omit details here, the essential ideas behind the proof of Theorem 1 are the same as in [45].

4. Regularized problem

For the efficient numerical solution of (7), we propose a domain decomposition method applied to a properly regularized problem.

For any positive number p, let us introduce the sets

$$U_1^p = \{\, v_1 \in V_1 \mid \|v_1\|_{V_1} \leq p \,\}, \quad U_2^p = \{\, v_2 \in V_2^g \mid \|v_2\|_{V_2} \leq p \,\},$$

$$\Lambda_c^p = \{\, \lambda_c \in L_2(\gamma_c) \mid 0 \leq \lambda_c \leq p \text{ on } \gamma_c \,\}, \quad \Lambda_g^p = \{\, \lambda_g \in L_2(\gamma_g) \mid -p \leq \lambda_g \leq p \text{ on } \gamma_g \,\},$$

and the Lagrange function L defined on $U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p$ by

$$L(v_1, v_2, \lambda_c, \lambda_g) = \Pi_1(v_1) + \Pi_2(v_2) + \int\limits_{\gamma_c} \lambda_c (v_1 - v_2) \, ds + \int\limits_{\gamma_g} \lambda_g (v_1 - v_2) \, ds.$$

Consider the family of saddle point problems depending on the parameter p:

$$\text{Find} \quad (u_1^p, u_2^p, \mu_c^p, \mu_g^p) \in U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p \quad \text{such that}$$

(10)
$$L(u_1^p, u_2^p, \lambda_c, \lambda_g) \leq L(u_1^p, u_2^p, \mu_c^p, \mu_g^p) \leq L(v_1, v_2, \mu_c^p, \mu_g^p)$$

$$\text{for all} \quad (v_1, v_2, \lambda_c, \lambda_g) \in U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p.$$

The existence theory of saddle points (see, e.g., [12, 18]) implies that, for every $p > 0$, the problem (10) has a solution $(u_1^p, u_2^p, \mu_c^p, \mu_g^p) \in U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p$.

Theorem 2. *There exists a constant $c > 0$ such that, for all $p > c$, the saddle point $(u_1^p, u_2^p, \mu_c^p, \mu_g^p)$ satisfies the system of variational equalities and inequalities*

$$\int_{\Omega_1} \nabla u_1^p \nabla v_1\, dx + \int_{\gamma_c} \mu_c^p v_1\, ds + \int_{\gamma_g} \mu_g^p v_1\, ds = \int_{\Omega_1} f v_1\, dx \quad \forall v_1 \in V_1, \tag{11}$$

$$\int_{\Omega_2} \nabla u_2^p \nabla v_2\, dx - \int_{\gamma_c} \mu_c^p v_2\, ds - \int_{\gamma_g} \mu_g^p v_2\, ds = \int_{\Omega_2} f v_2\, dx \quad \forall v_2 \in V_2^g, \tag{12}$$

$$\int_{\gamma_c} \lambda_c (u_1^p - u_2^p)\, ds \le \int_{\gamma_c} \mu_c^p (u_1^p - u_2^p)\, ds \quad \forall \lambda_c \in \Lambda_c^p, \tag{13}$$

$$\int_{\gamma_g} \lambda_g (u_1^p - u_2^p)\, ds \le \int_{\gamma_g} \mu_g^p (u_1^p - u_2^p)\, ds \quad \forall \lambda_g \in \Lambda_g^p. \tag{14}$$

Proof. It follows from [12] that (10) is equivalent to the following inequalities:

$$\begin{aligned} &\int_{\Omega_1} \nabla u_1^p (\nabla v_1 - \nabla u_1^p)\, dx + \int_{\Omega_2} \nabla u_2^p (\nabla v_2 - \nabla u_2^p)\, dx \\ &\quad + \int_{\gamma_c} \mu_c^p (v_1 - v_2 - (u_1^p - u_2^p)) ds + \int_{\gamma_g} \mu_g^p (v_1 - v_2 - (u_1^p - u_2^p))\, ds \\ &\qquad \ge \int_{\Omega_1} f (v_1 - u_1^p)\, dx + \int_{\Omega_2} f (v_2 - u_2^p)\, dx,\ \forall (v_1, v_2) \in U_1^p \times U_2^p, \end{aligned} \tag{15}$$

$$\int_{\gamma_c} \lambda_c (u_1^p - u_2^p)\, ds \le \int_{\gamma_c} \mu_c^p (u_1^p - u_2^p)\, ds \quad \forall \lambda_c \in \Lambda_c^p, \tag{16}$$

$$\int_{\gamma_g} \lambda_g (u_1^p - u_2^p)\, ds \le \int_{\gamma_g} \mu_g^p (u_1^p - u_2^p)\, ds \quad \forall \lambda_g \in \Lambda_g^p. \tag{17}$$

Substituting $v_1 = 0$ and $v_2 = 0$ into (15), we obtain

$$\|u_1^p\|_{V_1}^2 + \|u_2^p\|_{V_2}^2 + \int_{\gamma_c} \mu_c^p (u_1^p - u_2^p)\, ds + \int_{\gamma_g} \mu_g^p (u_1^p - u_2^p) ds \le \int_{\Omega_1} f u_1\, dx + \int_{\Omega_2} f u_2\, dx. \tag{18}$$

From (16) and (17) with $\lambda_c = 0$ and $\lambda_g = 0$, it follows that

$$\int_{\gamma_c} \mu_c^p (u_1^p - u_2^p)\, ds \ge 0,$$

$$\int_{\gamma_g} \mu_g^p (u_1^p - u_2^p)\, ds \ge 0.$$

Due to the Poincaré inequality, (18) yields the estimate

$$\|u_i^p\|_{V_i} \le c,\ i = 1, 2, \tag{19}$$

uniformly with respect to $p > 0$.

To prove the theorem, it is sufficient to show that the variational inequality (15) holds for all functions $(v_1, v_2) \in V_1 \times V_2$. Indeed, substituting $(\pm v_1 + u_1^p, u_2^p)$ and $(u_1^p, \pm v_2 + u_2^p)$ as test functions, we obtain (11) and (12).

Choose a positive number $p > c$, where c is the constant from estimate (19). Due to (19), (u_1^p, u_2^p) belongs to $U_1^c \times U_2^c \subset U_1^p \times U_2^p$. Let $\varepsilon \in (0, (p-c)/2)$; then the open set

$$U(u_1^p, u_2^p, \varepsilon) = \{\, (v_1, v_2) \in V_1 \times V_2 \mid \|v_1 - u_1^p\|_{V_1} < \varepsilon,\ \|v_2 - u_2^p\|_{V_2} < \varepsilon \,\}$$

is contained in $U_1^p \times U_2^p$.

Let $(v_1, v_2) \in V_1 \times V_2$ be any function; further, choose a positive number α such that $\alpha \|v_i - u_i^p\|_{V_i} < \varepsilon$, $i = 1, 2$. Then the pair $(u_1^p + \alpha(v_1 - u_1^p), u_2^p + \alpha(v_2 - u_2^p))$ belongs to $U(u_1^p, u_2^p, \varepsilon)$; therefore $(u_1^p + \alpha(v_1 - u_1^p), u_2^p + \alpha(v_2 - u_2^p))$ belongs to $U_1^p \times U_2^p$. Hence, it can be substituted into (15) as a test function. This yields that the variational inequality (15) holds for any test function $(v_1, v_2) \in V_1 \times V_2$. The theorem is proved. $\square$

Remark. Theorem 2 means that the saddle point of the Lagrangian L over the set $U_1^p \times U_2^p \times \Lambda_c^p \times \Lambda_g^p$ coincides with the saddle point of L over the set $V_1 \times V_2 \times \Lambda_c^p \times \Lambda_g^p$ for all $p > c$.

Applying the same arguments as in [45], we can prove the following theorem.

Theorem 3. *The sequence (u_1^p, u_2^p) is such that*

$$(u_1^p, u_2^p) \to (u_1, u_2) \quad \textit{strongly in} \quad V_1 \times V_2$$

as $p \to \infty$.

Thus, the regularized saddle point problem (10) is divided into two linear problem (11) and (12), which are connected with each other by the Lagrange multipliers μ_c^p and μ_g^p.

5. Domain decomposition algorithm I

In this section, we give the Uzawa method for solving (10), based on relations (11)–(14). We assume that $p > c$, where the constant c is defined in (19). Let $P_{\Lambda_c^p}$ and $P_{\Lambda_g^p}$ denote the projector operators onto the sets Λ_c^p and Λ_g^p, respectively. It is easy to see that

$$P_{\Lambda_c^p} v(x) = \begin{cases} 0, & v(x) \le 0, \\ v(x), & 0 < v(x) < p, \\ p, & v(x) \ge p, \end{cases}$$

$$P_{\Lambda_g^p} v(x) = \begin{cases} -p, & v(x) \le -p, \\ v(x), & -p < v(x) < p, \\ p, & v(x) \ge p, \end{cases}$$

The inequalities (13) and (14) are equivalent to the following relations (see [12, 14]):

$$\mu_c^p = P_{\Lambda_c^p}(\mu_c^p + \theta(u_1^p - u_2^p)), \tag{20}$$

$$\mu_c^p = P_{\Lambda_c^p}(\mu_c^p + \theta(u_1^p - u_2^p)) \tag{21}$$

for any $\theta > 0$. Basing on relations (20) and (21), we obtain the following algorithm.

Algorithm 1.

1. Initialization. $\mu_{c,0} \in \Lambda_c^p$ and $\mu_{g,0} \in \Lambda_g^p$ are given.
2. Iteration $k \ge 0$. Compute successively $u_{1,k}$, $u_{2,k}$ as follows:

– find $u_{1,k} \in V_1$ such that

$$\int_{\Omega_1} \nabla u_{1,k} \nabla v_1 dx + \int_{\gamma_c} \mu_{c,k} v_1 ds + \int_{\gamma_g} \mu_{g,k} v_1 ds = \int_{\Omega_1} f v_1 dx \quad \forall v_1 \in V_1.$$

– find $u_{2,k} \in V_2^g$ such that

$$\int_{\Omega_2} \nabla u_{2,k} \nabla v_2 dx - \int_{\gamma_c} \mu_{c,k} v_2 \, ds - \int_{\gamma_g} \mu_{g,k} v_2 ds = \int_{\Omega_2} f v_2 \, dx \quad \forall v_2 \in V_2^g. \tag{22}$$

– update the Lagrange multipliers

$$\mu_{c,k+1} = P_{\Lambda_c^p}(\mu_{c,k} + \theta(u_{1,k} - u_{2,k})),$$
$$\mu_{g,k+1} = P_{\Lambda_g^p}(\mu_{g,k} + \theta(u_{1,k} - u_{2,k})).$$

The function $u_{2,k}$ satisfying the variational equality (22) is a solution of a linear equilibrium problem for an elastic membrane with a rigid inclusion located on the external boundary. Despite the fact that this problem is linear, it is a constraint problem. By this reason, we need to detail Algorithm 1. In the next section, we specify the method of solving of (22).

6. Linear equilibrium problem for a membrane with a thin rigid inclusion

Let ω be a bounded domain with Lipschitz boundary $\partial\omega$, which is divided on two disjoint parts γ_N and γ_D. Let γ be a part of γ_N such that $\overline{\gamma} \cap \overline{\gamma}_D = \emptyset$. Denote by n the outward normal vector to $\partial\omega$.

In the domain ω, let us consider the boundary value problem with nonlocal boundary conditions. For given functions $f \in L_2(\omega)$ and $g \in H^{1/2}(\gamma_N)$, we have to find a function u and a real number A such that

$$-\Delta u = f \quad \text{in} \quad \omega, \tag{23}$$

$$u = 0 \quad \text{on} \quad \gamma_D, \tag{24}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \gamma_N \setminus \overline{\gamma}, \tag{25}$$

$$u = A \quad \text{on} \quad \gamma, \tag{26}$$

$$\int_{\gamma} \frac{\partial u}{\partial n} \, ds = \int_{\gamma} g \, ds. \tag{27}$$

The problem (23)–(27) corresponds to the equilibrium state of an elastic membrane occupying the domain ω and containing a thin rigid inclusion γ on the external boundary $\partial\omega$. The volume force f and the boundary traction g act on the membrane. In particular, the traction g acts on the rigid inclusion γ.

Let us give a weak formulation of (23)–(27). For this purpose, we introduce the set of admissible displacements

$$K = \{ \, v \in H^1_{\gamma_D}(\omega) \mid v \text{ is a constant on } \gamma \, \},$$

where

$$H^1_{\gamma_D}(\omega) = \{ \, v \in H^1(\omega) \mid \; v = 0 \text{ on } \gamma_D \, \}.$$

A function $u \in K$ is a weak solution of (23)–(27), if the variational equality

$$\int\limits_{\omega} \nabla u \nabla v \, dx = \int\limits_{\omega} f v dx + \int\limits_{\gamma_N} g v \, ds \quad \forall v \in K \tag{28}$$

holds.

The space $H^1_{\gamma_D}(\omega)$ is a Hilbert space with the inner product

$$(u, v) = \int\limits_{\omega} \nabla u \nabla v \, dx$$

in virtue of the Poincaré inequality. Let us consider the following problem:

$$\text{Find} \quad u_f \in H^1_{\gamma_D}(\omega) \quad \text{such that} \quad \int\limits_{\omega} \nabla u_f \nabla v \, dx = \int\limits_{\omega} f v \, dx + \int\limits_{\gamma_N} g v \, ds \tag{29}$$

for all $v \in H^1_{\gamma_D}(\omega)$.

There exists a unique solution u_f of the problem (29); therefore we can rewrite the variational equality (28) in the form

$$(u - u_f, v) = 0 \quad \forall v \in K,$$

which means that the function u is an orthogonal projection of u_f onto the set K.

Since K is a subspace of $H^1_{\gamma_D}(\omega)$, the decomposition

$$H^1_{\gamma_D}(\omega) = K \oplus K^{\perp}$$

holds, where $K^{\perp}$ is an orthogonal complement for K in $H^1_{\gamma_D}(\omega)$.

Let $u_f \in H^1_{\gamma_D}(\omega)$ be an arbitrary function. Describe the algorithm that allow us to find the decomposition $u_f = u + w$, where $u \in K$, $w \in K^{\perp}$. For this purpose, we give the description of the subspace $K^{\perp}$. Let $w \in K^{\perp}$ be an arbitrary function; then the equality

$$(w, v) = 0 \quad \forall v \in K \tag{30}$$

holds. Substituting $v \in C_0^{\infty}(\omega)$, we obtain the equation

$$-\Delta w = 0 \quad \text{in} \quad \omega, \tag{31}$$

which are fulfilled in the sense of distributions. Moreover, we have

$$w = 0 \quad \text{on} \quad \gamma_D. \tag{32}$$

Next, substituting a function $v \in K$ in (30) and applying the Green formula, we obtain

$$(w, v) = \int\limits_{\omega} \nabla w \nabla v \, dx = -\int\limits_{\omega} \Delta w v \, dx + \int\limits_{\partial\omega} \frac{\partial w}{\partial n} v \, ds.$$

Taking into account (31), (32) and the fact that v is constant on γ, we have

$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \gamma_N \setminus \overline{\gamma}, \tag{33}$$

$$\int\limits_{\gamma} \frac{\partial w}{\partial n} \, ds = 0. \tag{34}$$

Remark. Note the the condition (34) has the following interpretation:

$$\langle \frac{\partial w}{\partial n}, v\rangle_{1/2,\partial\omega} = 0 \quad \forall v \in H^{1/2}(\partial\omega) \text{ such that } v = \mathit{const} \text{ on } \gamma.$$

Thus, any function w from $K^{\perp}$ satisfies the nonlocal boundary value problem (31)–(34). Obviously, the converse is also true, i.e., any function satisfying (31)–(34) belongs to $K^{\perp}$.

Choose a real number $a \in \mathbb{R}$ and consider the variational problem:

$$\text{(35)} \quad \text{Find} \quad y^a \in H^1_{\gamma_D}(\omega) \quad \text{such that} \quad y^a = u_f|_{\gamma} - a \quad \text{on} \quad \gamma$$
$$\text{and} \quad \int\limits_{\omega} \nabla y^a \nabla v \, dx = 0 \quad \forall v \in H^1_{\gamma_D \cup \gamma}(\omega),$$

where

$$H^1_{\gamma_D \cup \gamma}(\omega) = \{v \in H^1(\omega) \ \mid \ v = 0 \text{ on } \gamma_D \cup \gamma\}.$$

For any a, the problem (35) has a unique solution. Furthermore, y^a is a weak solution to the boundary value problem

$$\begin{aligned} -\Delta y^a &= 0 \quad \text{in} \quad \omega, \\ y^a &= 0 \quad \text{on} \quad \gamma_D, \\ y^a &= u_f|_{\gamma} - a \quad \text{on} \quad \gamma, \\ \frac{\partial y^a}{\partial n} &= 0 \quad \text{on} \quad \gamma_N \setminus \overline{\gamma}. \end{aligned}$$

Denote by

$$h(a) = \int\limits_{\gamma} \frac{\partial y^a}{\partial n} \, ds,$$

and show that there exists $A \in \mathbb{R}$ such that $h(A) = 0$. It means that the function y^A satisfies the nonlocal condition (34). Since the solution y^a of (35) is an affine function of a, we have the decomposition

$$y^a = ay + y^0, \tag{36}$$

where y^0 is the solution of (35) corresponds to $a = 0$ and $y \in H^1_{\gamma_D}(\omega)$ satisfies the variational equality

$$(y, v) = 0 \quad \forall v \in H^1_{\gamma_D \cup \gamma}(\omega), \tag{37}$$

while $y = -1$ on γ. The differential formulation of the problem (37) reads as follows. We have to find a function y such that

$$\begin{aligned} -\Delta y &= 0 \quad \text{in} \quad \omega, \\ y &= 0 \quad \text{on} \quad \gamma_D, \\ y &= -1 \quad \text{on} \quad \gamma, \\ \frac{\partial y}{\partial n} &= 0 \quad \text{on} \quad \gamma_N \setminus \overline{\gamma}. \end{aligned}$$

In turn, the function h is an affine function of a, and the representation

$$h(a) = ka + b$$

takes place with

$$k = \int\limits_{\gamma} \frac{\partial y}{\partial n} \, ds, \quad b = \int\limits_{\gamma} \frac{\partial y^0}{\partial n} \, ds.$$

Due to the Green formula, we have

$$k = -(y, y) < 0, \quad b = -(y^0, y),$$

therefore the function h vanishes when $a = A := -b/k$.

Finally, we obtain that any function $u_f \in H^1_{\gamma_D}(\omega)$ can be decomposed into a sum of two functions u and w, where

$$w = y^A \in K^\perp, \quad u = u_f - y^A \in K$$

and y^A is the solution of (35) with $a = A$.

Remark. Due to (36), we have $y^A = Ay + y^0$. It means that we do not need to solve the problem (35) for $a = A$ again. It suffices to use the functions y and y^0.

7. Domain decomposition algorithm II

With the result of the previous section, we present a domain decomposition method to solve (7) numerically. Let us introduce the space

$$V_{2,\gamma} = \{\, v_2 \in V_2 \mid v_2 = 0 \text{ on } \gamma \,\} = \{\, v_2 \in H^1(\Omega_2) \mid v_2 = 0 \text{ on } \gamma \cup (\partial\Omega \cap \partial\Omega_2)\,\}.$$

Algorithm 2.

1. Initialization. $\mu_{c,0} \in \Lambda_c^p$ and $\mu_{g,0} \in \Lambda_g^p$ are given.
2. Iteration $k \geq 0$. Compute successively $u_{1,k}$, $u_{2,k}$ as follows:

– find $u_{1,k} \in V_1$ such that

$$\int\limits_{\Omega_1} \nabla u_{1,k} \nabla v_1 \, dx + \int\limits_{\gamma_c} \mu_{c,k} v_1 \, ds + \int\limits_{\gamma_g} \mu_{g,k} v_1 \, ds = \int\limits_{\Omega_1} f v_1 \, dx \quad \forall v_1 \in V_1.$$

– find $w_k \in V_2$ such that

$$\int\limits_{\Omega_2} \nabla w_k \nabla v_2 \, dx = \int\limits_{\Omega_2} f v_2 \, dx + \int\limits_{\gamma_c} \mu_{c,k} v_2 \, ds + \int\limits_{\gamma_g} \mu_{g,k} v_2 \, ds, \quad \forall v_2 \in V_2.$$

– find $y_k^0 \in V_2$ such that $y_k^0 = w_k$ on γ and

$$\int\limits_{\Omega_2} \nabla y_k^0 \nabla v_2 \, dx = 0 \quad \forall v_2 \in V_{2,\gamma}.$$

– find $y_k \in V_2$ such that $y_k = -1$ on γ and

$$\int\limits_{\Omega_2} \nabla y_k \nabla v_2 \, dx = 0 \quad \forall v_2 \in V_{2,\gamma}.$$

– compute the auxiliary real numbers A_k, b_k, c_k as follows:

$$b_k = -\int\limits_{\Omega_2} \nabla y_k \nabla y_k \, dx, \quad c_k = -\int\limits_{\Omega_2} \nabla y_k \nabla y_k^0 \, dx, \quad A_k = -c_k/b_k.$$

– set $y^{A_k} = A_k y_k + y_k^0$ and $u_{2,k} = w_k - y^{A_k}$.

– update the Lagrange multipliers

$$\mu_{c,k+1} = P_{\Lambda_c^p}(\mu_{c,k} + \theta(u_{1,k} - u_{2,k})),$$
$$\mu_{g,k+1} = P_{\Lambda_g^p}(\mu_{g,k} + \theta(u_{1,k} - u_{2,k})).$$

3. Iteration continue until the relative error

$$\varepsilon = \max\left(\frac{\|u_{1,k} - u_{1,k-1}\|^2_{V_1}}{\|u_{1,k}\|^2_{V_1}}, \frac{\|u_{2,k} - u_{2,k-1}\|^2_{V_2}}{\|u_{2,k}\|^2_{V_2}}\right)$$

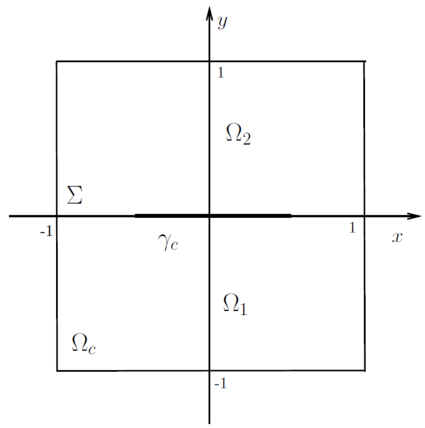

Рис. 1. Domain Ω_c with the delaminated thin rigid inclusion γ_c

becomes "sufficiently" small.

The convergence of Algorithm 1 and, therefore, Algorithm 2, follows from the general convergence theorem for the Uzawa algorithm (see [12, 18]). Moreover, note that the proof of the convergence is similar to that in the article [45].

Theorem 4. *There exists θ^* such that, for all $\theta \in (0, \theta^*)$, the sequence $(u_{1,k}, u_{2,k})$ generated in Algorithm 1 (and, respectively, in Algorithm 2) is such that*

$$(u_{1,k}, u_{2,k}) \to (u_1, u_2) \quad \textit{strongly in} \quad V_1 \times V_2$$

as $k \to \infty$.

8. Numerical experiments

Using piecewise linear finite elements ($P1$-Lagrange elements [1]), Algorithm 2 was implemented in FreeFEM++. The test problems with different external forces are used to illustrate the behavior of our algorithm. Dimensionless units are used, and deformed configurations are plotted with amplification factors.

In all numerical experiments, we set $\mu_{c,0} = 0$ and $\mu_{g,0} = 0$ for the initialization of the algorithm, $\varepsilon = 10^{-12}$ for the stopping criterion; the regularized parameter p equals 10^5 and the relaxation parameter θ equals 2.

We consider a special domain configuration. Let the domain Ω consist of two subdomains $\Omega_1 = (-1, 1) \times (-1, 0)$ and $\Omega_2 = (-1, 1) \times (0, 1)$ with the interface $\Sigma = (-1, 1) \times \{0\}$, while the crack is $\gamma_c = (-0.5, 0.5) \times \{0\}$.

The subdomains Ω_1 and Ω_2 are partitioned into 5700 triangles with 2971 nodes and into 5674 triangles with 2958 nodes, respectively. In particular, 80 nodes are on γ_c and γ_g for both domains. The maximal sizes of the triangles (in a neighborhood of Σ) are 0.077 and 0.078 in Ω_1 and Ω_2, respectively, while the minimal sizes are equal to 0.012 for both domains.

Next, we consider four types of the external force f.

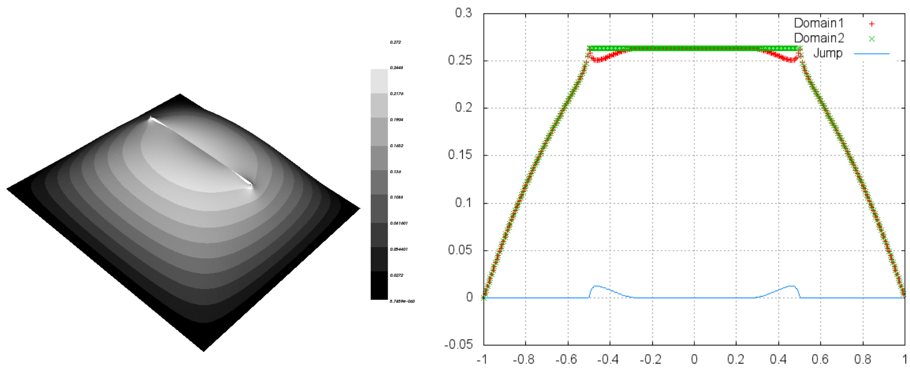

Рис. 2. Example 1. On the left: deformed configuration; On the right: displacements and jump on Σ.

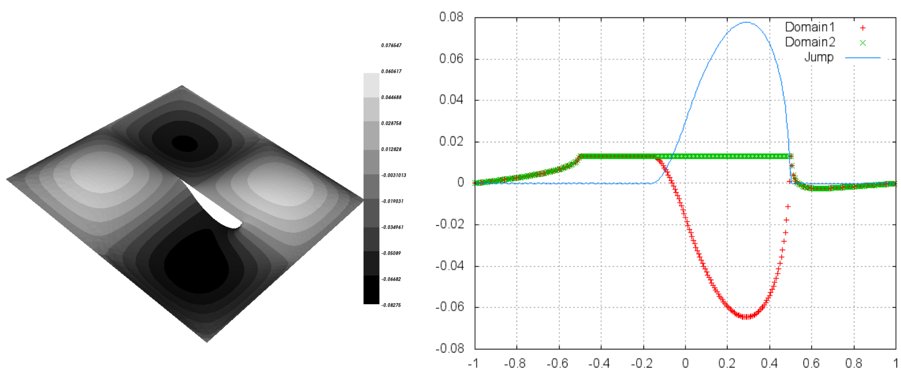

Рис. 3. Example 2. On the left: deformed configuration; On the right: displacements and jump on Σ.

Example 1. Let f is equal to 1 in Ω. The resulting deformations are shown in Figure 2 (left), the displacements along the interface Σ and the jump of the displacements on Σ are plotted in Figure 2 (right).

Example 2. Assume that f is given by

$$f(x_1, x_2) = \begin{cases} -1, & (x_1, x_2) \in \{(-1, 0) \times (0, 1)\} \cup \{(0, 1) \times (-1, 0)\}, \\ 1, & (x_1, x_2) \in \{(-1, 0) \times (-1, 0)\} \cup \{(0, 1) \times (0, 1)\}. \end{cases}$$

In this case, the crack faces contact with each other near the crack tip $(-1, 0)$.

The resulting deformations and plots of displacements and jump of u along the interface Σ are shown in Figure 3.

Example 3. Assume that f is given by

$$f(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in (-1, 1) \times (0, 1), \\ -1, & (x_1, x_2) \in (-1, 1) \times (-1, 0). \end{cases}$$

Such external force provides an opening mode of the crack γ_c.

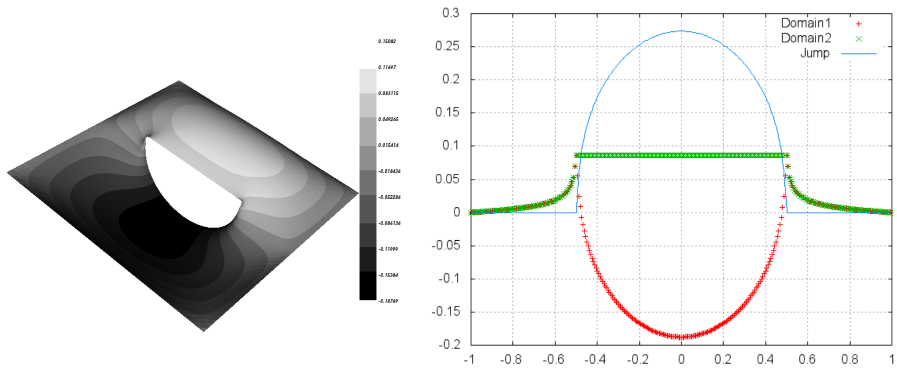

Рис. 4. Example 3. On the left: deformed configuration; On the right: displacements and jump on Σ.

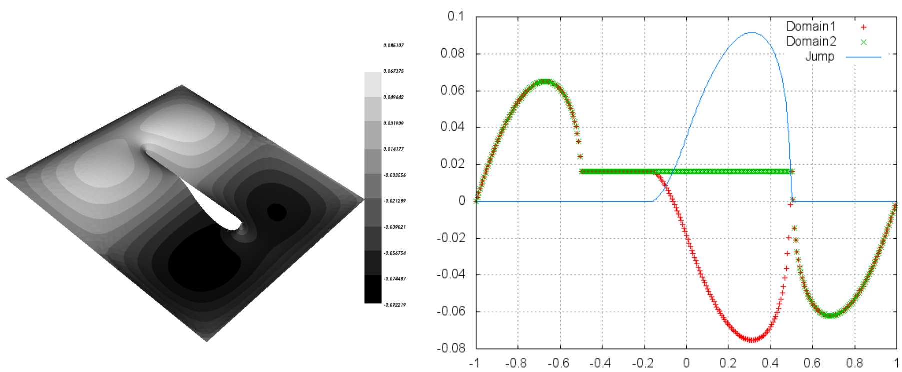

Рис. 5. Example 4. On the left: deformed configuration; On the right: displacements and jump on Σ.

The resulting deformations are shown in Figure 4 (left), the displacements along the interface Σ and the jump of the displacements on Σ are plotted in Figure 4 (right).

Example 4. If f is given by

$$f(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in (-1, 0) \times (-1, 1), \\ -1, & (x_1, x_2) \in (0, 1) \times (-1, 1), \end{cases}$$

then the crack faces contact near the crack tip $(-1, 0)$.

The resulting deformations are shown in Figure 5 (left). Figure 5 (right) shows plots of displacement along the interface Σ and the plot of the jump $[u]$ on Σ.

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