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ON THE DYNAMICS OF A CLASS OF KOLMOGOROV SYSTEMS

R. BOUKOUCHA

Abstract. In this paper we charaterize the integrability and the non-existence of limit cycles of Kolmogorov systems of the form

$$\begin{cases} x' = x\left(P(x,y) + \left(\frac{R(x,y)}{S(x,y)}\right)^{\lambda}\right), \\ y' = y\left(Q(x,y) + \left(\frac{R(x,y)}{S(x,y)}\right)^{\lambda}\right), \end{cases}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n, n, m, a respectively and $\lambda \in \mathbb{Q}^*$. Concrete example exhibiting the applicability of our result is introduced.

Keywords: Kolmogorov system, first integral, periodic orbits, limit cycle.

1. Introduction

The autonomous differential system on the plane given by

$$(1) \qquad \begin{cases} x' = \dfrac{dx}{dt} = xF(x,y), \\ y' = \dfrac{dy}{dt} = yG(x,y), \end{cases}$$

is known as Kolmogorov system, the derivatives are performed with respect to the time variable, and F, G are two functions in the variables x and y, is frequently used to model the iteration of two species occupying the same ecological niche [10, 14, 16]. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [12, 17, 18] chemical reactions, plasma physics [13], hydrodynamics [5], economics, etc. In the classical

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Lotka- Volterra-Gause model, F and G are linear and it is well known that there are no limit cycles. There can, of course, only be one critical point in the interior of the realistic quadrant $(x > 0, y > 0)$ in this case, but this can be a center; however, there are no isolated periodic solutions. We remind that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1). In the qualitative theory of planar dynamical systems [4, 7, 8, 9, 15], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly [1, 2, 3, 11].

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} xF(x,y) + \frac{\partial H(x,y)}{\partial y} yG(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait [6].

In this paper we are intersted in studying the integrability and the periodic orbits of the 2-dimensional Kolmogorov systems of the form

$$(2) \qquad \begin{cases} x' = x\left(P(x,y) + \left(\frac{R(x,y)}{S(x,y)}\right)^{\lambda}\right), \\ y' = y\left(Q(x,y) + \left(\frac{R(x,y)}{S(x,y)}\right)^{\lambda}\right), \end{cases}$$

where $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n, n, m, a respectively and $\lambda \in \mathbb{Q}^*$.

We define the trigonometric functions

$f_1(\theta) = P(\cos\theta, \sin\theta)\cos^2\theta + Q(\cos\theta, \sin\theta)\sin^2\theta,\ f_2(\theta) = \left(\frac{R(\cos\theta,\sin\theta)}{S(\cos\theta,\sin\theta)}\right)^{\lambda},$
$f_3(\theta) = Q(\cos\theta, \sin\theta)\cos\theta\sin\theta - P(\cos\theta, \sin\theta)\cos\theta\sin\theta.$

2. Main result

Our main result on the integrability and the periodic orbits of the Kolmogorov system (2) is the following.

Theorem 1. *Consider a Komogorov system* (2)*, then the following statements hold.*

(*a*) *If* $f_3(\theta) \neq 0$, $S(\cos\theta, \sin\theta) \neq 0$, $R(\cos\theta, \sin\theta)S(\cos\theta, \sin\theta) \geq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$ *and* $\lambda m - \lambda a \neq n$, *then system* (2) *has the first integral*

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{n-\lambda m-\lambda a}{2}} \exp\left((\lambda m - \lambda a - n)\int_{\omega_0}^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) + \\ &\quad (\lambda m - \lambda a - n)\int_{\omega_0}^{\arctan\frac{y}{x}} \exp\left((\lambda m - \lambda a - n)\int_{\omega_0}^{w} A(\omega)\,d\omega\right) B(w)\,dw, \end{aligned}$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$ *and* ω_0 *is a number from the interval* $\left(0, \frac{\pi}{2}\right)$.

Moreover, the system (2) *has no limit cycle.*

(b) If $f_3(\theta) \neq 0,\ S(\cos\theta, \sin\theta) \neq 0,\ R(\cos\theta, \sin\theta) S(\cos\theta, \sin\theta) \geq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$ *and* $\lambda m - \lambda a = n$*, then system* (2) *has the first integral*

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{1}{2}} \exp\left(-\int_{\omega_0}^{\arctan\frac{y}{x}} \left(A(\omega) + B(\omega)\right) d\omega\right).$$

Moreover, the system (2) *has no limit cycle.*

(c) If $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$*, then system* (2) *has the first integral* $H = \frac{y}{x}$*. Moreover, the system* (2) *has no limit cycle.*

Доказательство. In order to prove our results we write the polynomial differential system (2) in Polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (2) becomes

$$\left\{\begin{array}{l} r' = f_1(\theta) r^{n+1} + f_2(\theta) r^{\lambda m - \lambda a + 1}, \\ \theta' = f_3(\theta) r^n, \end{array}\right. \tag{3}$$

where the trigonometric functions $f_1(\theta),\ f_2(\theta),\ f_3(\theta),\ A(\theta),\ B(\theta)$ are given in introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

If $f_3(\theta) \neq 0,\ S(\cos\theta, \sin\theta) \neq 0,\ R(\cos\theta, \sin\theta) S(\cos\theta, \sin\theta) \geq 0$ for $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\lambda m - \lambda a \neq n$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta} = A(\theta) r + B(\theta) r^{1-n+\lambda m - \lambda a}, \tag{4}$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)},\ B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{n-\lambda m-\lambda a}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = (n - \lambda m - \lambda a)\left(A(\theta)\rho + B(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned}\rho(\theta) &= \exp\left((n - \lambda m - \lambda a)\int_{\omega_0}^{\theta} A(\omega)\, d\omega\right) \\ &\quad \left(\alpha + (n - \lambda m - \lambda a)\int_{\omega_0}^{\theta} \exp\left((\lambda m - \lambda a - n)\int_{\omega_0}^{w} A(\omega)\, d\omega\right) B(w)\, dw\right),\end{aligned}$$

where $\alpha \in \mathbb{R}$, which has the first integral

$$\begin{aligned}H(x, y) &= \left(x^2 + y^2\right)^{\frac{n-\lambda m-\lambda a}{2}} \exp\left((\lambda m - \lambda a - n)\int_{\omega_0}^{\arctan\frac{y}{x}} A(\omega)\, d\omega\right) \\ &+ (\lambda m - \lambda a - n)\int_{\omega_0}^{\arctan\frac{y}{x}} \exp\left((\lambda m - \lambda a - n)\int_{\omega_0}^{w} A(\omega)\, d\omega\right) B(w)\, dw.\end{aligned}$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$. The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), are written as

$$r(\theta)=\left(\begin{array}{c} h\exp\left((n-\lambda m-\lambda a)\int_{\omega_0}^{\theta}A(\omega)\,d\omega\right)+\\ (n-\lambda m-\lambda a)\exp\left((n-\lambda m-\lambda a)\int_{\omega_0}^{\theta}A(\omega)\,d\omega\right)\\ \int_{\omega_0}^{\theta}\exp\left((\lambda m-\lambda a-n)\int_{\omega_0}^{w}A(\omega)\,d\omega\right)B(w)\,dw \end{array}\right)^{\frac{1}{n-\lambda m-\lambda a}}.$$

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2+y^2=\left(\begin{array}{c} h\exp\left((n-\lambda m-\lambda a)\int_{\omega_0}^{\arctan\frac{y}{x}}A(\omega)\,d\omega\right)+\\ (n-\lambda m-\lambda a)\exp\left((n-\lambda m-\lambda a)\int_{\omega_0}^{\arctan\frac{y}{x}}A(\omega)\,d\omega\right)\\ \int_{\omega_0}^{\arctan\frac{y}{x}}\exp\left((\lambda m-\lambda a-n)\int_{\omega_0}^{w}A(\omega)\,d\omega\right)B(w)\,dw \end{array}\right)^{\frac{2}{n-\lambda m-\lambda a}},$$

where $h\in\mathbb{R}$.

Therefore the periodic orbit γ is contained in the curve

$$r(\theta)=\left(\begin{array}{c} h_\Gamma\exp\left((n-\lambda m-\lambda a)\int_{\omega_0}^{\theta}A(\omega)\,d\omega\right)+\\ (n-\lambda m-\lambda a)\exp\left((n-\lambda m-\lambda a)\int_{\omega_0}^{\theta}A(\omega)\,d\omega\right)\\ \int_{\omega_0}^{\theta}\exp\left((\lambda m-\lambda a-n)\int_{\omega_0}^{w}A(\omega)\,d\omega\right)B(w)\,dw \end{array}\right)^{\frac{1}{n-\lambda m-\lambda a}}.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $(x>0,y>0)$, because this curve has at most a unique point on every ray $\theta=\theta^*$ for all $\theta^*\in\left(0,\frac{\pi}{2}\right)$.

Hence statement (a) of Theorem 1 is proved.

Suppose now that $f_3(\theta)\neq 0$, $S(\cos\theta,\sin\theta)\neq 0$, $R(\cos\theta,\sin\theta)\,S(\cos\theta,\sin\theta)\geq 0$ for $\theta\in\left(0,\frac{\pi}{2}\right)$ and $\lambda m-\lambda a=n$.

Taking as independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta}=(A(\theta)+B(\theta))\,r. \tag{6}$$

The general solution of equation (6) is

$$r(\theta)=\alpha\exp\left(\int_{\omega_0}^{\theta}(A(\omega)+B(\omega))\,d\omega\right),$$

where $\alpha\in\mathbb{R}$, which has the first integral

$$H(x,y)=\left(x^2+y^2\right)^{\frac{1}{2}}\exp\left(-\int_{\omega_0}^{\arctan\frac{y}{x}}(A(\omega)+B(\omega))\,d\omega\right).$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the realistic quadrant $(x>0,y>0)$, and let $h_\Gamma=H(\gamma)$. The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (2), are written as

$$r(\theta)=h\exp\left(\int_{\omega_0}^{\arctan\frac{y}{x}}(A(\omega)+B(\omega))\,d\omega\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_{\omega_0}^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right)d\omega\right) = 0,$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$r(\theta) = h_\Gamma \exp\left(\int_{\omega_0}^{\arctan\frac{y}{x}} \left(A(\omega)+B(\omega)\right)d\omega\right).$$

But this curve cannot contain the periodic orbit Γ, and consequently no limit cycle contained in the realistic quadrant $(x > 0, y > 0)$, because this curve at most have a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in \left(0, \frac{\pi}{2}\right)$. Hence statement (b) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$, then from system (3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.

This completes the proof of statement (c) of Theorem 1. $\square$

3. Examples

The following example are given to illustrate our result

Example 1 If we take $P(x,y) = 3x^2y - xy^2$, $Q(x,y) = x^3 + 3x^2y$, $R(x,y) = 4x^4 + 6x^2y^2 + 3y^4$, $S(x,y) = x^2 + y^2$ and $\lambda = \frac{1}{2}$, then system (2) reads

$$\left\{\begin{array}{l} x' = x\left(3x^2y - xy^2 + \sqrt{\frac{4x^4+6x^2y^2+3y^4}{x^2+y^2}}\right), \\ y' = y\left(x^3 + 3x^2y + \sqrt{\frac{4x^4+6x^2y^2+3y^4}{x^2+y^2}}\right), \end{array}\right. \tag{7}$$

the Kolmogorov system (7) in Polar coordinates (r, θ) becomes

$$\left\{\begin{array}{l} r' = \left(3\cos^2\theta\sin\theta\right)r^4 + \sqrt{3+\cos^4\theta}\,r^2, \\ \theta' = \left(\cos^2\theta\sin\theta\right)r^3, \end{array}\right.$$

here $f_1(\theta) = 3\cos^2\theta\sin\theta$, $f_2(\theta) = \sqrt{3+\cos^4\theta}$ and $f_3(\theta) = \cos^2\theta\sin\theta$. In the realistic quadrant $(x > 0, y > 0)$ it is the case (a) of the Theorem 1, then the Kolmogorov system (7) has the first integral

$$H(x,y) = \left(x^2+y^2\right)\exp\left(-6\arctan\frac{y}{x}\right) - 2\int_{\omega_0}^{\arctan\frac{y}{x}} \exp(-6w)\frac{\sqrt{3+\cos^4 w}}{\cos^2 w\sin w}dw.$$

where ω_0 is a number from the interval $\left(0, \frac{\pi}{2}\right)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (7), in Cartesian coordinates are written as

$$x^2+y^2 = h\exp\left(6\arctan\frac{y}{x}\right) + 2\exp\left(6\arctan\frac{y}{x}\right)\int_{\omega_0}^{\arctan\frac{y}{x}} \exp(-6w)\frac{\sqrt{3+\cos^4 w}}{\cos^2 w\sin w}dw,$$

where $h \in \mathbb{R}$. Clearly the system (7) has no periodic orbits, and consequently no limit cycle

4. Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar rational differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories, this is a one of the classical tools in the classification of all trajectories of dynamical systems.

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Boukoucha Rachid,
Department of Technology, Faculty of Technology,
University of Bejaia,
06000 Bejaia, Algeria
E-mail address: rachid_boukecha@yahoo.fr