EMBEDDINGS OF DIFFERENTIAL GROUPOIDS INTO MODULES OVER COMMUTATIVE RINGS

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Abstract. As is well known, subreducts of modules over commutative rings in a given variety form a quasivariety. Stanovský proved that a differential mode is a subreduct of a module over a commutative ring if and only if it is abelian. In the present article, we consider a minimal variety of differential groupoids with nonzero multiplication and show that its abelian algebras form the least subquasivariety with nonzero multiplication.

Keywords: Differential groupoid, module over a commutative ring, term conditions, quasivariety.

1. Introduction

As is well known, every groupoid mode is a subreduct of a semimodule over a commutative semiring. However, there exist differential groupoids that cannot be represented as subreducts of modules over commutative rings, see [14].

By a differential groupoid we mean an algebra with one binary operation that satisfies the identities

(I) $x \cdot x = x$;

(E) $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$;

(D) $x \cdot (x \cdot y) = x$.

Each differential groupoid satisfies the following identities (see [14, Section 5.6]):
\[(L) \ (x \cdot y) \cdot z = (x \cdot z) \cdot y; \]
\[(R) \ x \cdot (y \cdot z) = x \cdot y. \]

In the sequel, by \(x_1 x_2 \ldots x_n\) we mean \(\ldots ((x_1 \cdot x_2) \cdot x_3) \ldots \cdot x_n\); by \(x y^n\) we mean \(x y \ldots y\) \(n\) times.

Let \(D_{0,2}\) denote the variety of differential groupoids defined by the additional identity
\[xy^2 = x.\]

Since differential groupoids satisfy identities (I) and (E), they form a variety of modes, see [14, Definition 5.3]. Notice that many authors use the term medial groupoid instead of entropic or binary mode, see [3]. Differential groupoids were introduced in [11] under the name of LIR-groupoids. The basis for their identities consisted of (L), (I), and (R). The structure of differential groupoids and their varieties was studied in [10, 11, 12]. According to [4], the lattice of quasivarieties of differential groupoids is very complicated. The name of differential groupoids and their relation to differential groups were introduced in [13].

In the theory of modes, differential groupoids are often used as examples or counterexamples. In [8], the variety \(D_{0,2}\) is mentioned as one of the four classes of algebras having exactly \(n\) essentially \(n\)-ary term operations for each \(n \geq 1\). In [7], the variety of ternary differential modes serves as a source of modes that do not embed as subreducts into semimodules over commutative semirings.

The lattice of varieties of differential groupoids is described by [11, Theorem 5.3] (see also [14, Theorem 8.4.14]). In particular, \(D_{0,2}\) covers the least nontrivial variety \(D_{0,1}\) of differential groupoids (defined by the identity \(xy = x\)). By [6], there exist \(2^2\) subquasivarieties of \(D_{0,2}\).

By a recent result of Stanovský [15], a differential groupoid is a subreduct of a module over a commutative ring if and only if this groupoid is abelian in the sense of commutator theory [1], i.e., satisfies the quasi-identity
\[\forall x \forall u \forall y \forall z \ (t(x, y) = t(x, z) \Rightarrow t(u, y) = t(u, z))\]
for every \((n + 1)\)-ary term \(t\). In the sequel, we call such quasi-identities term conditions and denote by (TC) the collection of conditions (2), where \(t\) ranges over the set of all groupoid terms.

As is known, differential groupoids in \(D_{0,2}\) that are subreducts of modules over commutative rings form a quasivariety. We denote this quasivariety by \(K\). As is shown in [15], \(K\) is not a variety. In the present article, we prove that \(K\) covers \(D_{0,1}\) in the lattice of quasivarieties of differential groupoids. For more detail on minimal quasivarieties with nonzero multiplication, the reader is referred to [5].

Recall [2] that each quasivariety is defined by a set of quasi-identities. On the other hand, a quasivariety is generated by a finite groupoid \(G\) if every groupoid \(H\) in this quasivariety is approximated by copies of \(G\). This means that \(H\) is a subdirect product of copies of \(G\) or (which is equivalent), for all \(x, y \in G\) with \(x \neq y\), there exists a homomorphism \(\varphi : H \to G\) with \(\varphi(x) \neq \varphi(y)\).

As usual, we denote structures by calligraphic letters and their universes by the corresponding italic letters.
2. A DIFFERENTIAL GROUPOID EMBEDDABLE INTO A MODULE OVER A COMMUTATIVE RING

We first recall the definition of an \( L_{z}^{2} \)-sum.

A groupoid \( G \) is called an \( L_{z}^{2} \)-sum (of orbits \( G_{i} \) over a groupoid \( T \)) satisfying the left normal law if there exists a partition \( G = \bigcup_{i \in I} G_{i} \), and, for every pair \( (i, j) \in I^{2} \), there exists a mapping \( h_{ij}^{i} : G_{i} \to G_{j} \) such that the following conditions hold:

(i) \( h_{i}^{i} \) is the identity mapping for every \( i \in I \);
(ii) \( h_{i}^{j}(h_{j}^{i}(x)) = h_{i}^{i}(h_{i}^{j}(x)) \) for all \( i, j, k \in I \) and \( x \in G_{i} \);
(iii) \( a_{i} \cdot a_{j} = h_{i}^{j}(a_{i}) \) for all \( i, j \in I \), \( a_{i} \in G_{i} \), and \( a_{j} \in G_{j} \).

By [11, Theorem 2.2], \( G \) is a differential groupoid if and only if \( G \) can be represented as an \( L_{z}^{2} \)-sum satisfying the left normal law, see also [9, 10, 12].

Let \( G_{i} = \{0, 1\} \), where \( i \in \{0, 1\} \). Let \( C \) be the \( L_{z}^{2} \)-sum of the orbits \( G_{0} \) and \( G_{1} \), where the mappings \( h_{i}^{i} \) are the identity mappings and the other mappings are cyclic shifts, i.e.,

\[
h_{i}^{i-1}(j) = (1 - j)_{i}
\]

for all \( i, j \in \{0, 1\} \). As is proven in [5], the quasivariety generated by \( C \) covers the variety \( D_{0,1} \) in the lattice of quasivarieties of differential groupoids; moreover, for each subquasivariety \( Q \) with \( D_{0,1} \subsetneq Q \subseteq D_{0,2} \), we have \( C \subseteq Q \).

**Proposition 1.** The groupoid \( C \) is embeddable into a commutative ring.

**Proof.** Consider the ring

\[
Z = \mathbb{Z}[d, e]/(d^{2} = 0, d + e = 1, 2d = 0).
\]

In fact, \( Z \) is the affinisation ring of \( D_{0,2} \) (see [14, Chapter 7] for more detail on affinisation).

We define a mapping \( \varphi \) from \( C \) to \( Z \). For \( k \in \{0, 1\} \), we put

\[
\varphi(k_{0}) = kd, \quad \varphi(k_{1}) = 1 - kd = e^{k}.
\]

The elements 0, 1, \( d \), and \( e \) are pairwise distinct; hence, \( \varphi \) is a one-to-one mapping. On \( Z \), define a binary operation by putting \( x \cdot y = xe + yd \). We prove that \( C \) is isomorphic to the subring of \( Z \) generated by the set \( \{0, 1, d, e\} \).

Since \( d + e = 1 \), we have \( x \cdot x = x \) for each \( x \in C \).

If \( \{x, y\} = G_{0} \) then \( xy = x \) and \( \varphi(xy) = xe + yd = xe \) because \( d^{2} = 0 \). Since \( de = d(1 - d) = d - d^{2} = d \), we have \( \varphi(x) = \varphi(xy) \).

If \( \{x, y\} = G_{1} \) then \( \varphi(0, 1_{1}) = e + d = 1 = \varphi(0_{1}) \) and \( \varphi(1_{1}0_{1}) = e^{2} + d = 1 + d = e \).

If \( x \in G_{0} \) and \( y \in G_{1} \) then \( \varphi(0_{1}0_{1}) = d = \varphi(1_{0}) \), \( \varphi(0_{1}1_{1}) = de = d = \varphi(1_{0}) \) and \( \varphi(1_{0}1_{1}) = de + d = d = 0 = \varphi(0_{1}) \).

Finally, if \( x \in G_{1} \) and \( y \in G_{0} \) then \( \varphi(0_{1}0_{0}) = e = \varphi(1_{1}) \), \( \varphi(0_{1}1_{0}) = e + d^{2} = \varphi(1_{1}) \) and \( \varphi(1_{1}0_{0}) = e^{2} + 1 = \varphi(0_{1}) \), \( \varphi(1_{1}0_{1}) = e^{2} + d^{2} = 1 = \varphi(0_{1}) \).

Since \( C \) is embeddable into a (module over a) commutative ring, we conclude that the quasivariety generated by \( C \) is a subquasivariety of \( K \).
3. A GRAPH ASSOCIATED WITH A DIFFERENTIAL GROUPOID

Since $D_{0,2}$ is a locally finite variety, it suffices to describe finite groupoids in $K$.
In the sequel, we assume that $G$ is a finite differential groupoid and $G \in K$.

Let $V = \{g_0, g_1, \ldots, g_r\}$ be a minimal set of generators of $G$. By the orbit of $g_i$
we mean the set

$$O(g_i) = \{g_ig_{i_1}\cdots g_{i_m} : g_{i_k} \in V \setminus \{g_i\}\}.$$  

Since the set of generators is minimal, we have $O(g_i) \cap O(g_j) \neq \emptyset$ if and only
if $i = j$. More information on representation of differential groupoids in terms of
orbits and graphs can be found in [10].

In [6], a construction was introduced for obtaining differential groupoids from
graphs. In the present section, we consider a version of the reverse construction,
i.e., we construct graphs from differential groupoids.

For each groupoid $G \in K$, we define a graph $H(G)$. Its vertex set is $V$ and $g_i$ is
adjacent to $g_j$ if and only if $g_i g_j \neq g_j$. For adjacent vertices, we write $(g_i, g_j) \in E$.

By (I), we have $(g_i, g_j) \notin E$ for each $g_i \in V$, i.e., $H(G)$ is a loopless graph.

**Lemma 2.** For every $G \in K$, the graph $H(G)$ is undirected, i.e., $(g_i, g_j) \in E$ if and
only if $(g_j, g_i) \in E$.

**Proof.** Consider the term condition

$$\forall x \forall u (xu = xx \rightarrow uu = ux)$$

and put $x = g_i$ and $u = g_j$. □

**Corollary 3.** If $|V| = 2$ then either $G$ is isomorphic to $C$ or $G$ is a left zero band.

We consider triangles in $H(G)$.

**Proposition 4.** If $x, y, z \in V$ are pairwise distinct vertices then one of the three
following conditions holds.

- The triangle is empty, i.e., none of $(x, y), (x, z), (y, z)$ belongs to $E$.
- The triangle is bipartite, i.e., exactly two of $(x, y), (x, z), (y, z)$ belong to $E$.
- The triangle is complete, i.e., each of $(x, y), (x, z), (y, z)$ belongs to $E$.

**Proof.** Let $(x, y) \notin E$. Consider the term condition

$$\forall x \forall z \forall y (xx = xy \rightarrow zz = yz).$$

We find that $(x, z) \in E$ if and only if $(y, z) \in E$.

Therefore, if at least one of $(x, y), (x, z), (y, z)$ does not belong to $E$ then the
triangle is either empty or bipartite. □

**Corollary 5.** If $G$ is not a left zero band then $H(G)$ is a connected graph.

**Proof.** If each connected component of $H(G)$ is a singleton then $G$ is a left zero
band. If there is a connected component $V_1$ with $|V_1| > 1$ then consider arbitrary
$u, v \in V_1$ with $(u, v) \in E$. For every $w \in V$ with $(u, w) \notin E$, we have $(v, w) \in E$ by
Proposition 4. □

**Corollary 6.** If $G$ is not a left zero band and $(x, y) \notin E$ then there exists $z \in V$
such that $x, y, z$ and $z$ form a bipartite triangle.

**Proof.** By Corollary 5, $H(G)$ is a connected graph and $|V| > 1$. Consider arbitrary
$u, v \in V$ with $(u, v) \in E$. By Proposition 4, we have $(z, x) \in E$ for some $z \in \{u, v\}$
and, consequently, $(z, y) \in E$. □
4. Bipartite triangles and approximation

Throughout the section, we assume that \( x, y, z \in V \) form a bipartite triangle and \((x, y) \notin E\).

We show that, in some sense, the elements \( x \) and \( y \) “duplicate each other.”

**Lemma 7.** We have \( ux = uy \) for every \( u \in G \) and
\[
xy_{i_1} \cdots y_{i_r} = xy_{j_1} \cdots y_{j_s} \iff yg_{i_1} \cdots g_{i_r} = yg_{j_1} \cdots g_{j_s}.
\]

**Proof.** The first assertion is immediate from the term condition
\[
\forall x \forall u \forall y (xx = xy \rightarrow ux = uy).
\]

We prove the second assertion. Notice that the equality \( xy_{i_1} \cdots y_{i_r} = xy_{j_1} \cdots y_{j_s} \) is equivalent to an equality of the form \( xy_{i_1} \cdots y_{i_r} = x \). Indeed, it suffices to multiply both parts by each of \( g_{j_1}, \ldots, g_{j_s} \) and take into account identities (L) and (1). If \( m \) is even then \( x = xy^m \); if \( m \) is odd then \( x = xy^{m-1} \). By (TC), we have either
\[
xy_{i_1} \cdots y_{i_r} = xy^m \rightarrow yg_{i_1} \cdots g_{i_r} = yy^m = y
\]
(if \( m \) is even) or
\[
xy_{i_1} \cdots y_{i_r} = xy^{m-1} \rightarrow yg_{i_1} \cdots g_{i_r} = xxy^m = xy = y
\]
(if \( m \) is odd).

\( \square \)

**Corollary 8.** Let \( \theta \) denote the principal congruence on \( G \) generated by the pair \((x, y)\). Then \( \theta = \{(xg_{i_1} \cdots g_{i_m}, yg_{i_1} \cdots g_{i_m}) : g_{i_k} \in V\} \) and the quotient \( G/\theta \) is isomorphic to the subgroupoid of \( G \) whose universe is \( G \setminus O(y) \).

We denote by \( \varphi \) the homomorphism whose kernel is \( \theta \). We denote by \( \psi \) the homomorphism whose kernel is the principal congruence generated by the pair \((x, yz)\). Repeating the arguments above, we conclude that \( \psi(a) = \psi(b) \) if and only if \( a = xg_{i_1} \cdots g_{i_m} \) and \( b = yzg_{i_1} \cdots g_{i_m} \) for some \( g_{i_1}, \ldots, g_{i_m} \in V \); moreover, the quotients \( G/\ker \psi \) and \( G/\ker \psi \) are isomorphic. We notice that \( yzg_{i_1} \cdots g_{i_m} \neq yg_{i_1} \cdots g_{i_m} \). Indeed, applying term conditions of the form
\[
yzg_{i_1} \cdots g_{i_k} = ygz_{i_1} \cdots g_{i_k} \rightarrow yzg_{i_1} \cdots g_{i_{k-1}}y = ygz_{i_1} \cdots g_{i_{k-1}}y,
\]
we find that \( yz = y \), a contradiction.

**Proposition 9.** The groupoid \( G \) is approximated by copies of its subgroupoid \( G' \) such that every three vertices of \( \mathcal{H}(G') \) form a complete triangle.

**Proof.** For every bipartite triangle let \( \varphi \) and \( \psi \) be endomorphisms defined above. Take \( a, b \in G \) with \( a \neq b \). If \( a = xg_{i_1} \cdots g_{i_m} \) and \( b = yg_{i_1} \cdots g_{i_m} \) for some \( g_{i_1}, \ldots, g_{i_m} \in V \) then \( \psi(a) = \psi(b) \); otherwise, \( \varphi(a) \neq \varphi(b) \).

Applying this procedure to each \((x, y) \notin E\), we obtain a subgroupoid whose universe is \( G \setminus \bigcup_{y \in Y} O(y) \), where \( Y \) corresponds to the sequence of “removed” orbits.

\( \square \)
5. Complete triangles and embedding

In the sequel, we assume that $\mathcal{G} \in \mathbb{K}$, the graph $\mathcal{H}(\mathcal{G})$ is complete, its vertex set
is $V = \{g_0, g_1, \ldots, g_r\}$, and $g_i g_j \neq g_i$ if $i \neq j$.

For every subset $X$ of $V \setminus \{g_0\}$, let $D(X)$ denote the set of all equalities of the form
(3) $g_0 g_i = g_0 g_k g_m$

that are true in $\mathcal{G}$, where $\{g_i, g_k, \ldots, g_m\} \subseteq X$ and all these elements are pairwise
distinct. A subset $X$ is said to be independent if $D(X) = \emptyset$. By a basis we mean a set of the form $X \cup \{g_0\}$, where $X$ is a maximal independent subset.

We fix a basis $\{g_0, g_1, \ldots, g_n\}$. We denote $X = \{g_1, \ldots, g_n\}$. By definition, we have
$g_0 g_i \neq g_0 g_{i_1} \cdots g_{i_m}$
if $g_i \in X$ and $g_{i_1}, \ldots, g_{i_m} \in X \setminus \{g_i\}$. Since $X$ is a maximal independent subset, we conclude that, for every $g_i \notin X \cup \{g_0\}$, there exist $g_{i_1}, \ldots, g_{i_m} \in X$ such that equality (3) is true in $\mathcal{G}$. By (1) and the definition of an independent subset, we conclude that, for every $g_i \notin X$, such an equality is unique up to a permutation of the elements $g_{i_1}, \ldots, g_{i_m}$. We denote this equality by $E_i$.

Let $\Delta = \{E_i : g_i \notin X \cup \{g_0\}\}$. We show that $\mathcal{G}$ is determined, in $\mathbb{K}$, by
the generators $V$ and the defining relations $\Delta$. For more detail on defining relations, the reader is referred, for example, to [2, Section 2.1].

**Proposition 10.** The differential groupoid $\mathcal{G}$ is determined, in $\mathbb{K}$, by the set of
generators $V$ and the set of defining relations $\Delta$.

**Proof.** By the choice of $V$, the groupoid $\mathcal{G}$ is generated by $V$. By the choice of $\Delta$, each equality in $\Delta$ holds in $\mathcal{G}$.

Let an equality of the form
(4) $g_0 = g_0 g_{j_1} \cdots g_{j_s}$
hold in $\mathcal{G}$, where $g_{j_1}, \ldots, g_{j_s} \in V \setminus \{g_0\}$. Since $X$ is an independent set, we have
$g_{j_k} \notin X$, where $1 \leq k \leq s$. Without loss of generality, we may assume that $k = s$.
By (1), we conclude that (4) is equivalent, in $\mathbb{K}$, to the equality
(5) $g_0 g_{j_s} = g_0 g_{j_1} \cdots g_{j_{s-1}}$.
If $g_{j_1}, \ldots, g_{j_{s-1}} \in X$ then we obtain an equality in $\Delta$. Otherwise, we have $g_{j_k} \notin X$, where $1 \leq k < s$. Without loss of generality, we may assume that $k = 1$. Consider the corresponding equality $E_1$, i.e., the equality
(6) $g_0 g_{j_1} = g_0 g_{i_1} \cdots g_{i_m}$.
Substituting (6) into (5), we obtain

$g_0 g_{j_s} = g_0 g_{i_1} \cdots g_{i_m} g_{j_2} \cdots g_{j_{s-1}}$.

Repeating this procedure for each $g_{j_k} \notin X$, we obtain an equality in $\Delta$. We conclude
that equality (4) is a consequence of $\Delta$ in $\mathbb{K}$.

Let $i \neq 0$ and let an equality of the form
$g_i = g_i g_{j_1} \cdots g_{j_s}$
hold in $\mathcal{G}$.

If $s$ is even then $g_i = g_i g_{j_1} = g_i g_{j_1} \cdots g_{j_s}$. By (TC), the latter equality is equivalent
to the equality $g_0 g_{j_1} = g_0 g_{j_1} \cdots g_{j_s}$, i.e., it is equivalent to an equality of the form (4).
If \( s \) is odd then \( g_i = g_ig_ig_i^{-1} = g_ig_j \ldots g_j \). By (TC), the latter equality is equivalent to the equality \( g_0g_0g_0^{-1} = g_0g_j \ldots g_j \). By (1), this equality is equivalent to the equality \( g_0 = g_0g_j \ldots g_j \), i.e., it is equivalent to an equality of the form (4).

By (1), each equality \( p = q \), where \( p \) and \( q \) are terms, is a consequence, in \( K \), of the set \( \Delta \). By [2, Proposition 2.1.1], the groupoid \( G \) is determined, in \( K \), by the generators \( V \) and the defining relations \( \Delta \).

Let \( n = |X| \). We now prove that the differential groupoid determined, in \( K \), by the generators \( V \) and the defining relations \( \Delta \) is embeddable into \( C^n \).

As above, we assume that \( X = \{ g_0, \ldots, g_n \} \), \( V = \{ g_0, \ldots, g_r \} \), \( n \leq r \), and \( \Delta = \{ E_i : g_i \notin X \cup \{ g_0 \} \} \).

We define a mapping \( \alpha \) from the differential groupoid \( G \) determined, in \( K \), by the generators \( V \) and the defining relations \( \Delta \) into the groupoid \( C^n \). We put

\[
\alpha(g_0)(i) = 0_0 \text{ for all } i,
\]

\[
\alpha(g_i)(k) = 0_1 \text{ if } k = i \text{ and } 0_0 \text{ otherwise, } \quad 0 < i \leq n.
\]

For \( g_i \notin X \cup \{ g_0 \} \), we consider the corresponding equality \( E_i \) of the form \( g_0g_i = g_0g_{i_1}g_{i_m} \) and observe that, for distinct \( g_j, g_k \in X \), we have

\[
g_0g_jg_k(l) = 1_0 \text{ if } l \in \{ i, j \} \text{ and } 0_0 \text{ otherwise.}
\]

We conclude that

\[
g_0g_i(l) = \begin{cases} 1_0 & \text{if } l \in \{ i_1, \ldots, i_m \}, \\ 0_0 & \text{otherwise} \end{cases}
\]

and

\[
g_i(l) = \begin{cases} 0_1 & \text{if } l \in \{ i_1, \ldots, i_m \}, \\ 0_0 & \text{otherwise.} \end{cases}
\]

In view of identities (L) and (R), we calculate

\[
(7) \quad g_0g_{j_1} \ldots g_j \cdot g_0g_{i_1} \ldots g_i = g_0g_{j_1} \ldots g_j, g_i = g_0g_{j_1} \ldots g_j, \ldots
\]

If \( g_i \notin X \cup \{ g_0 \} \) then we substitute \( E_i \) into (7). Thus, each element of the orbit of \( g_0 \) is represented as \( g_0g_{i_1} \ldots g_{i_m} \), where \( g_{i_1}, \ldots, g_{i_m} \in X \setminus \{ g_0 \} \). In particular, if \( g_{i_1}, \ldots, g_{i_m} \in X \setminus \{ g_0 \} \) then \( g_0 \neq g_0g_{i_1} \ldots g_{i_m} \), i.e., for all \( x, y \in O(g_0) \), we have \( \alpha(x) \neq \alpha(y) \) provided \( x \neq y \).

By the above, we find that \( C^n \) satisfies all equalities in \( \Delta \). Hence, there exists a homomorphism from \( G \) into \( C^n \). If \( \alpha(x) = \alpha(y) \) then, repeating arguments from the proof of Proposition 10, we conclude that we may assume that \( x, y \in O(g_0) \); hence, \( x = y \). Thus, the homomorphism from \( G \) into \( C^n \) is an embedding.

We obtain the following main result.

**Theorem 11.** Each differential groupoid in \( D_{0,2} \) is a subreduct of a semimodule over a commutative semiring. A differential groupoids in \( D_{0,2} \) is a subreduct of a module over a commutative ring if and only if it belongs to the least subquasivariety of \( D_{0,2} \) with nonzero multiplication.

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References