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EMBEDDINGS OF DIFFERENTIAL GROUPOIDS INTO MODULES OVER COMMUTATIVE RINGS

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Abstract. As is well known, subreducts of modules over commutative rings in a given variety form a quasivariety. Stanovský proved that a differential mode is a subreduct of a module over a commutative ring if and only if it is abelian. In the present article, we consider a minimal variety of differential groupoids with nonzero multiplication and show that its abelian algebras form the least subquasivariety with nonzero multiplication.

Keywords: Differential groupoid, module over a commutative ring, term conditions, quasivariety.

1. Introduction

As is well known, every groupoid mode is a subreduct of a semimodule over a commutative semiring. However, there exist differential groupoids that cannot be represented as subreducts of modules over commutative rings, see [14].

By a *differential groupoid* we mean an algebra with one binary operation that satisfies the identities

(I) $x \cdot x = x$;
(E) $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$;
(D) $x \cdot (x \cdot y) = x$.

Each differential groupoid satisfies the following identities (see [14, Section 5.6]):

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(L) $(x \cdot y) \cdot z = (x \cdot z) \cdot y$;
(R) $x \cdot (y \cdot z) = x \cdot y$.

In the sequel, by $x_1 x_2 \dots x_n$ we mean $(\dots((x_1 \cdot x_2) \cdot x_3) \dots) \cdot x_n$; by xy^n we mean $x \underbrace{y \dots y}_{n \text{ times}}$.

Let $\mathbf{D}_{0,2}$ denote the variety of differential groupoids defined by the additional identity

$$xy^2 = x. \tag{1}$$

Since differential groupoids satisfy identities (I) and (E), they form a variety of *modes*, see [14, Definition 5.3]. Notice that many authors use the term *medial* groupoid instead of *entorpic* or *binary* mode, see [3]. Differential groupoids were introduced in [11] under the name of LIR-*groupoids*. The basis for their identities consisted of (L), (I), and (R). The structure of differential groupoids and their varieties was studied in [10, 11, 12]. According to [4], the lattice of quasivarieties of differential groupoids is very complicated. The name of *differential groupoids* and their relation to differential groups were introduced in [13].

In the theory of modes, differential groupoids are often used as examples or counterexamples. In [8], the variety $\mathbf{D}_{0,2}$ is mentioned as one of the four classes of algebras having exactly n essentially n-ary term operations for each $n \geqslant 1$. In [7], the variety of ternary differential modes serves as a source of modes that do not embed as subreducts into semimodules over commutative semirings.

The lattice of varieties of differential groupoids is described by [11, Theorem 5.3] (see also [14, Theorem 8.4.14]). In particular, $\mathbf{D}_{0,2}$ covers the least nontrivial variety $\mathbf{D}_{0,1}$ of differential groupoids (defined by the identity $xy = x$). By [6], there exist 2^ω subquasivarieties of $\mathbf{D}_{0,2}$.

By a recent result of Stanovský [15], a differential groupoid is a subreduct of a module over a commutative ring if and only if this groupoid is abelian in the sense of commutator theory [1], i.e., satisfies the quasi-identity

$$\forall x \, \forall u \, \forall \overline{y} \, \forall \overline{z} \; (t(x, \overline{y}) = t(x, \overline{z}) \to t(u, \overline{y}) = t(u, \overline{z})) \tag{2}$$

for every $(n+1)$-ary term t. In the sequel, we call such quasi-identities *term conditions* and denote by (TC) the collection of conditions (2), where t ranges over the set of all groupoid terms.

As is known, differential groupoids in $\mathbf{D}_{0,2}$ that are subreducts of modules over commutative rings form a quasivariety. We denote this quasivariety by $\mathbf{K}$. As is shown in [15], $\mathbf{K}$ is not a variety. In the present article, we prove that $\mathbf{K}$ covers $\mathbf{D}_{0,1}$ in the lattice of quasivarieties of differential groupoids. For more detail on minimal quasivarieties with nonzero multiplication, the reader is referred to [5].

Recall [2] that each quasivariety is defined by a set of quasi-identities. On the other hand, a quasivariety is generated by a finite groupoid $\mathcal{G}$ if every groupoid $\mathcal{H}$ in this quasivariety is approximated by copies of $\mathcal{G}$. This means that $\mathcal{H}$ is a subdirect product of copies of $\mathcal{G}$ or (which is equivalent), for all $x, y \in G$ with $x \neq y$, there exists a homomorphism $\varphi : \mathcal{H} \to \mathcal{G}$ with $\varphi(x) \neq \varphi(y)$.

As usual, we denote structures by calligraphic letters and their universes by the corresponding italic letters.

2. A DIFFERENTIAL GROUPOID EMBEDDABLE INTO A MODULE OVER A COMMUTATIVE RING

We first recall the definition of an **Lz**-**Lz**-sum.

A groupoid $\mathcal{G}$ is called an **Lz**-**Lz**-*sum* (of orbits $\mathcal{G}_i$ over a groupoid $\mathcal{I}$) *satisfying the left normal law* if there exists a partition $G = \bigcup_{i\in I} G_i$ and, for every pair $(i,j) \in I^2$, there exists a mapping $h_i^j : G_i \to G_i$ such that the following conditions hold:

(i) h_i^i is the identity mapping for every $i \in I$;
(ii) $h_i^j\big(h_i^k(x)\big) = h_i^k\big(h_i^j(x)\big)$ for all $i,j,k \in I$ and $x \in G_i$;
(iii) $a_i \cdot a_j = h_i^j(a_i)$ for all $i,j \in I$, $a_i \in G_i$, and $a_j \in G_j$.

By [11, Theorem 2.2], $\mathcal{G}$ is a differential groupoid if and only if $\mathcal{G}$ can be represented as an **Lz**-**Lz**-sum satisfying the left normal law, see also [9, 10, 12].

Let $G_i = \{0_i, 1_i\}$, where $i \in \{0,1\}$. Let $\mathcal{C}$ be the **Lz**-**Lz**-sum of the orbits G_0 and G_1, where the mappings h_i^i are the identity mappings and the other mappings are cyclic shifts, i.e.,

$$h_i^{1-i}(j_i) = (1-j)_i$$

for all $i,j \in \{0,1\}$. As is proven in [5], the quasivariety generated by $\mathcal{C}$ covers the variety $\mathbf{D}_{0,1}$ in the lattice of quasivarieties of differential groupoids; moreover, for each subquasivariery $\mathbf{Q}$ with $\mathbf{D}_{0,1} \subsetneq \mathbf{Q} \subseteq \mathbf{D}_{0,2}$, we have $\mathcal{C} \in \mathbf{Q}$.

Proposition 1. *The groupoid $\mathcal{C}$ is embeddable into a commutative ring.*

Proof. Consider the ring

$$\mathcal{Z} = \mathbb{Z}[d,e]/\langle d^2 = 0,\ d+e = 1,\ 2d = 0\rangle.$$

In fact, $\mathcal{Z}$ is the affinisation ring of $\mathbf{D}_{0,2}$ (see [14, Chapter 7] for more detail on affinisation).

We define a mapping φ from C to Z. For $k \in \{0,1\}$, we put

$$\varphi(k_0) = kd, \quad \varphi(k_1) = 1 - kd = e^k.$$

The elements 0, 1, d, and e are pairwise distinct; hence, φ is a one-to-one mapping. On Z, define a binary operation by putting $x \cdot y = xe + yd$. We prove that $\mathcal{C}$ is isomorphic to the subring of $\mathcal{Z}$ generated by the set $\{0,1,d,e\}$.

Since $d + e = 1$, we have $x \cdot x = x$ for each $x \in C$.

If $\{x,y\} = G_0$ then $xy = x$ and $\varphi(xy) = xe + yd = xe$ because $d^2 = 0$. Since $de = d(1-d) = d - d^2 = d$, we have $\varphi(x) = \varphi(xy)$.

If $\{x,y\} = G_1$ then $\varphi(0_1 1_1) = e + ed = e + d = 1 = \varphi(0_1)$ and $\varphi(1_1 0_1) = e^2 + d = 1 + d = e$.

If $x \in G_0$ and $y \in G_1$ then $\varphi(0_0 0_1) = d = \varphi(1_0)$, $\varphi(0_0 1_1) = ed = d = \varphi(1_0)$ and $\varphi(1_0 0_1) = de + d = d + d = 0 = \varphi(0_0)$, $\varphi(1_0 1_1) = de + ed = d + d = 0 = \varphi(0_0)$.

Finally, if $x \in G_1$ and $y \in G_0$ then $\varphi(0_1 0_0) = e = \varphi(1_1)$, $\varphi(0_1 1_0) = e + d^2 = \varphi(1_1)$ and $\varphi(1_1 0_0) = e^2 = 1 = \varphi(0_1)$, $\varphi(1_1 1_0) = e^2 + d^2 = 1 = \varphi(0_1)$. □

Since $\mathcal{C}$ is embeddale into a (module over a) commutative ring, we conclude that the quasivariety generated by $\mathcal{C}$ is a subquasivariety of $\mathbf{K}$.

3. A graph associated with a differential groupoid

Since $\mathbf{D}_{0,2}$ is a locally finite variety, it suffices to describe finite groupoids in $\mathbf{K}$. In the sequel, we assume that $\mathcal{G}$ is a finite differential groupoid and $\mathcal{G} \in \mathbf{K}$.

Let $V = \{g_0, g_1, \ldots, g_r\}$ be a minimal set of generators of $\mathcal{G}$. By the *orbit* of g_i we mean the set

$$O(g_i) = \{g_i g_{i_1} \ldots g_{i_m} : g_{i_k} \in V \setminus \{g_i\}\}.$$

Since the set of generators is minimal, we have $O(g_i) \cap O(g_j) \neq \varnothing$ if and only if $i = j$. More information on representation of differential groupoids in terms of orbits and graphs can be found in [10].

In [6], a construction was introduced for obtaining differential groupoids from graphs. In the present section, we consider a version of the reverse construction, i.e., we construct graphs from differential groupoids.

For each groupoid $\mathcal{G} \in \mathbf{K}$, we define a graph $\mathcal{H}(\mathcal{G})$. Its vertex set is V and g_i is adjacent to g_j if and only if $g_j g_i \neq g_j$. For adjacent vertices, we write $(g_i, g_j) \in E$.

By (I), we have $(g_i, g_i) \notin E$ for each $g_i \in V$, i.e., $\mathcal{H}(\mathcal{G})$ is a loopless graph.

Lemma 2. *For every $\mathcal{G} \in \mathbf{K}$, the graph $\mathcal{H}(\mathcal{G})$ is undirected, i.e, $(g_i, g_j) \in E$ if and only if $(g_j, g_i) \in E$.*

Proof. Consider the term condition

$$\forall x \, \forall u \, (xu = xx \to uu = ux)$$

and put $x = g_i$ and $u = g_j$. □

Corollary 3. *If $|V| = 2$ then either $\mathcal{G}$ is isomorphic to $\mathcal{C}$ or $\mathcal{G}$ is a left zero band.*

We consider triangles in $\mathcal{H}(\mathcal{G})$.

Proposition 4. *If $x, y, z \in V$ are pairwise distinct vertices then one of the three following conditions holds.*

- *The triangle is empty, i.e., none of (x, y), (x, z), (y, z) belongs to E.*
- *The triangle is bipartite, i.e., exactly two of (x, y), (x, z), (y, z) belong to E.*
- *The triangle is complete, i.e., each of (x, y), (x, z), (y, z) belongs to E.*

Proof. Let $(x, y) \notin E$. Consider the term condition

$$\forall x \, \forall z \, \forall y \, (xx = xy \to zx = zy).$$

We find that $(x, z) \in E$ if and only if $(y, z) \in E$.

Therefore, if at least one of (x, y), (x, z), (y, z) does not belong to E then the triangle is either empty or bipartite. □

Corollary 5. *If $\mathcal{G}$ is not a left zero band then $\mathcal{H}(\mathcal{G})$ is a connected graph.*

Proof. If each connected component of $\mathcal{H}(\mathcal{G})$ is a singleton then $\mathcal{G}$ is a left zero band. If there is a connected component V_1 with $|V_1| > 1$ then consider arbitrary $u, v \in V_1$ with $(u, v) \in E$. For every $w \in V$ with $(u, w) \notin E$, we have $(v, w) \in E$ by Proposition 4. □

Corollary 6. *If $\mathcal{G}$ is not a left zero band and $(x, y) \notin E$ then there exists $z \in V$ such that x,y, and z form a bipartite triangle.*

Proof. By Corollary 5, $\mathcal{H}(\mathcal{G})$ is a connected graph and $|V| > 1$. Consider arbitrary $u, v \in V$ with $(u, v) \in E$. By Proposition 4, we have $(z, x) \in E$ for some $z \in \{u, v\}$ and, consequently, $(z, y) \in E$. □

4. Bipartite triangles and approximation

Throughout the section, we assume that $x, y, z \in V$ form a bipartite triangle and $(x, y) \notin E$.

We show that, in some sense, the elements x and y "duplicate each other."

Lemma 7. *We have $ux = uy$ for every $u \in G$ and*

$$xg_{i_1} \dots g_{i_r} = xg_{j_1} \dots g_{j_s} \iff yg_{i_1} \dots g_{i_r} = yg_{j_1} \dots g_{j_s}.$$

Proof. The first assertion is immediate from the term condition

$$\forall x \, \forall u \, \forall y \, (xx = xy \to ux = uy).$$

We prove the second assertion. Notice that the equality $xg_{i_1} \dots g_{i_r} = xg_{j_1} \dots g_{j_s}$ is equivalent to an equality of the form $xg_{i_1} \dots g_{i_m} = x$. Indeed, it suffices to multiply both parts by each of $g_{j_1}, \dots, g_{j_s}$ and take into account identities (L) and (1). If m is even then $x = xy^m$; if m is odd then $x = xxy^{m-1}$. By (TC), we have either

$$xg_{i_1} \dots g_{i_m} = xy^m \to yg_{i_1} \dots g_{i_m} = yy^m = y$$

(if m is even) or

$$xg_{i_1} \dots g_{i_m} = xxy^{m-1} \to yg_{i_1} \dots g_{i_m} = yxy^m = yx = y$$

(if m is odd). □

Corollary 8. *Let θ denote the principal congruence on $\mathcal{G}$ generated by the pair (x, y). Then $\theta = \{(xg_{i_1} \dots g_{i_m}, yg_{i_1} \dots g_{i_m}) : g_{i_k} \in V\}$ and the quotient $\mathcal{G}/\theta$ is isomorphic to the subgroupoid of $\mathcal{G}$ whose universe is $G \setminus O(y)$.*

We denote by φ the homomorphism whose kernel is θ. We denote by ψ the homomorphism whose kernel is the principal congruence generated by the pair (x, yz). Repeating the arguments above, we conclude that $\psi(a) = \psi(b)$ if and only if $a = xg_{i_1} \dots g_{i_m}$ and $b = yzg_{i_1} \dots g_{i_m}$ for some $g_{i_1}, \dots, g_{i_m} \in V$; moreover, the quotients $\mathcal{G}/\ker\varphi$ and $\mathcal{G}/\ker\psi$ are isomorphic. We notice that $yzg_{i_1} \dots g_{i_m} \neq yg_{i_1} \dots g_{i_m}$. Indeed, applying term conditions of the form

$$yzg_{i_1} \dots g_{i_k} = yyg_{i_1} \dots g_{i_k} \to yzg_{i_1} \dots g_{i_{k-1}}y = yyg_{i_1} \dots g_{i_{k-1}}y,$$

we find that $yz = y$, a contradiction.

Proposition 9. *The groupoid $\mathcal{G}$ is approximated by copies of its subgroupoid $\mathcal{G}'$ such that every three vertices of $\mathcal{H}(\mathcal{G}')$ form a complete triangle.*

Proof. For every bipartite triangle let φ and ψ be endomorphisms defined above. Take $a, b \in G$ with $a \neq b$. If $a = xg_{i_1} \dots g_{i_m}$ and $b = yg_{i_1} \dots g_{i_m}$ for some $g_{i_1}, \dots, g_{i_m} \in V$ then $\psi(a) \neq \psi(b)$; otherwise, $\varphi(a) \neq \varphi(b)$.

Applying this procedure to each $(x, y) \notin E$, we obtain a subgroupoid whose universe is $G \setminus \bigcup_{y \in Y} O(y)$, where Y corresponds to the sequence of "removed" orbits. □

5. Complete triangles and embedding

In the sequel, we assume that $\mathcal{G} \in \mathbf{K}$, the graph $\mathcal{H}(\mathcal{G})$ is complete, its vertex set is $V = \{g_0, g_1, \dots, g_r\}$, and $g_i g_j \neq g_i$ if $i \neq j$.

For every subset X of $V \setminus \{g_0\}$, let $D(X)$ denote the set of all equalities of the form

$$g_0 g_i = g_0 g_{i_1} g_{i_m} \tag{3}$$

that are true in $\mathcal{G}$, where $\{g_i, g_{i_1}, \dots, g_{i_m}\} \subseteq X$ and all these elements are pairwise distinct. A subset X is said to be *independent* if $D(X) = \varnothing$. By a *basis* we mean a set of the form $X \cup \{g_0\}$, where X is a maximal independent subset.

We fix a basis $\{g_0, g_1, \dots, g_n\}$. We denote $X = \{g_1, \dots, g_n\}$. By definition, we have

$$g_0 g_i \neq g_0 g_{i_1} \cdots g_{i_m}$$

if $g_i \in X$ and $g_{i_1}, \dots, g_{i_m} \in X \setminus \{g_i\}$. Since X is a maximal independent subset, we conclude that, for every $g_i \notin X \cup \{g_0\}$, there exist $g_{i_1}, \dots, g_{i_m} \in X$ such that equality (3) is true in $\mathcal{G}$. By (1) and the definition of an independent subset, we conclude that, for every $g_i \notin X$, such an equality is unique up to a permutation of the elements $g_{i_1}, \dots, g_{i_m}$. We denote this equality by E_i.

Let $\Delta = \{E_i : g_i \notin X \cup \{g_0\}\}$. We show that $\mathcal{G}$ is determined, in $\mathbf{K}$, by the generators V and the defining relations Δ. For more detail on defining relations, the reader is referred, for example, to [2, Section 2.1].

Proposition 10. *The differential groupoid $\mathcal{G}$ is determined, in $\mathbf{K}$, by the set of generators V and the set of defining relations Δ.*

Proof. By the choice of V, the groupoid $\mathcal{G}$ is generated by V. By the choice of Δ, each equality in Δ holds in $\mathcal{G}$.

Let an equality of the form

$$g_0 = g_0 g_{j_1} \cdots g_{j_s} \tag{4}$$

hold in $\mathcal{G}$, where $g_{j_1}, \dots, g_{j_s} \in V \setminus \{g_0\}$. Since X is an independent set, we have $g_{j_k} \notin X$, where $1 \leqslant k \leqslant s$. Without loss of generality, we may assume that $k = s$. By (1), we conclude that (4) is equivalent, in $\mathbf{K}$, to the equality

$$g_0 g_{j_s} = g_0 g_{j_1} \cdots g_{j_{s-1}}. \tag{5}$$

If $g_{j_1}, \dots, g_{j_{s-1}} \in X$ then we obtain an equality in Δ. Otherwise, we have $g_{j_k} \notin X$, where $1 \leqslant k < s$. Without loss of generality, we may assume that $k = 1$. Consider the corresponding equality E_{j_1}, i.e., the equality

$$g_0 g_{j_1} = g_0 g_{i_1} \cdots g_{i_m}. \tag{6}$$

Substituting (6) into (5), we obtain

$$g_0 g_{j_s} = g_0 g_{i_1} \cdots g_{i_m} g_{j_2} \cdots g_{j_{s-1}}.$$

Repeating this procedure for each $g_{j_k} \notin X$, we obtain an equality in Δ. We conclude that equality (4) is a consequence of Δ in $\mathbf{K}$.

Let $i \neq 0$ and let an equality of the form

$$g_i = g_i g_{j_1} \cdots g_{j_s}$$

hold in $\mathcal{G}$.

If s is even then $g_i = g_i g_0^s = g_i g_{j_1} \dots g_{j_s}$. By (TC), the latter equality is equivalent to the equality $g_0 g_0^s = g_0 g_{j_1} \dots g_{j_s}$, i.e., it is equivalent to an equalty of the form (4).

If s is odd then $g_i = g_i g_i g_0^{s-1} = g_i g_{j_1} \dots g_{j_s}$. By (TC), the latter equality is equivalent to the equality $g_0 g_i g_0^{s-1} = g_0 g_{j_1} \dots g_{j_s}$. By (1), this equality is equivalent to the equality $g_0 = g_0 g_i g_{j_1} \dots g_{j_s}$, i.e., it is equivalent to an equalty of the form (4).

By (1), each equality $p = q$, where p and q are terms, is a consequence, in $\mathbf{K}$, of the set Δ. By [2, Proposition 2.1.1], the groupoid G is determined, in $\mathbf{K}$, by the generators V and the defining relations Δ. $\square$

Let $n = |X|$. We now prove that the differential groupoid determined, in $\mathbf{K}$, by the generators V and the defining relations Δ is embeddable into $\mathcal{C}^n$.

As above, we assume that $X = \{g_1, \dots, g_n\}$, $V = \{g_0, \dots, g_r\}$, $n \leqslant r$, and $\Delta = \{E_i : g_i \notin X \cup \{g_0\}\}$.

We define a mapping α from the differential groupoid $\mathcal{G}$ determined, in $\mathbf{K}$, by the generators V and the defining relations Δ into the groupoid $\mathcal{C}^n$. We put

$$\alpha(g_0)(i) = 0_0 \text{ for all } i,$$
$$\alpha(g_i)(k) = 0_1 \text{ if } k = i \text{ and } 0_0 \text{ otherwise}, \quad 0 < i \leqslant n.$$

For $g_i \notin X \cup \{g_0\}$, we consider the corresponding equality E_i of the form $g_0 g_i = g_0 g_{i_1} g_{i_m}$ and observe that, for distinct $g_j, g_k \in X$, we have

$$g_0 g_j g_k(l) = 1_0 \text{ if } l \in \{i, j\} \text{ and } 0_0 \text{ otherwise.}$$

We conclude that

$$g_0 g_i(l) = \begin{cases} 1_0 & \text{if } l \in \{i_1, \dots, i_m\}, \\ 0_0 & \text{otherwise} \end{cases}$$

and

$$g_i(l) = \begin{cases} 0_1 & \text{if } l \in \{i_1, \dots, i_m\}, \\ 0_0 & \text{otherwise.} \end{cases}$$

In view of identities (L) and (R), we calculate

$$g_0 g_{j_1} \dots g_{j_s} \cdot g_i g_{i_1} \dots g_{i_m} = g_0 g_{j_1} \dots g_{j_s} g_i = g_0 g_i g_{j_1} \dots g_{j_s}. \tag{7}$$

If $g_i \notin X \cup \{g_0\}$ then we substitute E_i into (7). Thus, each element of the orbit of g_0 is represented as $g_0 g_{i_1} \dots g_{i_m}$, where $g_{i_1}, \dots, g_{i_m} \in X \setminus \{g_0\}$. In particular, if $g_{i_1}, \dots, g_{i_m} \in X \setminus \{g_0\}$ then $g_0 \neq g_0 g_{i_1} \dots g_{i_m}$, i.e., for all $x, y \in O(g_0)$, we have $\alpha(x) \neq \alpha(y)$ provided $x \neq y$.

By the above, we find that $\mathcal{C}^n$ satisfies all equalities in Δ. Hence, there exists a homomorphism from $\mathcal{G}$ into $\mathcal{C}^n$. If $\alpha(x) = \alpha(y)$ then, repeating arguments from the proof of Proposition 10, we conclude that we may assume that $x, y \in O(g_0)$; hence, $x = y$. Thus, the homomorphism from $\mathcal{G}$ into $\mathcal{C}^n$ is an embedding.

We obtain the following main result.

Theorem 11. *Each differential groupoid in $\mathbf{D}_{0,2}$ is a subreduct of a semimodule over a commutative semiring. A differential groupoids in $\mathbf{D}_{0,2}$ is a subreduct of a module over a commutative ring if and only if it belongs to the least subquasivariety of $\mathbf{D}_{0,2}$ with nonzero multiplication.*

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