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LIGHT NEIGHBORHOODS OF 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5

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Abstract. In 1940, in attempts to solve the Four Color Problem, Henry Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $\mathbf{P_5}$ of 3-polytopes with minimum degree 5.

Given a 3-polytope P, by $w(P)$ ($h(P)$) we denote the minimum degree-sum (minimum of the maximum degrees) of the neighborhoods of 5-vertices in P.

A 5^*-vertex is a 5-vertex adjacent to four 5-vertices. It is known that if a polytope P in $\mathbf{P_5}$ has a 5^*-vertex, then $h(P)$ can be arbitrarily large.

For each P without vertices of degrees from 6 to 9 and 5^*-vertices in $\mathbf{P_5}$, it follows from Lebesgue's Theorem that $w(P) \leq 44$ and $h(P) \leq 14$.

In this paper, we prove that every such polytope P satisfies $w(P) \leq 42$ and $h(P) \leq 12$, where both bounds are tight.

Keywords: planar map, planar graph, 3-polytope, structural properties, height, weight.

1. Introduction

The degree of a vertex or face x in a convex finite 3-dimensional polytope (called a *3-polytope*) is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. A k^+-*vertex* (k^--*vertex*) is one of degree at least k (at most k). Similar notation is used for the faces. A 3-polytope with minimum degree δ is denoted by P_δ. The *weight* of a subgraph S of a 3-polytope is the sum of degrees of the vertices of S in the 3-polytope. The *height* of a subgraph S of a 3-polytope is the maximum degree of the vertices of S in the 3-polytope. A k-star $S_k(v)$ is *minor* if its center v has

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degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By $w(S_k)$ and $h(S_k)$ we denote the minimum weight and height, respectively, of minor k-stars in a given 3-polytope.

In 1904, Wernicke [18] proved that every P_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [11] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, Lebesgue [17, p.36] gave an approximate description of the neighborhoods of 5-vertices in P_5's. In particular, this description implies the results in [18, 11] and shows that there is a 5-vertex with three 7^--neighbors.

For P_5's, the bounds $w(S_1) \leq 11$ (Wernicke [18]) and $w(S_2) \leq 17$ (Franklin [11]) are tight. It was proved by Lebesgue [17] that $w(S_3) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [14] to the sharp bound $w(S_3) \leq 23$. Furthermore, Jendrol' and Madaras [14] gave a precise description of minor 3-stars in P_5's. Lebesgue [17] proved $w(S_4) \leq 31$, which was strengthened by Borodin and Woodall [3] to the tight bound $w(S_4) \leq 30$. Note that $w(S_3) \leq 23$ easily implies $w(S_2) \leq 17$ and immediately follows from $w(S_4) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, we obtained a precise description of 4-stars in P_5's [7].

For arbitrary 3-polytopes, that is for P_3's, the following results concerning $(d-2)$-stars at d-vertices, $d \leq 5$, are known. Van den Heuvel and McGuinness [13] proved (in particular) that either $w(S_1(v)) \leq 14$ with $d(v) = 3$, or $w(S_2(v)) \leq 22$ with $d(v) = 4$, or $w(S_3(v)) \leq 29$ with $d(v) = 5$. Balogh et al. [1] proved that there is a 5^--vertex adjacent to at most two 11^+-vertices. Harant and Jendrol' [12] strengthened these results by proving (in particular) that either $w(S_1(v)) \leq 13$ with $d(v) = 3$, or $w(S_2(v)) \leq 19$ with $d(v) = 4$, or $w(S_3(v)) \leq 23$ with $d(v) = 5$. Recently, we obtained a precise description of $(d-2)$-stars in P_3's [6].

For P_3's, the problem of describing $(d-1)$-stars at d-vertices, $d \leq 5$, called *pre-complete stars*, appears difficult. As follows from the double n-pyramid, the minimum weight $w(S_{d-1})$ of pre-complete stars in P_4's can be arbitrarily large. Even when $w(S_{d-1})$ is restricted by appropriate conditions, the tight upper bounds on it are unknown. Borodin et al. [4, 5] proved (in particular) that if a planar graph with $\delta \geq 3$ has no edge joining two 4^--vertices, then there is a star $S_{d-1}(v)$ with $w(S_{d-1}(v)) \leq 38 + d(v)$, where $d(v) \leq 5$ (see [5, Theorem 2.A]). Jendrol' and Madaras [15] proved that if the weight $w(S_1)$ of every edge in an P_3 is at least 9, then there is a pre-complete star of height at most 20, where the bound of 20 is best possible.

The more general problem of describing d-stars at d-vertices, $d \leq 5$, called *complete stars*, at the moment seems untractable for arbitrary 3-polytopes and difficult even for P_5's.

Lebesgue [17] proved that if a P_5 has no 5-vertices adjacent to two 5-vertices and two 6^--vertices, then $w(S_5) \leq 68$ and $h(S_5) \leq 41$. Recently, Borodin, Ivanova, and Jensen [9] lowered these bounds to $w(S_5) \leq 55$ and $h(S_5) \leq 28$, and then Borodin and Ivanova [8] to 51 and 23.

A 5^*-vertex is a 5-vertex adjacent to four 5-vertices. Jendrol' and Madaras [14] showed that if a polytope P in $\mathbf{P_5}$ has a 5^*-vertex, then $h(P)$ can be arbitrarily large.

For each P_5 with neither vertices of degrees from 6 to 9 nor 5^*-vertices, it follows from Lebesgue's Theorem that $w(P_5) \le 44$ and $h(P_5) \le 14$. It is known that if 6-vertices are allowed in P_5, then $h(P_5)$ can be arbitrarily large (see [9]), and if only 8-vertices are allowed, then $h(P_5)$ can reach 14 (see [10]).

The purpose of this note is to prove the following fact.

Theorem 1. *Every 3-polytope P with minimum degree 5 having neither vertices of degrees from 6 to 9 nor 5-vertices adjacent to four 5-vertices satisfies $w(P) \le 42$ and $h(P) \le 12$, where both bounds are tight.*

2. Proof of Theorem 1

The tightness of the bounds 42 and 12.

We put a 5-vertex into each 5-face of dodecahedron. This yields a triangulation with only vertices of degrees five and six such that each 5-vertex is surrounded by 6-vertices. The vertex-face dual of this triangulation is a cubic graph in which every 5-face is surrounded by 6-faces. We now replace each 3-vertex by a small 3-face, so as to make each k-face into a $2k$-face. Finally, we put a vertex inside each 10^+-face and join it to the boundary vertices of this face.

In the resulting triangulation, every 5-vertex has a 12-neighbor, a 10-neighbor, and three 5-neighbors, as desired.

Discharging.

It suffices to prove the theorem for plane graphs in which no 4^+-face is incident with two non-consecutive 10^+-vertices, since adding a diagonal between such vertices cannot create a forbidden 5-star, nor can it reduce the weight or height of any existing minor 5-star.

So suppose that a 3-polytope P_5, with its sets of vertices, edges, and faces denoted by V, E, and F, respectively, is a counterexample to the main statement of Theorem 1.

By assumption, each minor 5-star in P_5 either is of weight at least 43 or contains a 13^+-vertex. Also, recall that no 5-vertex has four 5-neighbors.

Euler's formula $|V| - |E| + |F| = 2$ for P_5 implies

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of P_5 as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12.

The *final charge* $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R6 below (see Fig. 1).

We put $\xi(v) = \frac{2}{5}$ if $d(v) = 10$, $\xi(v) = \frac{1}{2}$ if $11 \le d(v) \le 12$, and $\xi(v) = \frac{11}{20}$ if $d(v) \ge 13$. For a vertex v let $v_1, \dots, v_{d(v)}$ be the vertices adjacent to v in a fixed cyclic order. If f is a face, then $v_1, \dots, v_{d(f)}$ are the vertices incident with f in the same cyclic order. A vertex is *simplicial* if it is completely surrounded by 3-faces.

R1. *Each 4^+-face f gives the following charge to a 5-vertex v_2:*
(a) $\frac{1}{2}$ if $d(v_1) = d(v_3) = 5$ or
(b) $\frac{3}{4}$ if $d(v_1) \geq 10$.

R2. *If $f = v_1v_2v_3\dots$ is a face such that $d(v_1) = 5$ and v_2 satisfies $d(v_2) \in \{10, 12, 14^+\}$, then v_2 gives v_1 the following charge through f:*
(a) $\frac{\xi(v)}{2}$ if $d(v_3) = 5$ or
(b) $\xi(v)$ if $d(v_3) \geq 10$.

If v with $d(v) \in \{11, 13\}$ is non-simplicial, then one of its incident 4^+-faces is declared *special for v*, and the other 3^+-faces are *non-special for v*.

R3. *If $f = v_1v_2v_3\dots$ is a non-special face such that $d(v_1) = 5$ and non-simplicial v_2 satisfies $d(v_2) \in \{11, 13\}$, then v_2 gives v_1 the following charge through f:*
(a) $\frac{\xi(v)}{2}$ if $d(v_3) = 5$ or
(b) $\xi(v)$ if $d(v_3) \geq 10$.

R4. *If v is a simplicial vertex with $d(v) \in \{11, 13\}$ such that $d(v_2) = 5$, then v gives v_2 the following charge through the face v_1vv_2:*
(a) $\frac{1}{4}$ if $d(v_1) \geq 10$ and $d(v_3) \geq 10$,
(b) $\xi(v)$ if $d(v_1) \geq 10$ and $d(v_3) = 5$,
(c) $\frac{\xi(v)}{2}$ if $d(v_1) = d(v_3) = 5$.

R5. *If v is a simplicial vertex with $d(v) \in \{11, 13\}$ such that $d(v_i) = 5$, whenever $1 \leq i \leq d(v)$, then v receives the following charge:*
(a) $\frac{1}{4}$ from each of v_1 and v_2 if there is a 4^+-face $v_1v_2\dots$,
(b) $\frac{1}{2}$ from v_2 if v_2 is simplicial and adjacent to three 10^+-vertices.

We precede stating our last rule of discharging by two definitions (see Fig. 1, R6).

A simplicial 5-vertex v is *poor* if $d(v_1) = d(v_3) = d(v_4) = 5$, $d(v_2) = 10$, and $d(v_5) \geq 13$. The latter inequality follows from $w(P_5) \geq 43$ and $h(P_5) \geq 13$.

Now let x be the neighbor of v_4 different from v and next to v_3 (around v_4). If $d(x) = 5$ or $d(x) \geq 13$, then v_4 is *rich* for v. If $10 \leq d(x) \leq 12$, then v_3 is *rich* for v. Hence every poor vertex has precisely one rich neighbor.

R6. *Each poor vertex receives $\frac{1}{20}$ from its rich neighbor.*

Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$ except for 5-vertices.

Case 1. Suppose f is a 4^+-face. If f has at least one 10^+-vertex, then f gives nothing to it, and so we have $\mu'(f) \geq 2d(f) - 6 - 2 \times \frac{3}{4} - (d-3) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \geq 0$ by R1. Otherwise, $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \geq 0$.

Case 2. $v \in V$.

Subcase 2.1. $d(v) \in \{10, 12, 14^+\}$. Note that v gives $\xi(v)$ through each incident face by R2. Namely, $d(v) = 10$ implies $\mu'(v) \geq 10 - 6 - 10 \times \frac{2}{5} = 0$, for $d(v) = 12$ we have $\mu'(v) \geq 6 - 12 \times \frac{1}{2} = 0$, and if $d(v) \geq 14$, then $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{11}{20} = \frac{9(d-14)+6}{20} > 0$.

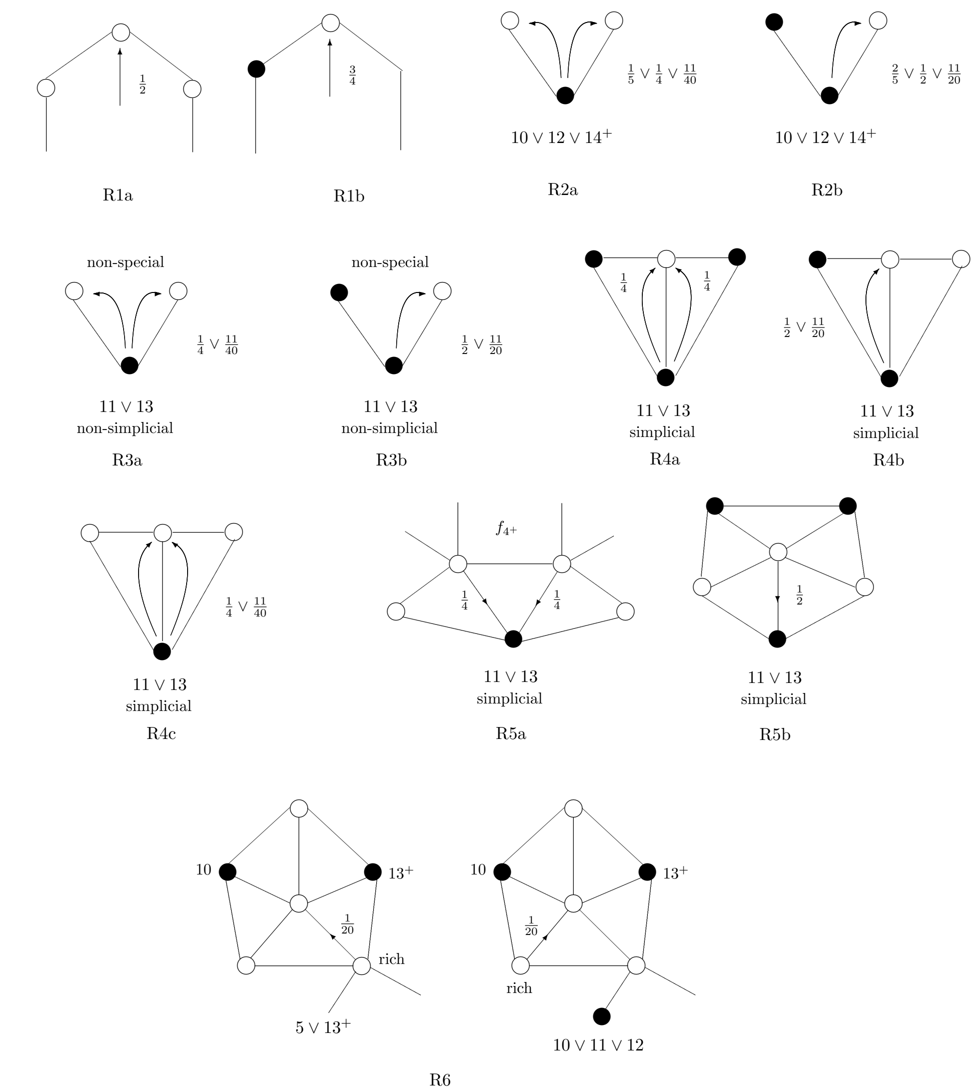

Рис. 1. Rules of discharging.

Subcase 2.2. $d(v) \in \{11, 13\}$. Note that v gives at most $\xi(v)$ through each incident face by R3 and R4. If v is non-simplicial, then there is a special face at v that receives nothing from v; so we have $\mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0$ for an 11-vertex and $\mu'(v) \geq 7 - 12 \times \frac{11}{20} > 0$ for a 13-vertex.

Now suppose that v is simplicial. If v has two consecutive 10^{+}-neighbors, then again $\mu'(v) \geq 0$ by the formulas in the previous paragraph. Suppose otherwise. Now if $d(v_1) \geq 10$, $d(v_2) = 5$, and $d(v_3) \geq 10$, then we have $\mu'(v) \geq 5 - 2 \times \frac{1}{4} - 9 \times \frac{1}{2} = 0$ for an 11-vertex and $\mu'(v) \geq 7 - 2 \times \frac{1}{4} - 11 \times \frac{11}{20} > 0$ for a 13-vertex according to R4a.

So we can assume from now one that every two consecutive 10^+-neighbors of v round v are separated from each other by at least two 5-neighbors of v. If $d(v_1) \geq 10$, $d(v_2) = d(v_3) = 5$, then v gives nothing to v_2 through v_2vv_3 and $\xi(v)$ through v_1vv_2 by R4b, and at most $\frac{\xi(v)}{2}$ to v_3 through v_2vv_3 by R4c.

This means that v sends away at most $\frac{3\xi(v)}{2}$ through the faces v_1vv_2 and v_2vv_3 together. By symmetry, v also sends at most $\frac{3\xi(v)}{2}$ through the two faces from the other side of the edge vv_1 round v. This implies that $\mu'(v) \geq 5 - 2 \times \frac{3}{4} - 7 \times \frac{1}{2} = 0$ for an 11-vertex and $\mu'(v) \geq 7 - 2 \times \frac{33}{40} - 9 \times \frac{11}{20} > 0$ for a 13-vertex.

Lemma 1. *If a simplicial vertex v of degree 11 or 13 is completely surrounded by 5-vertices, then $\mu'(v) \geq 0$.*

Proof. If the edge v_1v_2 is incident with a 4^+-face, then v receives $\frac{1}{4} + \frac{1}{4}$ from v_1 and v_2 by R5a, so we have $\mu'(v) \geq 5 + 2 \times \frac{1}{4} - 11 \times \frac{1}{2} = 0$ for $d(v) = 11$ and $\mu'(v) \geq 7 + 2 \times \frac{1}{4} - 13 \times \frac{11}{20} > 0$ for $d(v) = 13$.

Suppose there exist 3-faces $v_iw_iv_{i+1}$ (addition modulo $d(v)$) such that $w_i \neq v$ whenever $1 \leq i \leq d(v)$. Note that there are w_i and w_{i+1} with $d(w_i) \geq 10$ and $d(w_{i+1}) \geq 10$ due to the absence of 5^*-vertices in our counterexample P_5 combined with the oddness of $d(v)$. Recall that v_{i+1} is simplicial as mentioned above. Hence, v receives $\frac{1}{2}$ from v_{i+1} by R5b, so we are done as in the previous paragraph. ■

Checking $\mu'(v) \geq 0$ for a 5-vertex v.

Case 1. v gives $\frac{1}{2}$ by R5b. This means that v is simplicial with $d(v_1) = d(v_3) = 5$, $d(v_4) \geq 10$, $d(v_5) \geq 10$, and $d(v_2) \in \{11, 13\}$. Note that v is neither poor nor giving $\frac{1}{20}$ by R6 to v_1 or v_3. Hence v receives at least $\frac{1}{2}$ from each of v_4 and v_5 by R2b, R3b, R4b, and $\frac{1}{2}$ from v_2 by R3a, R4c, and as a result we have $\mu'(v) \geq -1 + 3 \times \frac{1}{2} - \frac{1}{2} = 0$.

Case 2. v gives $\frac{1}{4}$ by R5a. This means that v is not simplicial with $d(v_1) = d(v_3) = 5$, $d(v_2) \in \{11, 13\}$, and edge vv_1 lies in the boundary of 4^+-face $f = v_1vv_5\ldots$.

Subcase 2.1. $d(v_5) = 5$. Now v receives $\frac{1}{2}$ from f by R1a and $d(v_4) \geq 10$. Again v is neither poor nor giving $\frac{1}{20}$ by R6, and hence receives $\frac{1}{2}$ from v_2. Note that v_4 gives at least $\frac{1}{2}$ to v also and can take back $\frac{1}{4}$ from v by R5a. Hence $\mu'(v) \geq -1 + 3 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0$.

Subcase 2.2. $d(v_5) \geq 10$. Now our v receives $\frac{3}{4}$ from f by R1b and still receives $\frac{1}{2}$ from v_2. Furthermore, v receives at least $\frac{1}{5}$ from v_5 by our rules, and gives away at most $2 \times \frac{1}{20}$ by R6, and so $\mu'(v) \geq -1 + \frac{3}{4} + \frac{1}{2} + \frac{1}{5} - \frac{1}{4} - 2 \times \frac{1}{20} > 0$.

Thus we can assume due to Cases 1 and 2 that v does not participate in R5 in what follows.

Case 3. v is incident with at least three 4^+-faces. Here, $\mu'(v) > -1 + 3 \times \frac{1}{2} - 3 \times \frac{1}{20} > 0$ by R1 and R6.

Case 4. v is incident with precisely two 4^+-faces f_1, f_2. If one of f_1 and f_2 is incident with a 10^+-neighbor of v, then $\mu'(v) > -1 + \frac{3}{4} + \frac{1}{2} - 3 \times \frac{1}{20} > 0$ by R1 and R6. Otherwise, v has two 10^+-neighbors adjacent to each other, hence does not give charge by R6; so $\mu'(v) > -1 + 2 \times \frac{1}{2} = 0$.

Case 5. v is incident with precisely one 4^{+}-face f. If f is incident with a 10^{+}-neighbor of v, then v receives at least $\frac{2}{5}$ from the other 10^{+}-neighbor by R2–R4, so $\mu'(v) > -1 + \frac{3}{4} + \frac{2}{5} - 3 \times \frac{1}{20} > 0$ due to R1 and R6. Otherwise, each of at least two 10^{+}-neighbors of v gives at least $\frac{2}{5}$ to it by R2–R4, while f gives $\frac{1}{2}$ by R1, and we have $\mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{2}{5} - 3 \times \frac{1}{20} > 0$.

Case 6. v is simplicial.

Subcase 6.1. v has at least three 10^{+}-neighbors. Here $\mu'(v) \geq -1 + 3 \times \frac{2}{5} - 2 \times \frac{1}{20} > 0$ by R2, R4, and R6.

Subcase 6.2. v has precisely two 10^{+}-neighbors. First suppose $d(v_1) \geq 10$ and $d(v_3) \geq 10$. If $d(v_1) = 10$, then $d(v_3) \geq 13$ and v is poor. Here, $\mu'(v) = -1 + 2 \times \frac{1}{5} + 2 \times \frac{11}{40} + \frac{1}{20} = 0$ by R2a, R4c, and R6, as desired.

Suppose $d(v_1) \geq 11$ and $d(v_3) \geq 11$. Note that v_2 is not poor and each of v_1, v_3 gives at least $\frac{1}{2}$ to v. If $d(v_1) \in \{11, 12\}$ and $d(v_3) \in \{11, 12\}$, then neither v_4 nor v_5 is poor, so $\mu'(v) = -1 + 2 \times \frac{1}{2} = 0$. If $d(v_1) \in \{11, 12\}$ and $d(v_3) \geq 13$, then v_5 is not poor, and we have $\mu'(v) \geq -1 + \frac{1}{2} + \frac{11}{20} - \frac{1}{20} = 0$. Finally, if $d(v_1) \geq 13$ and $d(v_3) \geq 13$, then $\mu'(v) \geq -1 + 2 \times \frac{11}{20} - 2 \times \frac{1}{20} = 0$.

Next suppose $d(v_1) \geq 10$ and $d(v_2) \geq 10$. Note that v_4 is not poor. If $d(v_1) = 10$, then $d(v_2) \geq 13$ and $\mu'(v) \geq -1 + \frac{1}{5} + \frac{2}{5} + \frac{11}{20} - 3 \times \frac{1}{20} = 0$ by R2b, R4b, and R6.

Now suppose $d(v_1) \geq 11$ and $d(v_2) \geq 11$. If $d(v_1) \in \{11, 12\}$ and $d(v_2) \in \{11, 12\}$, then neither v_3 nor v_5 is poor, so $\mu'(v) = -1 + 2 \times \frac{1}{2} = 0$. If $d(v_1) \in \{11, 12\}$ and $d(v_2) \geq 13$, then v_5 is not poor, and we have $\mu'(v) \geq -1 + \frac{1}{2} + \frac{11}{20} - \frac{1}{20} = 0$. Finally, if $d(v_1) \geq 13$ and $d(v_3) \geq 13$, then $\mu'(v) \geq -1 + 2 \times \frac{11}{20} - 2 \times \frac{1}{20} = 0$.

Thus we have proved $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1.

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