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ASYMPTOTIC PROPERTIES OF SOLUTIONS TO A SYSTEM DESCRIBING THE SPREAD OF AVIAN INFLUENZA

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Abstract. In the present paper we consider a system of delay differential equations describing the spread of avian influenza between birds migrating between two territories. We study the asymptotic stability of the zero solution and the periodic solution corresponding to healthy birds. We establish estimates of solutions characterizing the rate of convergence to the zero solution, and also attraction domains and estimates of solutions characterizing the rate of convergence to the periodic solution. The results are obtained by the use of a solution to the special boundary value problem for the Lyapunov differential equation.

Keywords: birds' migration, avian influenza, delay differential equations, ordinary differential equations, Lyapunov differential equation, asymptotic stability, estimates of solutions, attraction domains.

1. Introduction

In the present paper we consider a system of delay differential equations describing the spread of avian influenza between birds migrating between two territories [1]:

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$$
\begin{cases}
\dfrac{d}{dt}S_w(t) = -[\mu_w^s + m_{wb}(t)]S_w(t) \\
\qquad\qquad +\alpha_{bw}^s m_{bw}(t-\tau_{bw})S_b(t-\tau_{bw}) - \dfrac{\beta_w S_w(t) I_w(t)}{S_w(t)+I_w(t)}, \\
\dfrac{d}{dt}S_b(t) = -[\mu_b^s + m_{bw}(t)]S_b(t) + b(t)S_b(t)\left(1 - \dfrac{S_b(t)}{K}\right) \\
\qquad\qquad +\alpha_{wb}^s m_{wb}(t-\tau_{wb})S_w(t-\tau_{wb}) - \dfrac{\beta_b S_b(t) I_b(t)}{S_b(t)+I_b(t)}, \\
\dfrac{d}{dt}I_w(t) = -[\mu_w^i + m_{wb}(t)]I_w(t) \\
\qquad\qquad +\alpha_{bw}^i m_{bw}(t-\tau_{bw})I_b(t-\tau_{bw}) + \dfrac{\beta_w S_w(t) I_w(t)}{S_w(t)+I_w(t)}, \\
\dfrac{d}{dt}I_b(t) = -[\mu_b^i + m_{bw}(t)]I_b(t) \\
\qquad\qquad +\alpha_{wb}^i m_{wb}(t-\tau_{wb})I_w(t-\tau_{wb}) + \dfrac{\beta_b S_b(t) I_b(t)}{S_b(t)+I_b(t)}.
\end{cases}
\tag{1}
$$

Here $S_w(t)$ and $S_b(t)$ are the numbers of susceptible birds at the winter and summer territories, $I_w(t)$ and $I_b(t)$ are the numbers of infected birds at the winter and summer territories, respectively, μ_w^s and μ_b^s are the death rates of the susceptible birds, μ_w^i and μ_b^i are the death rates of the infected birds in the winter and summer territories, respectively, $m_{bw}(t)$ is the migration rate from the summer territory to the winter territory, $m_{wb}(t)$ is the migration rate from the winter territory to the summer territory, the time delays $\tau_{bw} > 0$ and $\tau_{wb} > 0$ represent the time flying from one territory to another, α_{bw}^s and α_{wb}^s are survival probabilities of the susceptible birds during migrations, α_{bw}^i and α_{wb}^i are survival probabilities of the infected birds during migrations, $b(t)$ is the birth rate of the susceptible birds, K is the carrying capacity of the summer territory, β_w and β_b are disease transmission rates at the winter and summer territories, respectively.

It is assumed that $m_{wb}(t)$, $m_{bw}(t)$, and $b(t)$ are periodic functions with the period $T = 365$ days. Denote by t_0 the time when the birds begin to fly to the summer territory in a particular year. Let t_1 be the time when the birds at the winter territory stop their spring migration to the summer territory. Suppose that the birds begin the autumn migration at time t_2 and stop at time t_3. Then $t_0 + \tau_{wb}$ and $t_1 + \tau_{wb}$ are the beginning and the end of birds' arrival at the summer territory, $t_2 + \tau_{bw}$ and $t_3 + \tau_{bw}$ are the beginning and the end of birds' arrival at the winter territory. Suppose

$$t_0 < t_0 + \tau_{wb} < t_1 < t_1 + \tau_{wb} < t_2 < t_2 + \tau_{bw} < t_3 < t_3 + \tau_{bw} < t_0 + T.$$

It is assumed that the functions $m_{wb}(t)$, $m_{bw}(t)$, and $b(t)$ are piecewise constant:

$$m_{wb}(t) = \begin{cases} M_{wb}, & t \in [t_0, t_1], \\ 0, & t \in (t_1, t_0 + T), \end{cases} \qquad m_{bw}(t) = \begin{cases} M_{bw}, & t \in [t_2, t_3], \\ 0, & t \in (t_3, t_2 + T), \end{cases}$$

$$b(t) = \begin{cases} b_0, & t \in [t_0 + \tau_{wb}, t_2], \\ 0, & t \in (t_2, t_0 + \tau_{wb} + T). \end{cases}$$

System (1) without disease ($I_w(t) \equiv I_b(t) \equiv 0$) has the form [2, 3]:

$$\begin{cases} \dfrac{d}{dt}S_w(t) = -[\mu_w^s + m_{wb}(t)]S_w(t) \\ \qquad\qquad +\alpha_{bw}^s m_{bw}(t-\tau_{bw})S_b(t-\tau_{bw}), \\ \dfrac{d}{dt}S_b(t) = -[\mu_b^s + m_{bw}(t)]S_b(t) + b(t)S_b(t)\left(1 - \dfrac{S_b(t)}{K}\right) \\ \qquad\qquad +\alpha_{wb}^s m_{wb}(t-\tau_{wb})S_w(t-\tau_{wb}). \end{cases} \tag{2}$$

For this system in [3] it was obtained conditions of the asymptotic stability of the zero solution, and also conditions of the existence of periodic solution $(S_w(t), S_b(t)) = (S_w^*(t)S_b^*(t))$ and its asymptotic stability (see Theorem 1 in Section 2).

Using the results [3], in [1] it was obtained sufficient conditions of the asymptotic stability of the zero solution and periodic solution $(S_w(t), S_b(t), I_w(t), I_b(t)) = (S_w^*(t)S_b^*(t), 0, 0)$ to system (1), corresponding to healthy population (see Theorem 2 in Section 2).

Our aim is to establish estimates of the rate of convergence of solutions to the zero solution and the periodic solution to system (1).

2. Stability of solutions

2.1. The model of healthy birds

At first we consider system (2). In [4] it was shown that this system is equivalent to a system of ordinary differential equations.

Lemma 1 (**[4]**). *The vector-function $(S_w(t), S_b(t))$ is a solution to system (2) if and only if it is a solution to the system of ordinary differential equations*

$$\frac{d}{dt}x = A_0(t)x + F(t,x), \tag{3}$$

where

$$x(t) = \begin{pmatrix} S_w(t) \\ S_b(t) \end{pmatrix}, \quad F(t,x) = -\begin{pmatrix} 0 \\ \frac{b(t)}{K}S_b^2 \end{pmatrix},$$

$$A_0(t) = \begin{pmatrix} -[\mu_w^s + m_{wb}(t)] & \alpha_{bw}^s m_{bw}(t-\tau_{bw})\xi_{bw}^s(t) \\ \alpha_{wb}^s m_{wb}(t-\tau_{wb})\xi_{wb}^s(t) & -[\mu_b^s + m_{bw}(t)] + b(t) \end{pmatrix},$$

$\xi_{wb}^s(t)$ and $\xi_{bw}^s(t)$ are T-periodic functions,

$$\xi_{wb}^s(t) = \begin{cases} \exp\left([\mu_w^s + M_{wb}]\tau_{wb}\right), \quad t \in [t_0 + \tau_{wb}, t_1], \\ \exp\left([\mu_w^s + M_{wb}]\tau_{wb} - M_{wb}(t - t_1)\right), \quad t \in [t_1, t_1 + \tau_{wb}], \\ 0, \quad t \in (t_1 + \tau_{wb}, t_0 + \tau_{wb} + T), \end{cases}$$

$$\xi_{bw}^s(t) = \begin{cases} \exp\left([\mu_b^s + M_{bw}]\tau_{bw}\right), \quad t \in [t_2 + \tau_{bw}, t_3], \\ \exp\left([\mu_b^s + M_{bw}]\tau_{bw} - M_{bw}(t - t_3)\right), \quad t \in [t_3, t_3 + \tau_{bw}], \\ 0, \quad t \in (t_3 + \tau_{bw}, t_2 + \tau_{bw} + T). \end{cases}$$

For system (3) we specify initial conditions

$$S_w(t_0) = S_w^0 \geq 0, \quad S_b(t_0) = S_b^0 \geq 0, \quad S_w^0 + S_b^0 > 0. \tag{4}$$

It is easy to show that a solution to the Cauchy problem (3), (4) is defined for all $t \geq t_0$ and $S_w(t) \geq 0$, $S_b(t) \geq 0$ for $t \geq t_0$. Now we formulate a result on the asymptotic stability following from [3].

Theorem 1 (**[3]**)**.** *Let $X(t)$ be a solution to the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}X = A_0(t)X, \\ \\ X(t_0) = I, \end{cases}$$

where I is identity matrix. Denote by $\rho(X(t_0+T))$ the spectral radius of matrix $X(t_0+T)$.
I) If $\rho(X(t_0+T)) < 1$, then the zero solution to system (3) is globally asymptotically stable.
II) If $\rho(X(t_0+T)) > 1$, then there exists a periodic solution to system (3), which is globally asymptotically stable.

Remark 1. Let $\rho(X(t_0+T)) > 1$ and $(S_w^*(t), S_b^*(t))$ is the periodic solution to system (3). The change of variables $z_w(t) = S_w(t) - S_w^*(t)$, $z_b(t) = S_b(t) - S_b^*(t)$ leads to the system

$$\frac{d}{dt}z = A_p(t)z + F(t,z),$$

where

$$z(t) = \begin{pmatrix} z_w(t) \\ z_b(t) \end{pmatrix}, \quad F(t,z) = -\begin{pmatrix} 0 \\ \frac{b(t)}{K} z_b^2 \end{pmatrix},$$

$$A_p(t) = \begin{pmatrix} -[\mu_w^s + m_{wb}(t)] & \alpha_{bw}^s m_{bw}(t-\tau_{bw})\xi_{bw}^s(t) \\ \\ \alpha_{wb}^s m_{wb}(t-\tau_{wb})\xi_{wb}^s(t) & -[\mu_b^s + m_{bw}(t)] + b(t)\left(1 - \frac{2S_b^*(t)}{K}\right) \end{pmatrix}.$$

Remark 2. Note that estimates of the rate of convergence of solutions to the zero solution and the periodic solution to system (3) were obtained in [5].

2.2. The model of avian influenza

Before we begin to study the properties of solutions to the system (1), we consider a system, which is obtained, if in the last two equations of system (1) we replace the last summands by $\beta_w I_w(t)$ and $\beta_b I_b(t)$ respectively:

$$\begin{cases} \dfrac{d}{dt}I_w(t) = -[\mu_w^i + m_{wb}(t)]I_w(t) \\ \qquad\qquad + \alpha_{bw}^i m_{bw}(t-\tau_{bw})I_b(t-\tau_{bw}) + \beta_w I_w(t), \\ \\ \dfrac{d}{dt}I_b(t) = -[\mu_b^i + m_{bw}(t)]I_b(t) \\ \qquad\qquad + \alpha_{wb}^i m_{wb}(t-\tau_{wb})I_w(t-\tau_{wb}) + \beta_b I_b(t). \end{cases} \tag{5}$$

At first we formulate an auxiliary statement.

Lemma 2 (**[4]**)**.** *The vector-function $(I_w(t), I_b(t))$ is a solution to system* (5) *if and only if it is a solution to the system of ordinary differential equations*

$$\frac{d}{dt}u = B_p(t)u, \tag{6}$$

where

$$u(t) = \begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix}, \quad B_p(t) = \begin{pmatrix} -[\mu_w^i - \beta_w + m_{wb}(t)] & \alpha_{bw}^i m_{bw}(t-\tau_{bw})\eta_{bw}^i(t) \\ \alpha_{wb}^i m_{wb}(t-\tau_{wb})\eta_{wb}^i(t) & -[\mu_b^i - \beta_b + m_{bw}(t)] \end{pmatrix},$$

$\eta_{wb}^i(t)$ *and* $\eta_{bw}^i(t)$ *are* T*-periodic functions,*

$$\eta_{wb}^i(t) = \begin{cases} \exp\left([\mu_w^i - \beta_w + M_{wb}]\tau_{wb}\right), & t \in [t_0 + \tau_{wb}, t_1], \\ \exp\left([\mu_w^i - \beta_w + M_{wb}]\tau_{wb} - M_{wb}(t - t_1)\right), & t \in [t_1, t_1 + \tau_{wb}], \\ 0, \quad t \in (t_1 + \tau_{wb}, t_0 + \tau_{wb} + T), \end{cases}$$

$$\eta_{bw}^i(t) = \begin{cases} \exp\left([\mu_b^i - \beta_b + M_{bw}]\tau_{bw}\right), & t \in [t_2 + \tau_{bw}, t_3], \\ \exp\left([\mu_b^i - \beta_b + M_{bw}]\tau_{bw} - M_{bw}(t - t_3)\right), & t \in [t_3, t_3 + \tau_{bw}], \\ 0, \quad t \in (t_3 + \tau_{bw}, t_2 + \tau_{bw} + T). \end{cases}$$

For system (6) we specify initial conditions

$$I_w(t_0) = I_w^0 \geq 0, \quad I_b(t_0) = I_b^0 \geq 0. \tag{7}$$

It is obvious that for a solution to the Cauchy problem (6), (7) we have $I_w(t) \geq 0$ and $I_b(t) \geq 0$ for all $t \geq t_0$. Let $U(t)$ be a matrizant of system (6):

$$\begin{cases} \dfrac{d}{dt}U = B_p(t)U, \\ U(t_0) = I. \end{cases}$$

It is well known that if $\rho(U(t_0 + T)) < 1$, then the zero solution to system (6) is globally asymptotically stable.

Now consider system (1). For this system specify initial conditions (4), (7). It is easy to show that a solution to the Cauchy problem (1), (4), (7) is defined for all $t \geq t_0$ and $S_w(t) \geq 0$, $S_b(t) \geq 0$, $I_w(t) \geq 0$, $I_b(t) \geq 0$ for $t \geq t_0$.

We formulate a result on the asymptotic stability following from [1, 3].

Theorem 2 (**[1, 3]**).
I) *If* $\rho(X(t_0 + T)) < 1$ *and* $\rho(U(t_0 + T)) < 1$, *then the zero solution to system* (1) *is globally asymptotically stable.*
II) *If* $\rho(X(t_0 + T)) > 1$, *then there exists a periodic solution to system (1) of the form* $(S_w(t), S_b(t), I_w(t), I_b(t)) = (S_w^*(t), S_b^*(t), 0, 0)$. *Moreover, if* $\rho(U(t_0+T)) < 1$, *then the periodic solution is globally asymptotically stable.*

In sections 3 and 4 we will obtain estimates of the rate of convergence of solutions to the zero solution and the periodic solution to system (1).

3. Estimates of the rate of convergence to the zero solution

In this section we assume that $\rho(X(t_0+T)) < 1$ and $\rho(U(t_0+T)) < 1$. Therefore, by Theorem 2 the zero solution to system (1) is globally asymptotically stable. We obtain estimates of solutions to system (1) characterizing the rate of convergence to the zero solution.

From the results formulated in the previous section it follows that obtaining estimates of solutions to system of delay differential equations (1) can be reduced to obtaining estimates of solutions to a system of ordinary differential equations with periodic coefficients in linear terms. In this case we can use the results [6, 7].

At first consider a linear system of ordinary differential equations with T-periodic coefficients

$$\frac{d}{dt}y = A(t)y, \quad A(t+T) \equiv A(t). \tag{8}$$

In [6] by the use of the special boundary value problem for the Lyapunov differential equation

$$\begin{cases} \dfrac{d}{dt}H + HA(t) + A^*(t)H = -I, \quad t \in [0,T], \\ H(0) = H(T) > 0, \end{cases} \tag{9}$$

it was obtained conditions for the asymptotic stability of the zero solution to system (8) and estimates of solutions characterizing the decay rate at infinity.

Theorem 3 (**[6]**). *The zero solution to system (8) is asymptotically stable if and only if there exists a solution $H(t) = H^*(t)$ to the special boundary value problem* (9)*. Moreover, $H(t) > 0$ for $t \in [0,T]$.*

Theorem 4 (**[6]**). *Assume that there exists a solution $H(t) = H^*(t)$ to problem* (9)*. Extend this matrix T-periodically to $\mathbb{R}_+$ keeping the same notation. Then the estimate holds*

$$\|y(t)\| \le \sqrt{\frac{\|H(0)\|}{h_{\min}(t)}} \exp\left(-\int_0^t \frac{1}{2\|H(s)\|}ds\right)\|y(0)\|,$$

where $h_{\min}(t)$ is the minimal eigenvalue of matrix $H(t)$.

Here and throughout the paper we use the spectral norm of a matrix.

Theorem 3 is an analogue to the Lyapunov criterion of the asymptotic stability in the case of constant matrix A. The estimate in Theorem 4 is an analogue to the Krein estimate in the case of constant matrix A. It is important to note that finding a solution to the boundary value problem (9) is well-conditioned and it is not required to calculate the spectrum of a monodromy matrix [6].

In [7] using $H(t)$ it was obtained estimates of solutions and attraction domains of the zero solution for nonlinear systems of ordinary differential equations

$$\frac{d}{dt}y = A(t)y + f(t,y), \quad A(t+T) \equiv A(t).$$

Apply the results [6, 7] to system (1). The following theorem takes place.

Theorem 5. *Assume that there exist a solution $H_0(t) = H_0^*(t)$ to problem (9) with the matrix $A(t) \equiv A_0(t)$ and a solution $G_p(t) = G_p^*(t)$ to problem (9) with the matrix $A(t) \equiv B_p(t)$. Extend them T-periodically to $\mathbb{R}_+$ keeping the same notation. Then the zero solution to system (1) is globally asymptotically stable, herewith for the solution $(x(t),u(t)) = (S_w(t), S_b(t), I_w(t), I_b(t))$ the following estimates hold*

$$\|x(t)\| \le \sqrt{\frac{\|H_0(t_0)\|}{h^0_{\min}(t)}} \exp\left(-\int_{t_0}^t \frac{1}{2\|H_0(s)\|}ds\right)\|x(t_0)\|, \tag{10}$$

$$\|u(t)\| \le \sqrt{\frac{\|G_p(t_0)\|}{g^p_{\min}(t)}} \exp\left(-\int_{t_0}^t \frac{1}{2\|G_p(s)\|}ds\right)\|u(t_0)\|, \tag{11}$$

where $h^0_{\min}(t)$ and $g^p_{\min}(t)$ are the minimal eigenvalues of matrices $H_0(t)$ and $G_p(t)$, respectively.

Proof. Let $(S_w(t), S_b(t), I_w(t), I_b(t))$ be a solution to the Cauchy problem (1), (4), (7) and $(\widehat{S}_w(t), \widehat{S}_b(t), \widehat{I}_w(t), \widehat{I}_b(t))$ be a solution to system

$$\begin{cases} \dfrac{d}{dt}\widehat{S}_w(t) = -[\mu_w^s + m_{wb}(t)]\widehat{S}_w(t) \\ \qquad\qquad +\alpha_{bw}^s m_{bw}(t-\tau_{bw})\widehat{S}_b(t-\tau_{bw}), \\ \dfrac{d}{dt}\widehat{S}_b(t) = -[\mu_b^s + m_{bw}(t)]\widehat{S}_b(t) + b(t)\widehat{S}_b(t) \\ \qquad\qquad +\alpha_{wb}^s m_{wb}(t-\tau_{wb})\widehat{S}_w(t-\tau_{wb}), \\ \dfrac{d}{dt}\widehat{I}_w(t) = -[\mu_w^i + m_{wb}(t)]\widehat{I}_w(t) \\ \qquad\qquad +\alpha_{bw}^i m_{bw}(t-\tau_{bw})\widehat{I}_b(t-\tau_{bw}) + \beta_w\widehat{I}_w(t), \\ \dfrac{d}{dt}\widehat{I}_b(t) = -[\mu_b^i + m_{bw}(t)]\widehat{I}_b(t) \\ \qquad\qquad +\alpha_{wb}^i m_{wb}(t-\tau_{wb})\widehat{I}_w(t-\tau_{wb}) + \beta_b\widehat{I}_b(t) \end{cases} \tag{12}$$

with initial conditions (4), (7) (the system is obtained from (1) if in the first two equations we delete all nonlinear summands and in the last two equations we replace the last summands by $\beta_w I_w(t)$ and $\beta_b I_b(t)$, respectively). We show that

$$S_w(t) \le \widehat{S}_w(t), \quad S_b(t) \le \widehat{S}_b(t), \quad I_w(t) \le \widehat{I}_w(t), \quad I_b(t) \le \widehat{I}_b(t), \quad t \ge t_0. \tag{13}$$

Indeed, from (1) and (12) it is easy to obtain

$$\begin{cases} \dfrac{d}{dt}\left(S_w(t) - \widehat{S}_w(t)\right) \le -[\mu_w^s + m_{wb}(t)]\left(S_w(t) - \widehat{S}_w(t)\right) \\ \qquad\qquad +\alpha_{bw}^s m_{bw}(t-\tau_{bw})\left(S_b(t-\tau_{bw}) - \widehat{S}_b(t-\tau_{bw})\right), \\ \dfrac{d}{dt}\left(S_b(t) - \widehat{S}_b(t)\right) \le -[\mu_b^s + m_{bw}(t) - b(t)]\left(S_b(t) - \widehat{S}_b(t)\right) \\ \qquad\qquad +\alpha_{wb}^s m_{wb}(t-\tau_{wb})\left(S_w(t-\tau_{wb}) - \widehat{S}_w(t-\tau_{wb})\right), \\ S_w(t_0) - \widehat{S}_w(t_0) = 0, \quad S_b(t_0) - \widehat{S}_b(t_0) = 0, \end{cases}$$

$$\begin{cases} \dfrac{d}{dt}\left(I_w(t) - \widehat{I}_w(t)\right) \le -[\mu_w^i - \beta_w + m_{wb}(t)]\left(I_w(t) - \widehat{I}_w(t)\right) \\ \qquad\qquad +\alpha_{bw}^i m_{bw}(t-\tau_{bw})\left(I_b(t-\tau_{bw}) - \widehat{I}_b(t-\tau_{bw})\right), \\ \dfrac{d}{dt}\left(I_b(t) - \widehat{I}_b(t)\right) \le -[\mu_b^i - \beta_b + m_{bw}(t)]\left(I_b(t) - \widehat{I}_b(t)\right) \\ \qquad\qquad +\alpha_{wb}^i m_{wb}(t-\tau_{wb})\left(I_w(t-\tau_{wb}) - \widehat{I}_w(t-\tau_{wb})\right), \\ I_w(t_0) - \widehat{I}_w(t_0) = 0, \quad I_b(t_0) - \widehat{I}_b(t_0) = 0. \end{cases}$$

Therefore,

$$\begin{cases} S_w(t) - \widehat{S}_w(t) \le \int\limits_{t_0}^{t} \exp\left(-\int\limits_{\xi}^{t} [\mu_w^s + m_{wb}(\eta)] d\eta \right) \\ \qquad\qquad \times \alpha_{bw}^s m_{bw}(\xi - \tau_{bw}) \left(S_b(\xi - \tau_{bw}) - \widehat{S}_b(\xi - \tau_{bw}) \right) d\xi, \\ S_b(t) - \widehat{S}_b(t) \le \int\limits_{t_0}^{t} \exp\left(-\int\limits_{\xi}^{t} [\mu_b^s + m_{bw}(\eta) - b(\eta)] d\eta \right) \\ \qquad\qquad \times \alpha_{wb}^s m_{wb}(\xi - \tau_{wb}) \left(S_w(\xi - \tau_{wb}) - \widehat{S}_w(\xi - \tau_{wb}) \right) d\xi, \end{cases}$$

$$\begin{cases} I_w(t) - \widehat{I}_w(t) \le \int\limits_{t_0}^{t} \exp\left(-\int\limits_{\xi}^{t} [\mu_w^i - \beta_w + m_{wb}(\eta)] d\eta \right) \\ \qquad\qquad \times \alpha_{bw}^i m_{bw}(\xi - \tau_{bw}) \left(I_b(\xi - \tau_{bw}) - \widehat{I}_b(\xi - \tau_{bw}) \right) d\xi, \\ I_b(t) - \widehat{I}_b(t) \le \int\limits_{t_0}^{t} \exp\left(-\int\limits_{\xi}^{t} [\mu_b^i - \beta_b + m_{bw}(\eta)] d\eta \right) \\ \qquad\qquad \times \alpha_{wb}^i m_{wb}(\xi - \tau_{wb}) \left(I_w(\xi - \tau_{wb}) - \widehat{I}_w(\xi - \tau_{wb}) \right) d\xi. \end{cases}$$

From here it directly follows (13).

By Lemmas 1 and 2 the vector-function $(\widehat{S}_w(t), \widehat{S}_b(t), \widehat{I}_w(t), \widehat{I}_b(t))$ is a solution to system of ordinary differential equations

$$\begin{cases} \dfrac{d}{dt} x = A_0(t) x, \\ \dfrac{d}{dt} u = B_p(t) u. \end{cases}$$

Therefore, it is sufficient to establish estimates (10) and (11) for solutions to this system. These estimates are proved easily by the use of the Lyapunov functions $\langle H_0(t)x, x\rangle$ and $\langle G_p(t)u, u\rangle$ by analogy with the proof of Theorem 4 [6].

Theorem is proved. □

Remark. Existence of solutions to the boundary value problem (9) with matrices $A(t) \equiv A_0(t)$ and $A(t) \equiv B_p(t)$ is equivalent to the conditions $\rho(X(t_0 + T)) < 1$ and $\rho(U(t_0 + T)) < 1$.

4. Estimates of the rate of convergence to the periodic solution

In this section we assume that $\rho(X(t_0+T)) > 1$ and $\rho(U(t_0+T)) < 1$. Therefore, by Theorem 2 there exists globally asymptotically stable periodic solution to system (1) of the form $(S_w(t), S_b(t), I_w(t), I_b(t)) = (S_w^*(t), S_b^*(t), 0, 0)$, which corresponds to healthy birds. We obtain estimates of solutions to system (1) characterizing the rate of convergence to this periodic solution.

Let $(S_w(t), S_b(t), I_w(t), I_b(t))$ be a solution to system (1) with initial conditions (4), (7). Denote

$$z(t) = \begin{pmatrix} z_w(t) \\ z_b(t) \end{pmatrix} = \begin{pmatrix} S_w(t) - S_w^*(t) \\ S_b(t) - S_b^*(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix}, \quad v(t) = \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}.$$

Remark. In virtue of (1) the vector-function $v(t)$ is a solution to system

$$\begin{cases} \dfrac{d}{dt} z_w(t) = -[\mu_w^s + m_{wb}(t)] z_w(t) \\ \qquad + \alpha_{bw}^s m_{bw}(t - \tau_{bw}) z_b(t - \tau_{bw}) - \beta_w \dfrac{(S_w^*(t) + z_w(t)) I_w(t)}{S_w^*(t) + z_w(t) + I_w(t)}, \\ \dfrac{d}{dt} z_b(t) = \left(-[\mu_b^s + m_{bw}(t)] + b(t) \left(1 - \dfrac{2 S_b^*(t)}{K} \right) \right) z_b(t) \\ \qquad + \alpha_{wb}^s m_{wb}(t - \tau_{wb}) z_w(t - \tau_{wb}) - \dfrac{b(t)}{K} z_b^2(t) - \beta_b \dfrac{(S_b^*(t) + z_b(t)) I_b(t)}{S_b^*(t) + z_b(t) + I_b(t)}, \\ \dfrac{d}{dt} I_w(t) = -[\mu_w^i + m_{wb}(t)] I_w(t) \\ \qquad + \alpha_{bw}^i m_{bw}(t - \tau_{bw}) I_b(t - \tau_{bw}) + \beta_w \dfrac{(S_w^*(t) + z_w(t)) I_w(t)}{S_w^*(t) + z_w(t) + I_w(t)}, \\ \dfrac{d}{dt} I_b(t) = -[\mu_b^i + m_{bw}(t)] I_b(t) \\ \qquad + \alpha_{wb}^i m_{wb}(t - \tau_{wb}) I_w(t - \tau_{wb}) + \beta_b \dfrac{(S_b^*(t) + z_b(t)) I_b(t)}{S_b^*(t) + z_b(t) + I_b(t)}. \end{cases} \tag{14}$$

The following theorem takes place.

Theorem 6. *Assume that there exist a solution*

$$H_p(t) = H_p^*(t) = \begin{pmatrix} h_{11}^p(t) & h_{12}^p(t) \\ h_{12}^p(t) & h_{22}^p(t) \end{pmatrix}$$

to problem (9) with the matrix $A(t) \equiv A_p(t)$ and a solution $G_p(t) = G_p^*(t)$ to problem (9) with the matrix $A(t) \equiv B_p(t)$. Let $\mathcal{H}_p(t) = \mathcal{H}_p^*(t)$ be a solution to problem (9) with the matrix

$$\mathcal{A}_p(t) = \begin{pmatrix} A_p(t) & -Q \\ 0 & B_p(t) \end{pmatrix}, \quad \text{where} \quad Q = \begin{pmatrix} \beta_w & 0 \\ 0 & \beta_b \end{pmatrix}.$$

Extend matrices $H_p(t)$, $G_p(t)$ and $\mathcal{H}_p(t)$ T-periodically to $\mathbb{R}_+$ keeping the same notation. Then the periodic solution $(S_w^*(t), S_b^*(t), 0, 0)$ to system (1) is asymptotically stable, herewith the following statements take place.
1) *Let $z_w(t_0) \geq 0$, $z_b(t_0) \geq 0$. Then the estimates hold*

$$\|z(t)\| \leq \sqrt{\frac{\|H_p(t_0)\|}{h_{\min}^p(t)}} \exp\left(-\int_{t_0}^{t} \frac{1}{2\|H_p(s)\|} ds \right) \|z(t_0)\|, \tag{15}$$

$$\|u(t)\| \leq \sqrt{\frac{\|G_p(t_0)\|}{g_{\min}^p(t)}} \exp\left(-\int_{t_0}^{t} \frac{1}{2\|G_p(s)\|} ds \right) \|u(t_0)\|, \tag{16}$$

where $h_{\min}^p(t)$ and $g_{\min}^p(t)$ are the minimal eigenvalues of matrices $H_p(t)$ and $G_p(t)$, respectively.

2) *Let* $z_w(t_0) \le 0$, $z_b(t_0) \le 0$. *If* $v(t_0) \in \{v^0 \in \mathbb{R}^4 : \langle \mathcal{H}_p(t_0)v^0, v^0\rangle < r_1^2\}$, *where*

$$r_1 = \left[1 - \exp\left(-\int\limits_{t_0}^{t_0+T} \frac{1}{2\|\mathcal{H}_p(s)\|} ds\right)\right]$$

$$\times \left(\int\limits_{t_0}^{t_0+T} \frac{b(\xi)}{K} \frac{1}{\sqrt{h_{22}^p(\xi)}} \exp\left(-\int\limits_{t_0}^{\xi} \frac{1}{2\|\mathcal{H}_p(s)\|} ds\right) d\xi\right)^{-1},$$

then the estimate holds

$$\|v(t)\| \le \sqrt{\frac{\|\mathcal{H}_p(t_0)\|}{\mathcal{H}^p_{\min}(t)}} \exp\left(-\int\limits_{t_0}^{t} \frac{1}{2\|\mathcal{H}_p(s)\|} ds\right) \|v(t_0)\|$$

$$\times \left(1 - \frac{\sqrt{\langle \mathcal{H}_p(t_0)v(t_0), v(t_0)\rangle}}{r_1}\right)^{-1}, \tag{17}$$

where $\mathcal{H}^p_{\min}(t)$ *is the minimal eigenvalue of matrix* $\mathcal{H}_p(t)$.

3) *Let* $z_w(t_0)z_b(t_0) < 0$. *If* $v(t_0) \in \{v^0 \in \mathbb{R}^4 : \langle \mathcal{H}_p(t_0)v^0, v^0\rangle < r_2^2\}$, *where*

$$r_2 = \left[1 - \exp\left(-\int\limits_{t_0}^{t_0+T} \frac{1}{2\|\mathcal{H}_p(s)\|} ds\right)\right]$$

$$\times \left(\int\limits_{t_0}^{t_0+T} \frac{b(\xi)}{K} \frac{\sqrt{h_{22}^p(\xi)}}{\mathcal{H}^p_{\min}(\xi)} \exp\left(-\int\limits_{t_0}^{\xi} \frac{1}{2\|\mathcal{H}_p(s)\|} ds\right) d\xi\right)^{-1},$$

then the estimate holds

$$\|v(t)\| \le \sqrt{\frac{\|\mathcal{H}_p(t_0)\|}{\mathcal{H}^p_{\min}(t)}} \exp\left(-\int\limits_{t_0}^{t} \frac{1}{2\|\mathcal{H}_p(s)\|} ds\right) \|v(t_0)\|$$

$$\times \left(1 - \frac{\sqrt{\langle \mathcal{H}_p(t_0)v(t_0), v(t_0)\rangle}}{r_2}\right)^{-1}. \tag{18}$$

Before turning to the proof of Theorem 6, we prove auxiliary statements.

Lemma 3. *Let* $(z_w(t), z_b(t), I_w(t), I_b(t))$ *be a solution to system* (14) *with initial conditions*

$$z_w(t_0) = z_w^0, \quad z_b(t_0) = z_b^0, \quad I_w(t_0) = I_w^0 \ge 0, \quad I_b(t_0) = I_b^0 \ge 0. \tag{19}$$

Denote by $(\bar z_w(t), \bar z_b(t), \bar I_w(t), \bar I_b(t))$ *a solution to system*

$$\begin{cases} \dfrac{d}{dt} z = A_p(t) z, \\ \dfrac{d}{dt} u = B_p(t) u \end{cases} \tag{20}$$

with initial conditions (19) and by $(\widetilde z_w(t), \widetilde z_b(t), \widetilde I_w(t), \widetilde I_b(t))$ *a solution to system*

$$\begin{cases} \dfrac{d}{dt} z = A_p(t) z + F(t,z) - Qu, \\ \dfrac{d}{dt} u = B_p(t) u \end{cases} \quad or \quad \frac{d}{dt} v = \mathcal{A}_p(t) v + \begin{pmatrix} F(t,z) \\ 0 \end{pmatrix} \tag{21}$$

with initial conditions (19). (*It is obvious that* $\widetilde{I}_w(t) = \bar{I}_w(t)$, $\widetilde{I}_b(t) = \bar{I}_b(t)$.) *Then*

$$\widetilde{z}_w(t) \le z_w(t) \le \bar{z}_w(t), \quad \widetilde{z}_b(t) \le z_b(t) \le \bar{z}_b(t), \quad I_w(t) \le \bar{I}_w(t), \quad I_b(t) \le \bar{I}_b(t).$$

Proof. By analogy with the proof of Theorem 5 it is easy to establish the inequalities

$$z_w(t) \le \bar{z}_w(t), \quad z_b(t) \le \bar{z}_b(t), \quad 0 \le I_w(t) \le \bar{I}_w(t), \quad 0 \le I_b(t) \le \bar{I}_b(t).$$

We prove the estimates

$$z_w(t) \ge \widetilde{z}_w(t), \quad z_b(t) \ge \widetilde{z}_b(t).$$

Denote

$$z^-(t) = \begin{pmatrix} z_w^-(t) \\ z_b^-(t) \end{pmatrix} = -\begin{pmatrix} z_w(t) \\ z_b(t) \end{pmatrix}$$

and rewrite system (14) in the form

$$\begin{cases} \dfrac{d}{dt} z_w^-(t) = -[\mu_w^s + m_{wb}(t)] z_w^-(t) \\ \qquad + \alpha_{bw}^s m_{bw}(t-\tau_{bw}) z_b^-(t-\tau_{bw}) + \beta_w \dfrac{(S_w^*(t) - z_w^-(t)) I_w(t)}{S_w^*(t) - z_w^-(t) + I_w(t)}, \\ \dfrac{d}{dt} z_b^-(t) = \left(-[\mu_b^s + m_{bw}(t)] + b(t) \left(1 - \dfrac{2S_b^*(t)}{K} \right) \right) z_b^-(t) \\ \qquad + \alpha_{wb}^s m_{wb}(t-\tau_{wb}) z_w^-(t-\tau_{wb}) + \dfrac{b(t)}{K} (z_b^-)^2(t) + \beta_b \dfrac{(S_b^*(t) - z_b^-(t)) I_b(t)}{S_b^*(t) - z_b^-(t) + I_b(t)}, \\ \dfrac{d}{dt} I_w(t) = -[\mu_w^i + m_{wb}(t)] I_w(t) \\ \qquad + \alpha_{bw}^i m_{bw}(t-\tau_{bw}) I_b(t-\tau_{bw}) + \beta_w \dfrac{(S_w^*(t) - z_w^-(t)) I_w(t)}{S_w^*(t) - z_w^-(t) + I_w(t)}, \\ \dfrac{d}{dt} I_b(t) = -[\mu_b^i + m_{bw}(t)] I_b(t) \\ \qquad + \alpha_{wb}^i m_{wb}(t-\tau_{wb}) I_w(t-\tau_{wb}) + \beta_b \dfrac{(S_b^*(t) - z_b^-(t)) I_b(t)}{S_b^*(t) - z_b^-(t) + I_b(t)}. \end{cases}$$

It is easy to show that the components of the vector-function $(z_w^-(t), z_b^-(t), I_w(t), I_b(t))$ are bounded above by the components of a solutions to system

$$\begin{cases} \dfrac{d}{dt} z_w^-(t) = -[\mu_w^s + m_{wb}(t)] z_w^-(t) + \alpha_{bw}^s m_{bw}(t-\tau_{bw}) z_b^-(t-\tau_{bw}) + \beta_w I_w(t), \\ \dfrac{d}{dt} z_b^-(t) = \left(-[\mu_b^s + m_{bw}(t)] + b(t) \left(1 - \dfrac{2S_b^*(t)}{K} \right) \right) z_b^-(t) \\ \qquad\qquad + \alpha_{wb}^s m_{wb}(t-\tau_{wb}) z_w^-(t-\tau_{wb}) + \dfrac{b(t)}{K} (z_b^-)^2(t) + \beta_b I_b(t), \\ \dfrac{d}{dt} I_w(t) = -[\mu_w^i + m_{wb}(t)] I_w(t) + \alpha_{bw}^i m_{bw}(t-\tau_{bw}) I_b(t-\tau_{bw}) + \beta_w I_w(t), \\ \dfrac{d}{dt} I_b(t) = -[\mu_b^i + m_{bw}(t)] I_b(t) + \alpha_{wb}^i m_{wb}(t-\tau_{wb}) I_w(t-\tau_{wb}) + \beta_b I_b(t) \end{cases}$$

with initial conditions (19) (the system is obtained from the original by replacement the last summands in all equations by $\beta_w I_w(t)$, $\beta_b I_b(t)$, $\beta_w I_w(t)$, $\beta_b I_b(t)$, respectively). By analogy with Lemmas 1 and 2 we reduce this system to system of ordinary differential equations:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} z_w^-(t) \\ z_b^-(t) \end{pmatrix} = A_p(t)\begin{pmatrix} z_w^-(t) \\ z_b^-(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{b(t)}{K}(z_b^-)^2(t) \end{pmatrix} \\ + \begin{pmatrix} \beta_w & -\alpha_{bw}^s m_{bw}(t-\tau_{bw})\xi_{bw}^s(t)\beta_b\gamma_b \\ -\alpha_{wb}^s m_{wb}(t-\tau_{wb})\xi_{wb}^s(t)\beta_w\gamma_w & \beta_b \end{pmatrix}\begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix}, \\ \dfrac{d}{dt}\begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix} = B_p(t)\begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix}, \end{cases}$$

where

$$\gamma_w = \begin{cases} \dfrac{\exp([\mu_w^i - \mu_w^s]\tau_{wb}) - 1}{\mu_w^i - \mu_w^s}, & \text{if} \quad \mu_w^i \neq \mu_w^s, \\ \tau_{wb}, \quad \text{if} \quad \mu_w^i = \mu_w^s, \end{cases}$$

$$\gamma_b = \begin{cases} \dfrac{\exp([\mu_b^i - \mu_b^s]\tau_{bw}) - 1}{\mu_b^i - \mu_b^s}, & \text{if} \quad \mu_b^i \neq \mu_b^s, \\ \tau_{bw}, \quad \text{if} \quad \mu_b^i = \mu_b^s. \end{cases}$$

In turn, the components of the solution to this system are bounded above by the components of a solution to system

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} z_w^-(t) \\ z_b^-(t) \end{pmatrix} = A_p(t)\begin{pmatrix} z_w^-(t) \\ z_b^-(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{b(t)}{K}(z_b^-)^2(t) \end{pmatrix} + \begin{pmatrix} \beta_w & 0 \\ 0 & \beta_b \end{pmatrix}\begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix}, \\ \dfrac{d}{dt}\begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix} = B_p(t)\begin{pmatrix} I_w(t) \\ I_b(t) \end{pmatrix} \end{cases}$$

with initial conditions (19). Denote the solution to this system by $(\widetilde{z}_w^-(t), \widetilde{z}_b^-(t), \widetilde{I}_w(t), \widetilde{I}_b(t))$. Then

$$z_w^-(t) \le \widetilde{z}_w^-(t), \quad z_b^-(t) \le \widetilde{z}_b^-(t), \quad I_w(t) \le \widetilde{I}_w(t), \quad I_b(t) \le \widetilde{I}_b(t).$$

Taking into account $\widetilde{z}_w^-(t) = -\widetilde{z}_w(t)$ and $\widetilde{z}_b^-(t) = -\widetilde{z}_b(t)$ we obtain the required inequalities.

Lemma is proved. □

Lemma 4. *Let the conditions of Theorem* 6 *hold. Denote*

$$\mathcal{H}_p(t) = \begin{pmatrix} \mathcal{H}_{11}^p(t) & \mathcal{H}_{12}^p(t) \\ (\mathcal{H}_{12}^p)^*(t) & \mathcal{H}_{22}^p(t) \end{pmatrix}.$$

Then:
1) $\mathcal{H}_{11}^p(t) = H_p(t)$ *and the entries of this matrix are nonnegative;*
2) *the entries of matrix* $\mathcal{H}_{22}^p(t)$ *are nonnegative;*
3) *the entries of matrix* $\mathcal{H}_{12}^p(t)$ *are nonpositive.*

Proof. The equality $\mathcal{H}_{11}^p(t) = H_p(t)$ follows directly from the definitions of matrices $\mathcal{H}_p(t)$ and $H_p(t)$. We prove that the entries of matrix $\mathcal{H}_p(t)$ have fixed sign.

By the conditions of Theorem 6 $\mathcal{H}_p(t)$ is a solution to the special boundary value problem

$$\begin{cases} \dfrac{d}{dt}\mathcal{H}_p + \mathcal{H}_p\mathcal{A}_p(t) + \mathcal{A}_p^*(t)\mathcal{H}_p = -I, \quad t \in [0,T], \\ \mathcal{H}_p(0) = \mathcal{H}_p(T) > 0. \end{cases}$$

The solution to this problem can be written in the form [6]

$$\mathcal{H}_p(t) = \int_t^\infty \left(\mathcal{Y}(s)\mathcal{Y}^{-1}(t)\right)^* \left(\mathcal{Y}(s)\mathcal{Y}^{-1}(t)\right) ds,$$

where $\mathcal{Y}(t)$ is a solution to the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\mathcal{Y} = \mathcal{A}_p(t)\mathcal{Y}, \\ \mathcal{Y}(t_0) = I. \end{cases}$$

Denote

$$\widetilde{\mathcal{Y}}(s,t) = \mathcal{Y}(s)\mathcal{Y}^{-1}(t) = \begin{pmatrix} \widetilde{\mathcal{Y}}_{11}(s,t) & \widetilde{\mathcal{Y}}_{12}(s,t) \\ \widetilde{\mathcal{Y}}_{21}(s,t) & \widetilde{\mathcal{Y}}_{22}(s,t) \end{pmatrix}.$$

It is obvious that matrix $\widetilde{\mathcal{Y}}(s,t)$ is a solution to the Cauchy problem

$$\begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Y}} = \mathcal{A}_p(s)\widetilde{\mathcal{Y}}, \\ \widetilde{\mathcal{Y}}(t,t) = I. \end{cases}$$

Taking into account the explicit form of matrix $\mathcal{A}_p(s)$ we obtain

$$\begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{11} = A_p(s)\widetilde{\mathcal{Y}}_{11} - Q\widetilde{\mathcal{Y}}_{21}, \\ \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{21} = B_p(s)\widetilde{\mathcal{Y}}_{21}, \\ \widetilde{\mathcal{Y}}_{11}(t,t) = I, \\ \widetilde{\mathcal{Y}}_{21}(t,t) = 0, \end{cases} \qquad \begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{12} = A_p(s)\widetilde{\mathcal{Y}}_{12} - Q\widetilde{\mathcal{Y}}_{22}, \\ \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{22} = B_p(s)\widetilde{\mathcal{Y}}_{22}, \\ \widetilde{\mathcal{Y}}_{12}(t,t) = 0, \\ \widetilde{\mathcal{Y}}_{22}(t,t) = I. \end{cases}$$

Hence,

$$\widetilde{\mathcal{Y}}_{21}(s,t) = 0, \quad \begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{11} = A_p(s)\widetilde{\mathcal{Y}}_{11}, \\ \widetilde{\mathcal{Y}}_{11}(t,t) = I, \end{cases} \quad \begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{22} = B_p(s)\widetilde{\mathcal{Y}}_{22}, \\ \widetilde{\mathcal{Y}}_{22}(t,t) = I, \end{cases}$$

$$\begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Y}}_{12} = A_p(s)\widetilde{\mathcal{Y}}_{12} - Q\widetilde{\mathcal{Y}}_{22}, \\ \widetilde{\mathcal{Y}}_{12}(t,t) = 0. \end{cases}$$

From the explicit form of matrices $A_p(s)$ and $B_p(s)$ it is easy to obtain that for $s \ge t$ the entries of matrices $\widetilde{\mathcal{Y}}_{11}(s,t)$ and $\widetilde{\mathcal{Y}}_{22}(s,t)$ are nonnegative. Now we show that for $s \ge t$ the entries of matrix $\widetilde{\mathcal{Y}}_{12}(s,t)$ are nonpositive. Indeed,

$$\widetilde{\mathcal{Y}}_{12}(s,t) = -\int\limits_t^s \left(\widetilde{\mathcal{Y}}_{11}(s,t)\widetilde{\mathcal{Y}}_{11}^{-1}(\xi,t)\right) Q\widetilde{\mathcal{Y}}_{22}(\xi,t)d\xi.$$

Consider matrix $\widetilde{\mathcal{Z}}(s,\xi,t) = \widetilde{\mathcal{Y}}_{11}(s,t)\widetilde{\mathcal{Y}}_{11}^{-1}(\xi,t)$. This matrix is a solution to the Cauchy problem

$$\begin{cases} \dfrac{d}{ds}\widetilde{\mathcal{Z}} = A_p(s)\widetilde{\mathcal{Z}}, \\ \widetilde{\mathcal{Z}}(\xi,\xi,t) = I. \end{cases}$$

Therefore, for $s \ge \xi$ the entries of matrix $\widetilde{\mathcal{Z}}(s,\xi,t)$ are nonnegative. So, for $s \ge t$ the entries of matrix $\widetilde{\mathcal{Y}}_{12}(s,t)$ are nonpositive.

Finally, consider matrix $\mathcal{H}_p(t)$:

$$\mathcal{H}_p(t) = \begin{pmatrix} \mathcal{H}_{11}^p(t) & \mathcal{H}_{12}^p(t) \\ (\mathcal{H}_{12}^p)^*(t) & \mathcal{H}_{22}^p(t) \end{pmatrix} = \int\limits_t^\infty \widetilde{\mathcal{Y}}^*(s,t)\widetilde{\mathcal{Y}}(s,t)ds$$

$$= \int\limits_t^\infty \begin{pmatrix} \widetilde{\mathcal{Y}}_{11}^*(s,t)\widetilde{\mathcal{Y}}_{11}(s,t) & \widetilde{\mathcal{Y}}_{11}^*(s,t)\widetilde{\mathcal{Y}}_{12}(s,t) \\ \widetilde{\mathcal{Y}}_{12}^*(s,t)\widetilde{\mathcal{Y}}_{11}(s,t) & \widetilde{\mathcal{Y}}_{12}^*(s,t)\widetilde{\mathcal{Y}}_{12}(s,t) + \widetilde{\mathcal{Y}}_{22}^*(s,t)\widetilde{\mathcal{Y}}_{22}(s,t) \end{pmatrix} ds.$$

From here it directly follows that the entries of matrices $\mathcal{H}_{11}^p(t)$ and $\mathcal{H}_{22}^p(t)$ are nonnegative, as well as the entries of matrix $\mathcal{H}_{12}^p(t)$ are nonpositive.

Lemma is proved. □

Now we prove Theorem 6.

Proof. 1) Let $(z_w(t), z_b(t), I_w(t), I_b(t))$ be a solution to system (14) with initial conditions

$$z_w(t_0) = z_w^0 \ge 0, \quad z_b(t_0) = z_b^0 \ge 0, \quad I_w(t_0) = I_w^0 \ge 0, \quad I_b(t_0) = I_b^0 \ge 0.$$

Then

$$z_w(t) \ge 0, \quad z_b(t) \ge 0, \quad I_w(t) \ge 0, \quad I_b(t) \ge 0, \quad t \ge t_0.$$

Taking into account Lemma 3 we also have

$$z_w(t) \le \bar{z}_w(t), \quad z_b(t) \le \bar{z}_b(t), \quad I_w(t) \le \bar{I}_w(t), \quad I_b(t) \le \bar{I}_b(t), \quad t \ge t_0,$$

where $(\bar{z}_w(t), \bar{z}_b(t), \bar{I}_w(t), \bar{I}_b(t))$ is a solution to system (20) with the same initial conditions. Therefore, it is sufficient to establish estimates (15) and (16) for solutions to system (20), that follows directly from Theorem 4.

2) Let $(z_w(t), z_b(t), I_w(t), I_b(t))$ be a solution to system (14) with initial conditions

$$z_w(t_0) = z_w^0 \le 0, \quad z_b(t_0) = z_b^0 \le 0, \quad I_w(t_0) = I_w^0 \ge 0, \quad I_b(t_0) = I_b^0 \ge 0.$$

Then

$$z_w(t) \le 0, \quad z_b(t) \le 0, \quad I_w(t) \ge 0, \quad I_b(t) \ge 0, \quad t \ge t_0.$$

Taking into account Lemma 3 we also have

$$z_w(t) \ge \widetilde{z}_w(t), \quad z_b(t) \ge \widetilde{z}_b(t), \quad I_w(t) \le \widetilde{I}_w(t), \quad I_b(t) \le \widetilde{I}_b(t), \quad t \ge t_0,$$

where $(\widetilde{z}_w(t), \widetilde{z}_b(t), \widetilde{I}_w(t), \widetilde{I}_b(t))$ is a solution to system (21) with the same initial conditions. So, it is sufficient to establish estimate (17) for solutions to system (21).

Let $\mathcal{H}_p(t) = \mathcal{H}_p^*(t)$ be a solution to problem (9) with matrix $\mathcal{A}_p(t)$. Then by Theorem 3 we have $\mathcal{H}_p(t) > 0$ for $t \in [0, T]$. Since matrix $\mathcal{H}_p(t)$ is T-periodic, we also have $\mathcal{H}_p(t) > 0$ for $t \geq t_0$. Consider the Lyapunov function

$$\mathcal{V}_p(t, v) = \langle \mathcal{H}_p(t)v, v \rangle.$$

Differentiating it along the solutions to system (21), we obtain

$$\frac{d}{dt}\mathcal{V}_p(t, v(t)) \equiv \left\langle \left[\frac{d}{dt}\mathcal{H}_p(t) + \mathcal{H}_p(t)\mathcal{A}_p(t) + \mathcal{A}_p^*(t)\mathcal{H}_p(t)\right] v(t), v(t) \right\rangle$$
$$+2\left\langle \mathcal{H}_p(t)v(t), \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle.$$

Since $\mathcal{H}_p(t)$ is a solution to the special boundary value problem (9), then

$$\frac{d}{dt}\mathcal{V}_p(t, v(t)) \equiv -\|v(t)\|^2 + 2\left\langle \mathcal{H}_p(t)v(t), \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle$$
$$\leq -\frac{1}{\|\mathcal{H}_p(t)\|}\mathcal{V}_p(t, v(t)) + 2\left\langle \mathcal{H}_p(t)v(t), \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle. \tag{22}$$

Now we obtain $\left\langle \mathcal{H}_p(t)v(t), \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle$. We have:

$$\left\langle \mathcal{H}_p(t)v(t), \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle = \left\langle \mathcal{H}_p^{1/2}(t)v(t), \mathcal{H}_p^{1/2}(t) \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle$$
$$\leq \sqrt{\langle \mathcal{H}_p(t)v(t), v(t) \rangle} \sqrt{\left\langle \mathcal{H}_p(t) \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix}, \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle}$$
$$= \mathcal{V}_p^{1/2}(t, v(t)) \sqrt{\langle \mathcal{H}_{11}^p(t)F(t, z(t)), F(t, z(t)) \rangle}$$
$$= \mathcal{V}_p^{1/2}(t, v(t)) \sqrt{\left\langle \begin{pmatrix} h_{11}^p(t) & h_{12}^p(t) \\ h_{21}^p(t) & h_{22}^p(t) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{b(t)}{K}z_b^2(t) \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{b(t)}{K}z_b^2(t) \end{pmatrix} \right\rangle}$$
$$= \mathcal{V}_p^{1/2}(t, v(t)) \sqrt{h_{22}^p(t)} \, \frac{b(t)}{K} z_b^2(t). \tag{23}$$

By Lemma 4 the entries of matrix $\mathcal{H}_{11}^p(t)$ are nonnegative. Therefore,

$$h_{22}^p(t)z_b^2(t) \leq h_{11}^p(t)z_w^2(t) + 2h_{12}^p(t)z_w(t)z_b(t) + h_{22}^p(t)z_b^2(t) = \langle \mathcal{H}_{11}^p(t)z(t), z(t) \rangle.$$

Next, since the entries of matrix $\mathcal{H}_{12}^p(t)$ are nonpositive and the entries of matrix $\mathcal{H}_{22}^p(t)$ are nonnegative, then

$$\langle \mathcal{H}_{11}^p(t)z(t), z(t) \rangle \leq \langle \mathcal{H}_{11}^p(t)z(t), z(t) \rangle + \langle \mathcal{H}_{12}^p(t)u(t), z(t) \rangle + \langle (\mathcal{H}_{12}^p)^*(t)z(t), u(t) \rangle$$
$$+ \langle \mathcal{H}_{22}^p(t)u(t), u(t) \rangle = \langle \mathcal{H}_p(t)v(t), v(t) \rangle = \mathcal{V}_p(t, v(t)).$$

From here and from (23) we obtain

$$\left\langle \mathcal{H}_p(t)v(t), \begin{pmatrix} F(t, z(t)) \\ 0 \end{pmatrix} \right\rangle \leq \frac{b(t)}{K} \, \frac{1}{\sqrt{h_{22}^p(t)}} \, \mathcal{V}_p^{3/2}(t, v(t)).$$

Taking into account estimate (22) we have

$$\frac{d}{dt}\mathcal{V}_p(t, v(t)) \leq -\frac{1}{\|\mathcal{H}_p(t)\|}\mathcal{V}_p(t, v(t)) + 2\frac{b(t)}{K} \, \frac{1}{\sqrt{h_{22}^p(t)}} \, \mathcal{V}_p^{3/2}(t, v(t)).$$

Under conditions of the theorem $\mathcal{V}_p(t_0, v(t_0)) = \langle \mathcal{H}_p(t_0)v(t_0), v(t_0)\rangle < r_1^2$. Therefore, using the Gronwall inequality (see, for example, [7, 8]) we obtain

$$\mathcal{V}_p(t, v(t)) \leq \mathcal{V}_p(t_0, v(t_0)) \exp\left(-\int\limits_{t_0}^{t} \frac{1}{\|\mathcal{H}_p(s)\|} ds\right)\left(1 - \frac{\sqrt{\mathcal{V}_p(t_0, v(t_0))}}{r_1}\right)^{-2}.$$

Now it is easy to obtain the required estimate.

3) Let $(z_w(t), z_b(t), I_w(t), I_b(t))$ be a solution to system (14) with initial conditions

$$z_w(t_0) = z_w^0, \quad z_b(t_0) = z_b^0, \quad z_w^0 z_b^0 < 0,$$

$$I_w(t_0) = I_w^0 \geq 0, \quad I_b(t_0) = I_b^0 \geq 0.$$

If $(\widetilde{z}_w(t), \widetilde{z}_b(t), \widetilde{I}_w(t), \widetilde{I}_b(t))$ is a solution to system (21) and $(\bar{z}_w(t), \bar{z}_b(t), \widetilde{I}_w(t), \widetilde{I}_b(t))$ is a solution to system (20) with the same initial conditions, then by Lemma 4

$$\widetilde{z}_w(t) \leq z_w(t) \leq \bar{z}_w(t), \quad \widetilde{z}_b(t) \leq z_b(t) \leq \bar{z}_b(t),$$

$$0 \leq I_w(t) \leq \widetilde{I}_w(t), \quad 0 \leq I_b(t) \leq \widetilde{I}_b(t) \quad t \geq t_0.$$

So, it is sufficient to establish estimate (18) for solutions to system (21). Consider the Lyapunov function

$$\mathcal{V}_p(t, v) = \langle \mathcal{H}_p(t)v, v\rangle.$$

Differentiating it along the solutions to system (21), from (22) and (23) we obtain

$$\frac{d}{dt}\mathcal{V}_p(t, v(t)) \leq -\frac{1}{\|\mathcal{H}_p(t)\|}\mathcal{V}_p(t, v(t)) + 2\frac{b(t)}{K}\sqrt{h_{22}^p(t)}\; z_b^2(t)\; \mathcal{V}_p^{1/2}(t, v(t)).$$

Since

$$z_b^2(t) \leq \|v(t)\|^2 \leq \frac{1}{\mathcal{H}_{\min}^p(t)}\langle \mathcal{H}_p(t)v(t), v(t)\rangle = \frac{1}{\mathcal{H}_{\min}^p(t)}\; \mathcal{V}_p(t, v(t)),$$

then we have the inequality

$$\frac{d}{dt}\mathcal{V}_p(t, v(t)) \leq -\frac{1}{\|\mathcal{H}_p(t)\|}\mathcal{V}_p(t, v(t)) + 2\frac{b(t)}{K}\;\frac{\sqrt{h_{22}^p(t)}}{\mathcal{H}_{\min}^p(t)}\; \mathcal{V}_p^{3/2}(t, v(t)).$$

It remains to apply the Gronwall inequality.

Theorem is proved. □

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References

[1] X.-S. Wang, J. Wu, *Periodic systems of delay differential equations and avian influenza dynamics*, J. Math. Sci., **201**:5 (2014), 693–704. Zbl 1314.34165

[2] L. Bourouiba, J. Wu, S. Newman, J. Takekawa, T. Natdorj, N. Batbayar, C.M. Bishop, L.A. Hawkes, P.J. Butler, M. Wikelski, *Spatial dynamics of bar-headed geese migration in the context of H5N1*, J. R. Soc. Interface, **7** (2010), 1627–1639.

[3] S. Gourley, R. Liu, J. Wu, *Spatiotemporal distributions of migratory birds: patchy models with delay*, SIAM J. Appl. Dyn. Syst., **9**:2 (2010), 589–610. Zbl 1195.34126

[4] X.-S. Wang, J. Wu, *Seasonal migration dynamics: periodicity, transition delay, and finite dimensional reduction*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **468**:2139 (2012), 634–650.

[5] M.A. Skvortsova, *Estimates of solutions to a system describing birds' migration*, J. Anal. Appl., **13**:1–2 (2015), 15–27. Zbl 1335.34132

[6] G.V. Demidenko, I.I. Matveeva, *On stability of solutions to linear systems with periodic coefficients*, Siberian Math. J., **42**:2 (2001), 282–296. Zbl 0977.34046

[7] G.V. Demidenko, I.I. Matveeva, *On stability of solutions to quasilinear periodic systems of differential equations*, Siberian Math. J., **45**:6 (2004), 1041–1052. Zbl 1105.34033

[8] Ph. Hartman, *Ordinary Differential Equations*, New York, London, Sydney, 1964. Zbl 0125.32102

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