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ON THE REALIZABILITY OF A GRAPH AS THE GRUENBERG–KEGEL GRAPH OF A FINITE GROUP

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Abstract. Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e. the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this graph the vertex set is $\pi(G)$ and different vertices p and q are adjacent if and only if $pq \in \omega(G)$. We say that a graph Γ with $|\pi(G)|$ vertices *is realizable as the Gruenberg–Kegel graph of a group G* if there exists a vertices marking of Γ by distinct primes from $\pi(G)$ such that the marked graph is equal to $\Gamma(G)$. A graph Γ *is realizable as the Gruenberg–Kegel graph of a group* if Γ is realizable as the Gruenberg–Kegel graph of an appropriate group G. We prove that a complete bipartite graph $K_{m,n}$ is realizable as the Gruenberg–Kegel graph of a group if and only if $m + n \leq 6$ and $(m, n) \neq (3, 3)$. Moreover, we describe all the groups G such that the graph $K_{1,5}$ is realizable as the Gruenberg–Kegel graph of G.

Keywords: finite group, Gruenberg–Kegel graph (prime graph), realizability of a graph, complete bipartite graph.

1. Introduction

Allover this article by «group» we mean «a finite group» and by «graph» «an undirected graph without loops and multiple edges».

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In the finite group theory many researchers are interested in various problems of the study of groups by their arithmetical properties. One of such problems is the problem of the study of a group by some properties of its Gruenberg–Kegel graph.

Let G be a group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e. the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this graph the vertex set is $\pi(G)$ and different vertices p and q are adjacent if and only if $pq \in \omega(G)$.

We say that a graph Γ with $|\pi(G)|$ vertices *is realizable as the Gruenberg–Kegel graph of a group* G if there exists a one-to-one correspondence ϕ between the vertex set of Γ and $\pi(G)$ such that the vertices x and y are adjacent in Γ if and only if $\phi(x)\phi(y) \in \omega(G)$. In other words, there exists a vertices marking of Γ by different primes from $\pi(G)$ such that the marked graph is equal to $\Gamma(G)$. A graph Γ *is realizable as the Gruenberg–Kegel graph of a group* if Γ is realizable as the Gruenberg–Kegel graph of an appropriate group G.

The following problem arises.

Problem. *Let Γ be a graph. Is Γ realizable as the Gruenberg–Kegel graph of a group*?

Of course, in general, the problem has a negative solution. For example, Gruenberg–Kegel Theorem (see Lemma 3) and the description of connected components of the Gruenberg–Kegel graphs for all simple non-abelian groups [23, 14] imply that a graph consisting of $n \geq 5$ pairwise non-adjacent vertices (n-coclique with $n \geq 5$) is not realizable as the Gruenberg–Kegel graph of a group (see Lemma 5).

There are just a few works devoted to this interesting problem.

In unpublished graduate work of I. N. Zharkov [25], who was a student of V. D. Mazurov, it was proved that a chain is realizable as the Gruenberg–Kegel graph of a group if and only if the length of this chain is at most 4.

In the paper [6] it was shown that any graph with at most five vertices except a 5-coclique is realizable as the Gruenberg–Kegel graph of a group.

In the paper [10] it was shown that a graph Γ is realizable as the Gruenberg–Kegel graph of a solvable group if and only if its complement is 3-colorable and triangle free. So, it's interesting to obtain solutions of Problem for groups whose Gruenberg–Kegel graphs contain 3-cocliques.

In this paper we give a solution of the mentioned problem for complete bipartite graphs $K_{m,n}$. We prove the following theorem.

Theorem. *Let Γ be a complete bipartite graph $K_{m,n}$, where $m \leq n$. Then the following statements hold*:

(1) Γ *is realizable as the Gruenberg–Kegel graph of a group if and only if* $m+n \leq 6$ *and* $(m, n) \neq (3, 3)$;

(2) *if* $m + n \leq 6$ *and* $(m, n) \neq (3, 3), (1, 5)$, *then there exist infinitely many sets* T *of primes such that* Γ *is realizable as the Gruenberg–Kegel graph of a group* G *and* $T = \pi(G)$;

(3) *if* $(m, n) = (1, 5)$ *and* Γ *is realizable as the Gruenberg–Kegel graph of a group* G, *then* $\pi(G) = \{2, 3, 7, 13, 19, 37\}$, $O_2(G) \neq 1$, *and* $G/O_2(G) \cong {}^2G_2(27)$.

2. Notation and Auxiliary Results

Our notation and terminology are mostly standard and can be found in [2, 4, 5, 8, 9, 11].

In particular, if G and H are groups and n is a natural number, then we use notation $\mathbb{C}_n$ for the cyclic group of order n and $G : H$ ($G \leftthreetimes H$) for a split extension (semidirect product) of G by (with, on) H.

Let π be a set of primes. Denote by π' the set of the primes not in π. Given a natural n, denote by $\pi(n)$ the set of its prime divisors. Then $\pi(|G|)$ is exactly $\pi(G)$ for any group G. If $|\pi(G)| = n$ then G is called *n-primary.* A natural number n with $\pi(n) \subseteq \pi$ is called a π-number, and a group G with $\pi(G) \subseteq \pi$ is called a π-group. A subgroup H of a group G is called a *π-Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$.

We will denote by $S(G)$ the *solvable radical* of a group G (i. e. the largest solvable normal subgroup of G), by $F(G)$ the *Fitting subgroup* of G (i. e. the largest nilpotent normal subgroup of G), by $Soc(G)$ the *socle* of G (i. e. the subgroup of G generated by the set of all non-trivial minimal normal subgroups of G) and by $O_p(G)$ the largest normal p-subgroup of G.

Recall, that a *n-clique* (resp. a *n-coclique*) is a graph with n vertices in which all the vertices are pairwise adjacent (resp. non-adjacent). A graph Γ is called *bipartite* with parts of sizes m and n if its vertices can be divided into two non-empty disjoint subsets V_m and V_n such that $|V_m| = m$, $|V_n| = n$ and the vertices from the same subset are non-adjacent. We will denote by $K_{m,n}$ a *complete bipartite graph* whose vertices are adjacent if and only if they belong to different subsets.

Denote the number of connected components of $\Gamma(G)$ by $s(G)$ and the set of connected components of $\Gamma(G)$ by $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$; for a group G of even order, we assume that $2 \in \pi_1(G)$.

Denote by $t(G)$ the greatest cardinality (number of verices) of a coclique in the Gruenberg–Kegel graph of a group G and by $t(r, G)$ the greatest cardinality of a coclique in the Gruenberg–Kegel graph of a group G containing the prime r.

To prove Theorem we will need the following two number-theoretical results.

Lemma 1 (See [7]). *Let p and q be primes such that $p^a - q^b = 1$ for some integer numbers $a \geq 0$ and $b \geq 0$. Then $(p^a, q^b) \in \{(3^2, 2^3), (2^a, q), (p, 2^b)\}$, where a is a prime and b is a power of 2.*

Lemma 2 (See [26]). *Let q and n be natural numbers, $q \geq 2$. There exists a prime that divides $q^n - 1$ and doesn't divide $q^i - 1$ for $1 \leq i \leq n-1$, except for the following two cases: $q = 2$ and $n = 6$; $q = 2^k - 1$ for some prime k and $n = 2$.*

Lemma 3 (Gruenberg–Kegel theorem, [23, Theorem A]). *If G is a group with disconnected Gruenberg–Kegel graph, then one of the following statements holds:*

(1) *G is a Frobenius group;*

(2) *G is a 2-Frobenius group;*

(3) *G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $S \trianglelefteq A \leq \mathrm{Aut}(S)$, S is a simple non-abelian group with $s(G) \leq s(S)$, and A/S is a $\pi_1(G)$-group.*

Lemma 4 ([17, Theorem 1]). *Let G be a finite group with $t(G) \geq 3$. Then G is non-solvable.*

Proof. Let G be a solvable group whose Gruenberg–Kegel graph $\Gamma(G)$ contains a 3-coclique $\{p_1, p_2, p_3\}$. Using Hall Theorem [8, Theorem 6.4.1] we conclude, that there exists a solvable $\{p_1, p_2, p_3\}$-Hall subgroup H of G, such that its Gruenberg–Kegel graph $\Gamma(H)$ is a 3-coclique. In view of Higman Theorem [12, Theorem 1] a solvable group in which every element has prime power order is at most 2-primary. A contradiction. □

Remark 1. *Lemma* 4 *was first proved in* 1999 *by M. S. Lusido* [17, Theorem 1]. *But it follows directly from earlier results by G. Higman and P. Hall mentioned above.*

Lemma 5. *Let Γ be a n-coclique, where $n \geq 5$. Then Γ is not realizable as the Gruenberg–Kegel graph of a group.*

Proof. Note, if Γ is realizable as the Gruenberg–Kegel graph of a group G, then $s(G) = n \geq 5$ and $t(G) = n \geq 5$. By Lemma 4 G is non-solvable. By Lemma 3 and [27, Lemma 3] there exists a simple non-abelian group S such that $S \trianglelefteq G/F(G) \leq Aut(S)$ and $s(S) \geq s(G) = n$. According to [14, 23] $n \leq s(S) \leq 6$. So, if $n \geq 5$ then S is isomorphic to either J_4 or $E_8(q)$ for $q \equiv 0, 1, 4(5)$, and consequently $|\pi(S)| > 6$. A contradiction. □

Lemma 6 ([20, Propositions 2,3])**.** *Let us assume, that $t(G) \geq 3$ and $t(2, G) \geq 2$ for some group G. Then the following conditions hold*:

(1) *there exists a simple non-abelian group S such that $S \trianglelefteq G/S(G) \leq Aut(S)$*;

(2) *if $\rho \subseteq \pi(G)$ is a coclique in $\Gamma(G)$ with $|\rho| \geq 3$, then at most one of the primes from ρ divides the product $|S(G)| \cdot |G/S(G) : S|$. In particular, $t(S) \geq t(G) - 1$*;

(3) *one of the following conditions holds*:

(a) *every $p \in \pi(G)$ which is non-adjacent to* 2 *in $\Gamma(G)$ doesn't divide the product $|S(G)| \cdot |G/S(G) : S|$. In particular, $t(2, S) \geq t(2, G)$*;

(b) *there exists $r \in \pi(S(G))$ which is non-adjacent to* 2 *in $\Gamma(G)$; in this case $t(G) = 3$, $t(2, G) = 2$ and $S \cong A_7$ or $PSL_2(q)$ for any odd q.*

Lemma 7. *Let K be a normal subgroup of a group L. Then the following conditions hold*:

(1) *if $r, s \in \pi(K) \setminus \pi(L/K)$ and r and s are non-adjacent to each other in $\Gamma(K)$, then they are also non-adjacent in $\Gamma(L)$*;

(2) *if $r, s \in \pi(L/K) \setminus \pi(K)$, and r and s are non-adjacent in $\Gamma(L/K)$, then they are also non-adjacent in $\Gamma(L)$.*

Proof. (1) See, for example, [18, Lemma 2].

(2) Follows directly from the Shur–Zassenhaus Theorem [8, Theorem 6.2.1]. □

Lemma 8 ([1, Theorem 2])**.** *Let G be an almost simple group with the socle S. If $\Gamma(G)$ is triangle free then one of the following conditions holds*:

(1) *S is isomorphic to one of the following groups*: *A_n, where $n \in \{5, 6, 7\}$*; *$PSL_2(q)$, where $q \in \{7, 2^3, 3^4, 11, 13, 17, 5^2, 7^2, 2^9\}$*; *$PSL_3(q)$, where $q \in \{3, 4, 5, 17\}$*; *$PSU_3(q)$, where $q \in \{3, 7\}$*; *$PSL_4(3)$, $PSU_4(q)$, where $q \in \{2, 3\}$*; *$G_2(3)$*; *${}^2F_4(2)'$*; *M_{11}*; *M_{22};*

(2) *G is isomorphic to one of the following groups*: *A_8*; *$PSL_2(q)$, where $q \in \{2^4, 2^6\}$*; *$PSL_3(q)$, where $q \in \{7, 8, 9\}$*; *$PSL_3(7) : 2$*; *$PSL_3(9) : 2$*; *$PSU_3(q)$, where $q \in \{4, 5, 8\}$, $PSU_3(5) : 2$, $PSU_3(8) : 3$, $PSU_5(2)$*;

(3) *$S \cong PSL_2(q)$, where $q \in \{5^3, 17^2\}$ and $PGL_2(q) \not\leq G$*;

(4) $S \cong PSL_2(q)$, *where* $q \in \{2^p, 3^p\}$, p *is an odd prime and* $|\pi(q-1)| \leq 2 \geq |\pi(q+1)|$;

(5) $G \cong PSL_2(p)$, *where* $p > 17$ *is a prime and* $|\pi(p-1)| \leq 2 \geq |\pi(p+1)|$;

(6) $G \cong PGL_2(p)$, *where* $p > 17$ *is a prime distinct from Mersenne or Fermat primes and* $|\pi(p^2-1)| = 3$;

(7) $G \cong PSL_2(q)$ *or* $PSL_2(q) : r$, *where* $q = p^r$, $p \in \{3, 5, 7, 17\}$, r *is a prime,* r *doesn't divide* $|S|$, $q \equiv \varepsilon 1 \ (mod\ 4)$, *where* $\varepsilon \in \{+, -\}$ *and* $|\pi(q - \varepsilon 1)| = |\pi((q + \varepsilon 1)/2)| = 2$;

(8) $G \cong PSU_3(q)$ *or* $PSU_3(q) : p$, *where* $q = 2^p$, $p \geq 5$, $q-1$ *and* $(q+1)/3$ *are primes,* $|\pi((q^2 - q + 1)/3)| = 1$ *and* p *doesn't divide* $|S|$;

(9) $S \cong PSL_3^{\varepsilon}(p)$, *where* $\varepsilon \in \{+, -\}$, $p \geq 11$ *is a prime distinct from Mersenne or Fermat primes,* $(p - \varepsilon 1)_3 = 3$, $|\pi(p^2 - 1)| = 3$ *and* $|\pi((p^2 + \varepsilon p + 1)/3)| = 1$;

(10) $S \cong Sz(2^f)$, *where either* $f = 9$ *or* f *is an odd prime and* $max\{|\pi(q-1)|, |\pi(q - \sqrt{2q} + 1)|, |\pi(q + \sqrt{2q} + 1)|\} \leq 2$;

(11) $G \cong {}^2G_2(q)$, *where* $q = 3^p$, p *is an odd prime,* $|\pi((q-1)/2)| = |\pi((q + 1)/4)| = 1$ *and* $|\pi(q - \sqrt{3q} + 1)| \leq 2 \geq |\pi(q + \sqrt{3q} + 1)|$.

Remark 2. *The following conditions hold*:

(1) *all the groups in items* (1), (2) *and* (6) *of Lemma 8 are at most* 4*-primary, excluding the groups with socles* $PSL_2(2^6)$, $PSL_2(2^9)$ *and* $PSL_3(9)$ *which are* 5*-primary*;

(2) *all the groups in items* $(1)-(3)$ *and* $(5)-(6)$ *of Lemma 8 are at most* 5*-primary*;

(3) *all simple groups in items* $(1)-(9)$ *of Lemma 8 are at most* 5*-primary*;

(4) *in the items* (1)–(3), (5)–(6), (9) *and* (11) *of Lemma 8 we have* $\pi(G) = \pi(S)$.

Lemma 9 ([19, Lemma 1])**.** *Let* G *be a group,* N *its normal subgroup such that* G/N *a Frobenius group with the kernel* F *and a cyclic complement* C. *If* $(|F|, |N|) = 1$ *and* $F \not\leq NC_G(N)/N$, *then* $s|C| \in \omega(G)$ *for any* $s \in \pi(N)$.

Corollary 1. *Let* G *be such group, that* $p \subseteq \pi(S(G)) \subseteq \{2, p\}$ *for some prime* p. *Let us suppose, that the quotient group* $G/S(G)$ *contains a subgroup* $H = F : C$ *which is a Frobenius group of odd order with a cyclic complement* C *and such, that* $p \notin \pi(H)$. *Then either* p *and* r *are adjacent in* $\Gamma(G)$ *for some* $r \in \pi(F)$ *or* p *and* s *are adjacent in* $\Gamma(G)$ *for some* $s \in \pi(C)$.

Lemma 10. [13, Proposition 1.4] *Let* q *be an odd prime,* $q - 1 \neq 2^w$ *and* G *a group of the form* $G = P \rtimes (T \rtimes \langle x \rangle)$, *where* P *is a non-trivial* $\{2, q\}'$*-group,* T *is a* 2*-group,* $|x| = q$ *and* $C_G(P) = Z(P)$. *If* $[T, x] \neq 1$ *then* $C_P(x) \neq 1$.

3. Proof of the main results

Let Γ be a bipartite graph with parts of sizes m and n. Assume, that the vertices of Γ are marked by distinct primes such that $\Gamma = \Gamma(G)$ for an appropriate group G. As a graph is bipartite if and only if every its simple cycle has an even length [11, Theorem 2.4], all non-abelian composition factors of G are exhausted by simple groups from Lemma 8. Further information about adjacency of vertices in Gruenberg–Kegel graphs of finite simple non-abelian groups is taken from [21] and [22].

Lemma 11. *Let* Γ *be a bipartite graph with parts* U *and* V, *where* $|U| = 1$ *and* $|V| = n \geq 1$. *If* Γ *is realizable as the Gruenberg–Kegel graph of a group, then*

$n \leq 5$. Moreover, if $n = 5$ and Γ is realizable as the Gruenberg–Kegel graph of some group G, then $U = \{2\}$, $|\pi(S(G)) \setminus \{2\}| \leq 1$, $\pi(S(G)) \subseteq \pi(G/S(G))$, and $G/S(G) \cong {}^2G_2(q)$.

PROOF. Suppose, that Γ is realizable as the Gruenberg–Kegel graph of a group G. By [6, Theorem] we can assume $n \geq 5$, so G is non-solvable according to Lemma 4. Thus, by Feit–Thompson Theorem $2 \in \pi(G)$, so $|\pi(G)| = n + 1$ and $\pi(G) = \{2, p_1, \dots p_n\}$.

First, let us assume, that $2 \notin U$ and without loss of generality assume $U = \{p_1\}$. Note, that $t(G) \geq t(2, G) \geq n \geq 5$. By Lemma 6 there exists a simple non-abelian group S such that $S \trianglelefteq G/S(G) \leq Aut(S)$ and none of the primes $p_2, \dots, p_n$ divides $|S(G)| \cdot |G/S(G) : S|$. So $2, p_2, \dots p_n \in \pi(S)$ and are pairwise non-adjacent in $\Gamma(S)$. Therefore, $t(S) \geq n \geq 5$ and $t(2, S) \geq n \geq 5$. However, if S is a simple group from Lemma 8, then in view of [21, Tables 2,3] and [22, Tables 2,3,4] $t(S) \leq 5$ and $t(S) = 5$ if and only if $S \cong {}^2G_2(q)$. But in view of [21, Table 7] if $S \cong {}^2G_2(q)$ then $t(2, S) = 3 < 5$. A contradiction.

We conclude, that $U = \{2\}$. In view of [16, Theorem 5] there exists a simple non-abelian group S such that $S \trianglelefteq G/S(G) \leq Aut(S)$. Note, that S is a simple group from Lemma 8.

Prove, that $|\pi(S(G)) \setminus \{2\}| \leq 1$. Let us suppose, on the contrary, that $p_1, p_2 \in \pi(S(G)) \setminus \{2\}$, $p_3 \in \pi(G) \setminus \{2, p_1, p_2\}$, and consider a subgroup $S(G)P$, where $P \in Syl_{p_3}(G)$. Obviously, $S(G)P$ is solvable, $p_1, p_2, p_3 \in \pi(S(G)P)$ and are pairwise non-adjacent in $\Gamma(S(G)P)$. We got a contradiction with the statement of Lemma 4. So, $|\pi(S(G)) \setminus \{2\}| \leq 1$.

We start with the case $\pi(S(G)) \setminus \pi(G/S(G)) \neq \emptyset$. Without loss of generality we can assume, that $p_1 \in \pi(S(G)) \setminus \pi(G/S(G))$. So $\pi(G) \setminus \pi(G/S(G)) \subseteq \{p_1\}$ and $|\pi(G/S(G))| \geq 5$. Note, that every two odd primes are non-adjacent in $\Gamma(G)$, so $t(G/S(G)) \geq 4$.

Next we consider separately each groups S from items $(1) - (11)$ of Lemma 8. We need to prove, that S is not isomorphic to the non-abelian composition factor of a group G such that Γ is realizable as the Gruenberg–Kegel graph of G.

If S is a group from the item (1) of Lemma 8 and $|\pi(A)| \geq 5$ for some group A such that $S \trianglelefteq A \leq Aut(S)$, then in view of Remark 2 $S \cong PSL_2(2^9)$. But 3 and 19 are adjacent in $\Gamma(PSL_2(2^9))$. A contradiction and S isn't a group from the item (1) of Lemma 8.

If S is a group from the item (2) of Lemma 8 and $|\pi(A)| \geq 5$ for some group A such that $S \trianglelefteq A \leq Aut(S)$, then in view of Remark 2 either $S \cong PSL_2(2^6)$ or $S \cong PSL_3(9)$. But 3 and 7 are adjacent in $\Gamma(PSL_2(2^6))$ and 7 and 13 are adjacent in $\Gamma(PSL_3(9))$. A contradiction. So S isn't a group from the item (2) of Lemma 8.

If S is a group from the items (3), (5) or (6) of Lemma 8, then $\pi(S) = \pi(Aut(S))$, thus $t(S) \geq 4$. But in view of [22, Table 2] we have $t(S) = 3$. A contradiction and S isn't a group from the items (3), (5) or (6) of Lemma 8.

Let S be a group from the item (4) of Lemma 8. If $q = 2^p$, where p is an odd prime, then 3 divides $q + 1$, so in view of Lemma 1 either $p = 3$ and $|\pi(S)| = 3$ or $p > 3$, 3 and $p_2 \in \pi(q + 1) \setminus \{3\}$ are adjacent in $\Gamma(S)$. A contradiction. If $q = 3^p$ then $|\pi(S)| \leq 4$, thus p divides $|G/S(G)|$, and by [9, 2.5.13, Proposition 4.9.1] the centralizer in S of any element of the order p from $G/S(G)$ is isomorphic to $PSL_2(3)$, so p and 3 are adjacent in $\Gamma(G/S(G))$. A contradiction, so S isn't a group from the item (4) of Lemma 8.

If S is a group from the item (7) of Lemma 8, then $p_2, p_3 \in \pi(\frac{q+\varepsilon 1}{2})$ are adjacent in $\Gamma(S)$. A contradiction and S isn't a group from the item (7) of Lemma 8.

Let S be a group from the item (8) of Lemma 8. Then $r \in \pi((q+1)/3)$, r and 3 are adjacent in $\Gamma(S)$, a contradiction. So S isn't a group from the item (8) of Lemma 8.

Let S be a group from the item (9) of Lemma 8. We have $p_1 \notin \pi(\frac{p^2+\varepsilon p+1}{3}) \cup \{3\}$ and by [3, Tables 8.3, 8.5] in S exists a subgroup $H \cong \frac{p^2+\varepsilon p+1}{3} : 3$. This is a Frobenius group of odd order, which by Lemma 3 and [8, Theorem 3.1] has a cyclic complement. Thus by Corollary 1 either p_1 and $r \in \pi(\frac{p^2+\varepsilon p+1}{3})$ are adjacent in $\Gamma(G)$ or p_1 and 3 are adjacent in $\Gamma(G)$. A contradiction and S isn't a group from the item (9) of Lemma 8.

Let S be a group from the item (10) of Lemma 8. Note, that if $f \geq 7$ then $|\pi(S)| \geq 5$ and by [15, Lemma 4] $\pi_1(S) = \{2\}$. So by Lemma 5 at least two odd numbers are adjacent in $\Gamma(S)$, a contradiction. If $f = 5$ then $|\pi(S)| = |\pi(Aut(S))| = 4$, again a contradiction. If $f = 3$, then $|\pi(S)| = 4$, thus 3 divides $|G/S(G)|$, and by [9, 2.5.13, Proposition 4.9.1] the centralizer in S of any element of the order 3 from $G/S(G)$ is isomorphic to $Sz(2)$. So 3 and 5 are adjacent in $\Gamma(G/S(G))$, a contradiction. Thus S isn't a group from the item (10) of Lemma 8.

Finally, let us suppose, that S is a group from the item (11) of Lemma 8 and $S \cong {}^2G_2(3^p)$, where $p \geq 3$ is a prime. We have $p_1 \notin \pi(3^p + 3^{\frac{p+1}{2}} + 1) \cup \{3\}$ and according to [3, Table 8.43] S has a subgroup $H \cong (3^p + 3^{\frac{p+1}{2}} + 1) : 3$. This is a Frobenius group of odd order, which by Lemma 3 and [8, Theorem 3.1] has a cyclic complement. Thus by Corollary 1 either p_1 and r are adjacent in $\Gamma(S)$ for some $r \in \pi(3^p + 3^{\frac{p+1}{2}} + 1)$ or p_1 and 3 are adjacent in $\Gamma(S)$. A contradiction, and S isn't a group from the item (11) of Lemma 8.

Consider the case $\pi(S(G)) \subset \pi(G/S(G))$. Note, that $\pi(G/S(G)) = \pi(G)$ and every two distinct odd primes are non-adjacent in $\Gamma(G)$. Thus, they are non-adjacent in $\Gamma(G/S(G))$. Therefore, $|\pi(G/S(G))| \geq 6$ and $t(G/S(G)) \geq n \geq 5$. As S is a simple group from Lemma 8, $|\pi(Aut(S)) \setminus \pi(S)| \leq 1$, $|\pi(S)| \geq 5$, and $t(S) \geq t(G/S(G)) - 1 \geq 4$.

In view of Remark 2, [21, Tables 2,3] and [22, Tables 2,3,4] S is a simple group from the items $(8) - (11)$ of Lemma 8. Considering once more the above arguments we can conclude, that S isn't a group from the items (8) or (10) of Lemma 8.

Let S be a group from the item (9) of Lemma 8. Then $\pi(S) = \pi(Aut(S))$, so $t(S) = t(G) \geq 5$. But according to [22, Table 2] $t(S) = 4$, a contradiction. So S isn't a group from the item (9) of Lemma 8.

If S is a group from the item (11) of Lemma 8, then $G/S(G) \cong S \cong {}^2G_2(q)$ and in view of [22, Table 4] $t(S) = 5$. Therefore, $n \leq 5$. □

Lemma 12. *Let $S = {}^2G_2(q)$, where $q = 3^m$ and $m \geq 5$ is odd. Then $|\pi(S)| \geq 7$.*

Proof. According to [4]

$$|S| = 3^{3m}(3^m - 1)(3^{3m} + 1) = 2^w \cdot 3^{3m} \cdot \frac{3^m - 1}{(3^m - 1)_2} \cdot \frac{3^m + 1}{(3^m + 1)_2} \cdot (3^{2m} - 3^m + 1)$$

$$= 2^w \cdot 3^{3m} \cdot \frac{3^m - 1}{(3^m - 1)_2} \cdot \frac{3^m + 1}{(3^m + 1)_2} \cdot (3^m - 3^{\frac{m+1}{2}} + 1) \cdot (3^m + 3^{\frac{m+1}{2}} + 1),$$

and the last six factors are pairwise coprime. It means, that $|\pi(S)| \geq 6$ and $|\pi(S)| = 6$ if and only if every of the mentioned factors is a prime power.

Assume m isn't a prime and p is a prime divisor of m. Then by Lemma 2 there exists a prime r_1 dividing $3^p - 1$, but not dividing $3^i - 1$ for $i < p$ and a prime r_2 dividing $3^m - 1$, but not dividing $3^i - 1$ for $i < m$. Thus $3^m - 1$ is not a prime power, so $|\pi(S)| \geq 7$ and $m = p$ is a prime.

If $p = 7$, then $\pi(S) = \{2, 3, 7, 43, 547, 1093, 2269\}$ and $|\pi(S)| = 7$. So we can assume, that $p \neq 7$.

Note, that $(a^b - 1, a^c - 1) = a^{(b,c)} - 1$ for any positive integers a, b and c. As 7 divides $3^3 + 1$ and $(3^p + 1, 3^3 + 1)$ divides $(3^{2p} - 1, 3^6 - 1) = 3^2 - 1 = 8$, it follows, that 7 doesn't divide $3^p + 1$. Moreover, 7 divides $3^6 - 1 = 728$, $3^6 - 1$ divides $3^{6p} - 1$ and $(3^6 - 1, 3^{3p} - 1) = 3^3 - 1 = 26$, so 7 doesn't divide $3^{3p} - 1$ and $(3^{6p} - 1)_7 = (3^{3p} + 1)_7$. Note, that $(3^{42} - 1)_7 > 7$ and $(3^{42} - 1, 3^{6p} - 1) = 3^{6(7,p)} - 1 = 3^6 - 1$, so $(3^{3p} + 1)_7 = (3^{6p} - 1)_7 = 7$. Thus, 7 divides one of the numbers $3^m - 3^{\frac{m+1}{2}} + 1$ and $3^m + 3^{\frac{m+1}{2}} + 1$.

So if $|\pi(S)| = 6$, then $3^m - 3^{\frac{m+1}{2}} + 1 = 7$ or $3^m + 3^{\frac{m+1}{2}} + 1 = 7$. But this is impossible when $p > 3$. □

Lemma 13. *Let G be a group such that $\{3\} \subseteq \pi(S(G)) \subseteq \{2, 3\}$ and $G/S(G) \cong {}^2G_2(27)$. Then 3 and 7 are adjacent in $\Gamma(G)$.*

Proof. Suppose that, on the contrary, 3 and 7 are non-adjacent in $\Gamma(G)$. Denote $G/S(G)$ by S. According to [3, Tables 8.7, 8.43] S has the following series of subgroups:

$$K \cong 2^3 : 7 < PSL_2(8) < PSL_2(8) : 3 < S = {}^2G_2(27),$$

where K is a Frobenius group with the kernel of order 2^3 and a cyclic complement of order 7.

Consider the pre-image H in G of the subgroup $K \cong 2^3 : 7$ and a $2'$-Hall subgroup T of H. If U is a Sylow 3-subgroup of T, then $U \leq T \cap S(G) \trianglelefteq T$ and $T \cap S(G)$ is a 3-group. Thus, $T \cap S(G) = O_3(T) = F(T)$ and $F(T)$ is a Sylow 3-subgroup of $S(G)$. Denote $F(T)$ by F.

Consider the subgroup $Q = N_H(F)$ and denote $O_2(Q/F)$ by L. The Shur-Zassenhaus Theorem [8, Theorem 6.2.1] yields, that $Q = F \leftthreetimes (L \leftthreetimes \langle x \rangle)$, where $\langle x \rangle$ is a $2'$-Hall subgroup of Q/F and $|x| = 7$.

In view of Frattini Argument $G = N_G(F)S(G)$ and $H = QS(G)$. Furthermore, $C_G(F) \trianglelefteq N_G(F)$, so $C_G(F)S(G) \trianglelefteq G$. Note, that$(C_G(F)S(G))/S(G)$ is a normal subgroup of $S \cong {}^2G_2(27)$, a simple non-abelian group. Thus, no element of order 7 from G centralizes any element of order 3 from $S(G)$ and $(C_G(F)S(G))/S(G) = \{1\}$. So $C_G(F) \leq S(G)$, yielding that $C_H(F) \leq S(G)$, $C_Q(F) \leq S(G) \cap Q$ and $C_Q(F) = Z(F) \leftthreetimes P$, where P is a Sylow 2-subgroup of $C_Q(F)$.

Now $O_2(C_Q(F))$ is a characteristic subgroup of $C_Q(F)$ and $C_Q(F) \trianglelefteq Q$. Let as denote $O_2(C_Q(F))$ by C and Q/C by $\bar{Q}$. Note, that $[Q, F] \cap C \leq F \cap C = 1$, so $C_{\bar{Q}}(\bar{F}) = C_Q(F)/C$ and $O_2(C_{\bar{Q}}(\bar{F})) = 1$. Finally, if $C_{\bar{Q}}(\bar{F}) \neq Z(\bar{F})$ and R is a Sylow 2-subgroup of $C_{\bar{Q}}(\bar{F})$, then $Z(R)$ is a subgroup of $Z(C_{\bar{Q}}(\bar{F}))$, so $O_2(C_{\bar{Q}}(\bar{F})) \neq 1$. This is a contradiction.

Thus, $\bar{Q} = \bar{F} \leftthreetimes (\bar{L} \leftthreetimes \langle \bar{x} \rangle)$, where $\bar{F} \cong F$ is a 3-group, $\bar{L}$ is a 2-group, $|\bar{x}| = 7$ and $C_{\bar{Q}}(\bar{F}) = Z(\bar{F})$.

Moreover, $K \cong H/S(G) \cong Q/(S(G) \cap Q) \cong \bar{Q}/((S(G) \cap Q)/C)$, so $[\bar{L}, \bar{x}] \neq 1$. In view of Lemma 10 we have $C_{\bar{F}}(\bar{x}) \neq 1$, so 3 and 7 are adjacent in $\Gamma(\bar{Q})$ and, therefore, they are adjacent in $\Gamma(G)$, a contradiction. □

Lemma 14. *Let Γ be a bipartite graph with parts V_1 and V_5 such that $|V_1| = 1$ and $|V_5| = 5$. If Γ is realizable as the Gruenberg–Kegel graph of a group G, then $\pi(G) = \{2, 3, 7, 13, 19, 37\}$ and $G/O_2(G) \cong {}^2G_2(27)$.*

Proof. If Γ is realizable as the Gruenberg–Kegel graph of a group G, then by Lemma 11 we have $V_1 = \{2\}$, $|\pi(S(G)) \setminus \{2\}| \leq 1$, $\pi(S(G)) \subset \pi(G/S(G))$ and $G/S(G) \cong {}^2G_2(q)$. Then by Lemma 12 we conclude, that $q = 27$. Note, that $\pi(G) = \pi({}^2G_2(27)) = \{2, 3, 7, 13, 19, 37\}$, where $V_1 = \{2\}$ and $V_5 = \{3, 7, 13, 19, 37\}$.

Prove, that $S(G) = O_2(G)$. Supposing that $2 \neq p_1 \in \pi(S(G))$, we have $p_1 \neq 3$ by Lemma 13.

According to [3, Table 8.43] S has subgroups $H_1 \cong 37 : 3$ and $H_2 \cong 19 : 3$ of S which are Frobenius groups of odd orders by Lemma 3 and have cyclic complements.

If $p_1 \in \{7, 13, 19\}$, then by Corollary 1 either p_1 and 3 are adjacent in $\Gamma(G)$ or p_1 and 37 are adjacent in $\Gamma(G)$. A contradiction.

If $p_1 = 37$ then by Corollary 1 either p_1 and 19 are adjacent in $\Gamma(G)$ or p_1 and 3 are adjacent in $\Gamma(G)$, a contradiction. Thus, $S(G) = O_2(G)$. □

Lemma 15. *Let Γ be a complete bipartite graph $K_{m,n}$, where $m \geq 3$ and $n \geq 3$. Then Γ is not realizable as the Gruenberg–Kegel graph of a group.*

Proof. Suppose, that Γ is realizable as the Gruenberg–Kegel graph of a group G. Note, that Γ contains a 3-coclique, so by Lemma 4 G is non-solvable. Thus, by Feit–Thompson Theorem $2 \in \pi(G)$, so $\pi(G) = \{2, p_1, \ldots p_{m+n-1}\}$.

Without loss of generality we can assume, that $U = \{2, p_1, \ldots, p_{m-1}\}$ is the part of Γ containing 2, while $V = \{p_m, \ldots, p_{m+n-1}\}$ are the remaining vertices of Γ. As $t(G) = max(m, n) \geq 3$ and $t(2, G) \geq min(m, n) \geq 3$, by Lemma 6 there exists a simple non-abelian group S such that $S \trianglelefteq G/S(G) \leq Aut(S)$. Moreover, the primes $p_1, \ldots, p_{m-1}$ don't divide $|S(G)| \cdot |G/S(G) : S|$ and at most one of the primes $p_m, \ldots, p_{m+n-1}$ divides the product $|S(G)| \cdot |G/S(G) : S|$ (without loss of generality we can denote it by p_{m+n-1}). Thus, there exist $p_{i_1}, p_{i_2} \in U \setminus \{2\}$ and $p_{j_1}, p_{j_2} \in V \setminus \{p_{m+n-1}\}$.

Note, that by Lemma 7 p_{i_1}, p_{j_1}, p_{i_2} and p_{j_2} form a cycle in $\Gamma(S)$. Finally, if S is a simple group whose Gruenberg–Kegel graph is triangle free, then $\Gamma(S)$ is disconnected and every its connected component is a tree by [1, Corollary]. A contradiction. □

Lemma 16. *Let Γ be a complete bipartite graph $K_{2,n}$ with parts U, V, where $|U| = 2$ and $|V| = n \geq 5$. Then Γ is not realizable as the Gruenberg–Kegel graph of a group.*

Proof. Suppose, that Γ is realizable as the Gruenberg–Kegel graph of some group G. As Γ contains a 3-coclique, G is non-solvable by Lemma 4. Thus, by Feit–Thompson Theorem $2 \in \pi(G)$ and $\pi(G) = \{2, p_1, \ldots p_{n+1}\}$.

$t(G) = n \geq 5$ and $t(2, G) \geq 2$, so by Lemma 6 there exists a simple non-abelian group S such that $S \trianglelefteq G/S(G) \leq Aut(S)$.

Let us assume, that $2 \in U$, $U = \{2, p_1\}$ and $V = \{p_2, \ldots, p_{n+1}\}$. By Lemma 6 p_1 doesn't divide $|S(G)| \cdot |G/S(G) : S|$ and at most one of the primes $p_2, \ldots, p_{n+1}$ divides the product $|S(G)| \cdot |G/S(G) : S|$ (without loss of generality denote it by

p_{n+1}). Thus, $2, p_1, \dots p_n \in \pi(S)$ and $|\pi(S)| \geq n+1 \geq 6$, so by Remark 2 S is a simple group from either item (10) or item (11) of Lemma 8. According to Lemma 7 $s(S) \leq 3$ and if $\pi_1(S) \neq \{2\}$, then $s(S) \leq 2$. But by Lemma [15, Lemma 4] $s(Sz(q)) = 4$, $s(^2G_2(q)) = 3$ and $\pi_1(G_2(q)) \neq \{2\}$, a contradiction.

Finally, let us assume, that $2 \in V$, $V = \{2, p_1, \dots, p_{n-1}\}$ and $U = \{p_n, p_{n+1}\}$. By Lemma 6 none of the primes $p_1, \dots, p_{n-1}$ divides the product $|S(G)| \cdot |G/S(G) : S|$. So $2, p_1, \dots, p_{n-1} \in \pi(S)$ and are pairwise non-adjacent in $\Gamma(S)$. Thus, $t(S) \geq n \geq 5$ and $t(2, S) \geq n \geq 5$. However, if S is a simple group from Lemma 8 then according to [21, Tables 2,3] and [22, Tables 2,3,4] $t(S) \leq 5$ and $t(S) = 5$ if and only if $S \cong {}^2G_2(q)$. But in view of [21, Table 7] if $S \cong {}^2G_2(q)$, then $t(2, S) = 3 < 5$. A contradiction. □

Proof of Theorem. Let $\Gamma = K_{m,n}$ be a complete bipartite graph with $m \leq n$.

Note, that the Gruenberg–Kegel graph of a solvable Frobenius group has exactly two connected components, and each connected component is a clique [24]. In view of [27, Proposition 1] if p_1 and p_2 are any distinct primes then there exists a solvable Frobenius group $F_{\{p_1,p_2\}}$ such that $\{p_1\}$ and $\{p_2\}$ are connected components of $\Gamma(F_{\{p_1,p_2\}})$.

If $m = n = 1$ then Γ is realizable as the Gruenberg–Kegel graph of a group $\mathbb{C}_{pr}$, where p and r are any distinct primes. Thus, there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

If $m = 1$ and $n = 2$ then Γ is realizable as the Gruenberg–Kegel graph of a group $\mathbb{C}_{p_1} \times F_{\{p_2,p_3\}}$, where (p_1, p_2, p_3) is any triple of pairwise distinct primes. Thus, there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

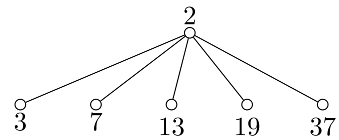
If $m = 1$ and $n = 3$ then Γ is realizable as the Gruenberg–Kegel graph of a group $\mathbb{C}_p \times A_5$, where $p > 5$ is a prime. Thus, there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

If $m = n = 2$ then Γ is realizable as the Gruenberg–Kegel graph of a group $F_{\{p_1,p_2\}} \times F_{\{p_3,p_4\}}$, where p_1, p_2, p_3 and p_4 are any four pairwise distinct primes. Thus, there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

If $m = 1$ and $n = 4$ then Γ is realizable as the Gruenberg–Kegel graph of a group $\mathbb{C}_p \times Sz(8)$, where $p \notin \{2, 5, 7, 13\}$ is a prime. Thus, there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

If $m = 2$ and $n = 3$ then Γ is realizable as the Gruenberg–Kegel graph of a group $F_{\{p_1,p_2\}} \times A_5$, where $p_1 > 5$ and $p_2 > 5$ are distinct primes. Thus, there again exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

If $m = 1$ and $n = 5$ then Γ is realizable as the Gruenberg–Kegel graph of the group $\mathbb{C}_2 \times {}^2G_2(27)$. By Lemma 14 if Γ is realizable as the Gruenberg–Kegel graph of a group G, then $\pi(G) = \{2, 3, 7, 13, 19, 37\}$, $G/O_2(G) \cong {}^2G_2(27)$. Therefore, the Gruenberg–Kegel graph of G is the following:



The Gruenberg–Kegel graph of ${}^2G_2(27)$ is disconnected, so $O_2(G) \neq 1$.

If $m = 2$ and $n = 4$ then Γ is realizable as the Gruenberg–Kegel graph of a group $F_{\{p_1,p_2\}} \times Sz(8)$, where $p_1 \notin \{2, 5, 7, 13\}$ and $p_2 \notin \{2, 5, 7, 13\}$ are distinct primes. Thus, there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$.

If either $m + n > 6$ or $(m, n) = (3, 3)$, then by Lemmas 11, 15 and 16 Γ is not realizable as the Gruenberg–Kegel graph of a group. □

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