LIGHT NEIGHBORHOODS OF 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5

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1. Introduction

The degree of a vertex or face $x$ in a convex finite 3-dimensional polytope (called a 3-polytope) is denoted by $d(x)$. A $k$-vertex is a vertex $v$ with $d(v) = k$. A $k^+$-vertex ($k^-$-vertex) is one of degree at least $k$ (at most $k$). Similar notation is used for the faces. A 3-polytope with minimum degree $\delta$ is denoted by $P_\delta$. The weight of a subgraph $S$ of a 3-polytope is the sum of degrees of the vertices of $S$ in the 3-polytope. The height of a subgraph $S$ of a 3-polytope is the maximum degree of the vertices of $S$ in the 3-polytope. A $k$-star $S_k(v)$ is minor if its center $v$ has...
degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars considered in this note are minor. By \( w(S_k) \) and \( h(S_k) \) we denote the minimum weight and height, respectively, of minor \( k \)-stars in a given 3-polytope.

In 1904, Wernicke [18] proved that every \( P_5 \) has a 5-vertex adjacent to a 6-vertex. This result was strengthened by Franklin [11] in 1922 to the existence of a 5-vertex with two 5-vertices. In 1940, Lebesgue [17, p.36] gave an approximate description of the neighborhoods of 5-vertices in \( P_5 \)'s. In particular, this description implies the results in [18, 11] and shows that there is a 5-vertex with three 7-neighbors.

For \( P_5 \)'s, the bounds \( w(S_1) \leq 11 \) (Wernicke [18]) and \( w(S_2) \leq 17 \) (Franklin [11]) are tight. It was proved by Lebesgue [17] that \( w(S_3) \leq 24 \), which was improved in 1996 by Jendrol' and Madaras [14] to the sharp bound \( w(S_3) \leq 23 \). Furthermore, Jendrol' and Madaras [14] gave a precise description of minor 3-stars in \( P_5 \)'s. Lebesgue [17] proved \( w(S_4) \leq 31 \), which was strengthened by Borodin and Woodall [3] to the tight bound \( w(S_4) \leq 30 \). Note that \( w(S_3) \leq 23 \) easily implies \( w(S_4) \leq 17 \) and immediately follows from \( w(S_4) \leq 30 \) (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, we obtained a precise description of 4-stars in \( P_5 \)'s [7].

For arbitrary 3-polytopes, that is for \( P_5 \)'s, the following results concerning \((d-2)\)-stars at \( d \)-vertices, \( d \leq 5 \), are known. Van den Heuvel and McGuinness [13] proved (in particular) that either \( w(S_1(v)) \leq 14 \) with \( d(v) = 3 \), or \( w(S_2(v)) \leq 22 \) with \( d(v) = 4 \), or \( w(S_3(v)) \leq 29 \) with \( d(v) = 5 \). Balogh et al. [1] proved that there is a 5-vertex adjacent to at most two 11-vertices. Harant and Jendrol' [12] strengthened these results by proving (in particular) that either \( w(S_1(v)) \leq 13 \) with \( d(v) = 3 \), or \( w(S_2(v)) \leq 19 \) with \( d(v) = 4 \), or \( w(S_3(v)) \leq 23 \) with \( d(v) = 5 \). Recently, we obtained a precise description of \((d-2)\)-stars in \( P_5 \)'s [6].

For \( P_5 \) the problem of describing \((d-1)\)-stars at \( d \)-vertices, \( d \leq 5 \), called pre-complete stars, appears difficult. As follows from the double \( n \)-pyramid, the minimum weight \( w(S_{d-1}) \) of pre-complete stars in \( P_5 \) can be arbitrarily large. Even when \( w(S_{d-1}) \) is restricted by appropriate conditions, the tight upper bounds on it are unknown. Borodin et al. [4, 5] proved (in particular) that if a planar graph with \( \delta \geq 3 \) has no edge joining two 4-vertices, then there is a star \( S_{d-1}(v) \) with \( w(S_{d-1}(v)) \leq 38 + d(v) \), where \( d(v) \leq 5 \) (see [5, Theorem 2.A]). Jendrol' and Madaras [15] proved that if the weight \( w(S_1) \) of every edge in an \( P_5 \) is at least 9, then there is a pre-complete star of height at most 20, where the bound of 20 is best possible.

The more general problem of describing \( d \)-stars at \( d \)-vertices, \( d \leq 5 \), called complete stars, at the moment seems intractable for arbitrary 3-polytopes and difficult even for \( P_5 \)’s.

Lebesgue [17] proved that if a \( P_5 \) has no 5-vertices adjacent to two 5-vertices and two 6-vertices, then \( w(S_5) \leq 68 \) and \( h(S_5) \leq 41 \). Recently, Borodin, Ivanova, and Jensen [9] lowered these bounds to \( w(S_5) \leq 55 \) and \( h(S_5) \leq 28 \), and then Borodin and Ivanova [8] to 51 and 23.

A 5-vertex is a 5-vertex adjacent to four 5-vertices. Jendrol' and Madaras [14] showed that if a polytope \( P \) in \( P_5 \) has a 5-vertex, then \( h(P) \) can be arbitrarily large.
For each $P_5$ with neither vertices of degrees from 6 to 9 nor 5*-vertices, it follows from Lebesgue’s Theorem that $w(P_5) \leq 44$ and $h(P_5) \leq 14$. It is known that if 6-vertices are allowed in $P_5$, then $h(P_5)$ can be arbitrarily large (see [9]), and if only 8-vertices are allowed, then $h(P_5)$ can reach 14 (see [10]).

The purpose of this note is to prove the following fact.

**Theorem 1.** Every 3-polytope $P$ with minimum degree 5 having neither vertices of degrees from 6 to 9 nor 5*-vertices adjacent to four 5-vertices satisfies $w(P) \leq 42$ and $h(P) \leq 12$, where both bounds are tight.

2. Proof of Theorem 1

The tightness of the bounds 42 and 12.

We put a 5-vertex into each 5-face of dodecahedron. This yields a triangulation with only vertices of degrees five and six such that each 5-vertex is surrounded by 6-vertices. The vertex-face dual of this triangulation is a cubic graph in which every 5-face is surrounded by 6-faces. We now replace each 3-vertex by a small 3-face, so as to make each $k$-face into a $2k$-face. Finally, we put a vertex inside each $10^+$-face and join it to the boundary vertices of this face.

In the resulting triangulation, every 5-vertex has a $12^+$-neighbor, a $10^+$-neighbor, and three 5-neighbors, as desired.

Discharging.

It suffices to prove the theorem for plane graphs in which no $4^+$-face is incident with two non-consecutive $10^+$-vertices, since adding a diagonal between such vertices cannot create a forbidden 5-star, nor can it reduce the weight or height of any existing minor 5-star.

So suppose that a 3-polytope $P_5$, with its sets of vertices, edges, and faces denoted by $V$, $E$, and $F$, respectively, is a counterexample to the main statement of Theorem 1.

By assumption, each minor 5-star in $P_5$ either is of weight at least 43 or contains a $13^+$-vertex. Also, recall that no 5-vertex has four 5-neighbors.

Euler’s formula $|V| - |E| + |F| = 2$ for $P_5$ implies

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \quad (1)$$

We assign an initial charge $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of $P_5$ as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to $-12$.

The final charge $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R6 below (see Fig. 1).

We put $\xi(v) = \frac{2}{5}$ if $d(v) = 10$, $\xi(v) = \frac{1}{5}$ if $11 \leq d(v) \leq 12$, and $\xi(v) = \frac{14}{20}$ if $d(v) \geq 13$. For a vertex $v$ let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to $v$ in a fixed
cyclic order. If $f$ is a face, then $v_1, \ldots, v_{d(f)}$ are the vertices incident with $f$ in the same cyclic order. A vertex is simplicial if it is completely surrounded by 3-faces.

**R1.** Each $4^+$-face $f$ gives the following charge to a 5-vertex $v_2$:
(a) $\frac{1}{2}$ if $d(v_1) = d(v_3) = 5$ or
(b) $\frac{3}{4}$ if $d(v_1) \geq 10$.

**R2.** If $f = v_1v_2v_3 \ldots$ is a face such that $d(v_1) = 5$ and $v_2$ satisfies $d(v_2) \in \{10, 12, 14^+\}$, then $v_2$ gives $v_1$ the following charge through $f$:
(a) $\frac{\xi(v)}{2}$ if $d(v_3) = 5$ or
(b) $\xi(v)$ if $d(v_3) \geq 10$.

If $v$ with $d(v) \in \{11, 13\}$ is non-simplicial, then one of its incident $4^+$-faces is declared special for $v$, and the other $3^+$-faces are non-special for $v$.

**R3.** If $f = v_1v_2v_3 \ldots$ is a non-special face such that $d(v_1) = 5$ and non-simplicial $v_2$ satisfies $d(v_2) \in \{11, 13\}$, then $v_2$ gives $v_1$ the following charge through $f$:
(a) $\frac{\xi(v)}{2}$ if $d(v_3) = 5$ or
(b) $\xi(v)$ if $d(v_3) \geq 10$.

**R4.** If $v$ is a simplicial vertex with $d(v) \in \{11, 13\}$ such that $d(v_2) = 5$, then $v$ gives $v_2$ the following charge through the face $v_1v_2$:
(a) $\frac{1}{2}$ if $d(v_1) \geq 10$ and $d(v_3) \geq 10$,
(b) $\xi(v)$ if $d(v_1) \geq 10$ and $d(v_3) = 5$,
(c) $\frac{\xi(v)}{2}$ if $d(v_1) = d(v_3) = 5$.

**R5.** If $v$ is a simplicial vertex with $d(v) \in \{11, 13\}$ such that $d(v_1) = 5$, whenever $1 \leq i \leq d(v)$, then $v$ receives the following charge:
(a) $\frac{1}{2}$ from each of $v_1$ and $v_2$ if there is a $4^+$-face $v_1v_2 \ldots$,
(b) $\frac{1}{2}$ from $v_2$ if $v_2$ is simplicial and adjacent to three $10^+$-vertices.

We precede stating our last rule of discharging by two definitions (see Fig. 1, R6).

A simplicial 5-vertex $v$ is poor if $d(v_1) = d(v_3) = d(v_4) = 5$, $d(v_2) = 10$, and $d(v_3) \geq 13$. The latter inequality follows from $w(P_5) \geq 43$ and $h(P_5) \geq 13$.

Now let $x$ be the neighbor of $v_4$ different from $v$ and next to $v_3$ (around $v_4$). If $d(x) = 5$ or $d(x) \geq 13$, then $v_4$ is rich for $v$. If $10 \leq d(x) \leq 12$, then $v_3$ is rich for $v$. Hence every poor vertex has precisely one rich neighbor.

**R6.** Each poor vertex receives $\frac{1}{20}$ from its rich neighbor.

**Checking $\mu'(x) \geq 0$ whenever $x \in V \cup F$ except for 5-vertices.**

**Case 1.** Suppose $f$ is a $4^+$-face. If $f$ has at least one $10^+$-vertex, then $f$ gives nothing to it, and so we have $\mu'(f) \geq 2d(f) - 6 - 2 \times \frac{3}{2} - (d - 3) \times \frac{1}{2} = \frac{3(d(f) - 4)}{2} \geq 0$ by R1. Otherwise, $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f) - 4)}{2} \geq 0$.

**Case 2.** $v \in V$.

**Subcase 2.1.** $d(v) \in \{10, 12, 14^+\}$. Note that $v$ gives $\xi(v)$ through each incident face by R2. Namely, $d(v) = 10$ implies $\mu'(v) \geq 10 - 6 - 10 \times \frac{3}{5} = 0$, for $d(v) = 12$ we have $\mu'(v) \geq 6 - 12 \times \frac{1}{2} = 0$, and if $d(v) \geq 14$, then $\mu'(v) \geq d(v) - 6 - d(v) \times \frac{11}{20} = \frac{9(d-14)+6}{20} > 0$. 
Subcase 2.2. \(d(v) \in \{11, 13\}\). Note that \(v\) gives at most \(\xi(v)\) through each incident face by R3 and R4. If \(v\) is non-simplicial, then there is a special face at \(v\) that receives nothing from \(v\); so we have \(\mu'(v) \geq 5 - 10 \times \frac{1}{2} = 0\) for an 11-vertex and \(\mu'(v) \geq 7 - 12 \times \frac{11}{20} > 0\) for a 13-vertex.

Now suppose that \(v\) is simplicial. If \(v\) has two consecutive \(10^+\)-neighbors, then again \(\mu'(v) \geq 0\) by the formulas in the previous paragraph. Suppose otherwise. Now if \(d(v_1) \geq 10\), \(d(v_2) = 5\), and \(d(v_3) \geq 10\), then we have \(\mu'(v) \geq 5 - 2 \times \frac{1}{4} - 9 \times \frac{1}{2} = 0\) for an 11-vertex and \(\mu'(v) \geq 7 - 2 \times \frac{1}{4} - 11 \times \frac{11}{20} > 0\) for a 13-vertex according to R4a.
So we can assume from now one that every two consecutive 10+-neighbors of \( v \) round \( v \) are separated from each other by at least two 5-neighbors of \( v \). If \( d(v_1) \geq 10, d(v_2) = d(v_3) = 5 \), then \( v \) gives nothing to \( v_2 \) through \( v_2v_3 \) and \( \xi(v) \) through \( v_1v_2 \) by R4b, and at most \( \frac{\xi(v)}{2} \) to \( v_3 \) through \( v_2v_3 \) by R4c.

This means that \( v \) sends away at most \( \frac{3\xi(v)}{2} \) through the faces \( v_1v_2 \) and \( v_2v_3 \) together. By symmetry, \( v \) also sends at most \( \frac{3\xi(v)}{2} \) through the two faces from the other side of the edge \( vv_1 \) round \( v \). This implies that \( \mu'(v) \geq 5 - 2 \times \frac{3}{4} - 7 \times \frac{1}{2} = 0 \) for an 11-vertex and \( \mu'(v) \geq 7 - 2 \times \frac{33}{50} - 9 \times \frac{11}{50} > 0 \) for a 13-vertex.

**Lemma 1.** If a simplicial vertex \( v \) of degree 11 or 13 is completely surrounded by 5-vertices, then \( \mu'(v) \geq 0 \).

**Proof.** If the edge \( v_1v_2 \) is incident with a 4+-face, then \( v \) gives \( \frac{1}{4} + \frac{1}{3} \) from \( v_1 \) and \( v_2 \) by R5a, so we have \( \mu'(v) \geq 5 + 2 \times \frac{1}{4} - 11 \times \frac{1}{2} = 0 \) for \( d(v) = 11 \) and \( \mu'(v) \geq 7 + 2 \times \frac{1}{4} - 13 \times \frac{11}{20} > 0 \) for \( d(v) = 13 \).

Suppose there exist 3-faces \( v_iw_{i+1}v_{i+1} \) (addition modulo \( d(v) \)) such that \( w_i \neq v \) whenever \( 1 \leq i \leq d(v) \). Note that there are \( w_i \) and \( w_{i+1} \) with \( d(w_i) \geq 10 \) and \( d(w_{i+1}) \geq 10 \) due to the absence of 5+-vertices in our counterexample \( P_5 \) combined with the oddness of \( d(v) \). Recall that \( v_{i+1} \) is simplicial as mentioned above. Hence, \( v \) receives \( \frac{1}{2} \) from \( v_{i+1} \) by R5b, so we are done as in the previous paragraph.  

**Checking \( \mu'(v) \geq 0 \) for a 5-vertex \( v \).**

**Case 1.** \( v \) gives \( \frac{1}{2} \) by R5b. This means that \( v \) is simplicial with \( d(v_1) = d(v_3) = 5, d(v_4) \geq 10, d(v_5) \geq 10, \) and \( d(v_2) \in \{11, 13, \} \). Note that \( v \) is neither poor nor giving \( \frac{1}{2} \) by R6 to \( v_1 \) or \( v_3 \). Hence \( v \) receives at least \( \frac{1}{2} \) from each of \( v_4 \) and \( v_5 \) by R2b, R3b, R4b, and \( \frac{1}{2} \) from \( v_2 \) by R3a, R4c, and as a result we have \( \mu'(v) \geq -1 + 3 \times \frac{1}{2} - \frac{1}{2} = 0 \).

**Case 2.** \( v \) gives \( \frac{1}{2} \) by R5a. This means that \( v \) is not simplicial with \( d(v_1) = d(v_3) = 5, d(v_2) \in \{11, 13, \} \), and edge \( vv_1 \) lies in the boundary of 4+-face \( f = vv_1 vv_3 \ldots \).

**Subcase 2.1.** \( d(v_5) = 5 \). Now \( v \) receives \( \frac{1}{2} \) from \( f \) by R1a and \( d(v_4) \geq 10 \). Again \( v \) is neither poor nor giving \( \frac{1}{2} \) by R6, and hence receives \( \frac{1}{2} \) from \( v_2 \). Note that \( v_4 \) gives at least \( \frac{1}{2} \) to \( v \) also and can take back \( \frac{1}{4} \) from \( v \) by R5a. Hence \( \mu'(v) \geq -1 + 3 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0 \).

**Subcase 2.2.** \( d(v_5) \geq 10 \). Now our \( v \) receives \( \frac{1}{2} \) from \( f \) by R1b and still receives \( \frac{1}{2} \) from \( v_2 \). Furthermore, \( v \) receives at least \( \frac{1}{4} \) from \( v_5 \) by our rules, and gives away at most \( 2 \times \frac{1}{50} \) by R6, and so \( \mu'(v) \geq -1 + \frac{3}{4} + \frac{1}{4} - \frac{1}{4} - 2 \times \frac{1}{20} > 0 \).

Thus we can assume due to Cases 1 and 2 that \( v \) does not participate in R5 in what follows.

**Case 3.** \( v \) is incident with at least three 4+-faces. Here, \( \mu'(v) > -1 + 3 \times \frac{1}{2} - 3 \times \frac{1}{20} > 0 \) by R1 and R6.

**Case 4.** \( v \) is incident with precisely two 4+-faces \( f_1, f_2 \). If one of \( f_1 \) and \( f_2 \) is incident with a 10+-neighbor of \( v \), then \( \mu'(v) > -1 + \frac{3}{2} + \frac{1}{2} - 3 \times \frac{1}{20} > 0 \) by R1 and R6. Otherwise, \( v \) has two 10+-neighbors adjacent to each other, hence does not give charge by R6; so \( \mu'(v) > -1 + 2 \times \frac{1}{2} = 0 \).
Case 5. \( v \) is incident with precisely one \( 4^+ \)-face. If \( f \) is incident with a \( 10^+ \)-neighbor of \( v \), then \( v \) receives at least \( \frac{2}{5} \) from the other \( 10^+ \)-neighbor by R2–R4, so \( \mu'(v) > -1 + \frac{3}{4} + \frac{2}{5} - 3 \times \frac{1}{20} > 0 \) due to R1 and R6. Otherwise, each of at least two \( 10^+ \)-neighbors of \( v \) gives at least \( \frac{x}{5} \) to it by R2–R4, while \( f \) gives \( \frac{1}{2} \) by R1, and we have \( \mu'(v) \geq -1 + \frac{1}{2} + 2 \times \frac{2}{5} - 3 \times \frac{1}{20} > 0 \).

Case 6. \( v \) is simplicial.

Subcase 6.1. \( v \) has at least three \( 10^+ \)-neighbors. Here \( \mu'(v) \geq -1 + 3 \times \frac{2}{5} - 2 \times \frac{1}{20} > 0 \) by R2, R4, and R6.

Subcase 6.2. \( v \) has precisely two \( 10^+ \)-neighbors. First suppose \( d(v_1) \geq 10 \) and \( d(v_3) \geq 10 \). If \( d(v_1) = 10 \), then \( d(v_3) \geq 13 \) and \( v \) is poor. Here, \( \mu'(v) = -1 + 2 \times \frac{1}{5} + 2 \times \frac{11}{20} + \frac{1}{20} = 0 \) by R2a, R4c, and R6, as desired.

Suppose \( d(v_1) \geq 11 \) and \( d(v_3) \geq 11 \). Note that \( v_2 \) is not poor and each of \( v_1, v_3 \) gives at least \( \frac{1}{2} \) to \( v \). If \( d(v_1) \in \{11,12\} \) and \( d(v_3) \in \{11,12\} \), then neither \( v_4 \) nor \( v_5 \) is poor, so \( \mu'(v) = -1 + 2 \times \frac{1}{2} = 0 \). If \( d(v_1) \in \{11,12\} \) and \( d(v_3) \geq 13 \), then \( v_5 \) is not poor, and we have \( \mu'(v) \geq -1 + \frac{1}{2} + \frac{11}{20} - \frac{1}{20} = 0 \). Finally, if \( d(v_1) \geq 13 \) and \( d(v_3) \geq 13 \), then \( \mu'(v) \geq -1 + 2 \times \frac{11}{20} - 2 \times \frac{1}{20} = 0 \).

Next suppose \( d(v_1) \geq 10 \) and \( d(v_2) \geq 10 \). Note that \( v_4 \) is not poor. If \( d(v_1) = 10 \), then \( d(v_2) \geq 13 \) and \( \mu'(v) \geq -1 + \frac{1}{5} + \frac{2}{5} + \frac{11}{20} - 3 \times \frac{1}{20} = 0 \) by R2b, R4b, and R6.

Now suppose \( d(v_1) \geq 11 \) and \( d(v_2) \geq 11 \). If \( d(v_1) \in \{11,12\} \) and \( d(v_2) \in \{11,12\} \), then neither \( v_3 \) nor \( v_5 \) is poor, so \( \mu'(v) = -1 + 2 \times \frac{1}{2} = 0 \). If \( d(v_1) \in \{11,12\} \) and \( d(v_3) \geq 13 \), then \( v_5 \) is not poor, and we have \( \mu'(v) \geq -1 + 1 + \frac{1}{2} + \frac{11}{20} - \frac{1}{20} = 0 \). Finally, if \( d(v_1) \geq 13 \) and \( d(v_3) \geq 13 \), then \( \mu'(v) \geq -1 + 2 \times \frac{11}{20} - 2 \times \frac{1}{20} = 0 \).

Thus we have proved \( \mu'(x) \geq 0 \) whenever \( x \in V \cup F \), which contradicts (1) and completes the proof of Theorem 1.

References

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