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KINETIC FORMULATION OF FORWARD-BACKWARD PARABOLIC EQUATIONS

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Abstract. We have proved that Dirichlet boundary value problem for nonlinear forward-backward parabolic equation has the unique entropy solution. The main difficulty is that initial and final conditions must be formulated in the form of inequalities. We have used here kinetic formulation of the boundary value problem.

Keywords: entropy solution, kinetic solution, forward-backward parabolic equation

1. Introduction.

It is known that forward-backward parabolic equations can play major role in separation and reversal of Prandtl's boundary layer [1], [2], [3]. In one dimensional case we give a model example of the boundary value problem for a forward-backward parabolic equation which describes the boundary layer reversal when a solution changes sign:

$$u(t,x)\partial_t u(t,x) = \partial_x^2 u(t,x), \quad (t,x) \in (0,T) \times (0,1),$$

$$u(t,0) = u(t,1) = 0, \quad t \in [0,T], \quad u(0,x) = u_0(x), \quad u(T,x) = u_T(x), \quad x \in (0,1).$$

Here we assume that solution u and initial and final data u_0 and u_T can be negative. In this case the above-mentioned boundary value problem is ill-posed.

In order to formulate well-posed boundary value problem we use methods that were invented to tackle hyperbolic partial differential equations. Entropy solutions were defined in [4] as unique weak solutions to the Cauchy problem for scalar hyperbolic conservation laws. The situation becomes more complicated when boundary

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conditions are involved. Here the main difficulty is that boundary conditions are formulated in the form of inequalities [5], [6], [7]. In [5], [6] it was assumed that an entropy solution belonged to BV and had a trace on the boundary. In [7] the existence and the uniqueness results were obtained without that assumption. But in [8], [9], [10], [11] it was proved that under genuine nonlinearity condition an entropy solution had a trace on a boundary even if it did not belong to BV. Moreover, these results can be used in a kinetic formulation of scalar conservation laws [12], [13], [14] and boundary conditions [15], [16].

Furthermore, it is well-known that above mentioned methods can be applied to higher order differential equations. Entropy solutions were obtained for degenerate parabolic boundary-value problems [17], [18] and ultra-parabolic equations [19], [20].

Forward backward parabolic equations with nonnegative quadratic form were analyzed in several papers [22], [21]. Here we apply results from [6], [15], [16] to formulate entropy solutions of these equations. Since there are traces in L^1, see [21], there is no need in 'boundary entropy-flux pairs' proposed in [7].

In the present paper we analyze non-degenerate forward-backward parabolic equation

$$\partial_t a(u) - \Delta_x u + \mathrm{div}_x \boldsymbol{\varphi}(u) = 0 \tag{1a}$$

in the domain $(t, \boldsymbol{x}) \in G_T = (0, T) \times \Omega$, $\Omega \subset \mathbb{R}^d$, $|\Omega| < \infty$, $\partial\Omega$ is smooth surface. On functions $a \in C^2(\mathbb{R})$ and $\boldsymbol{\varphi} \in C^1(\mathbb{R})^d$ we impose the condition.

Condition 1. *Let $a \in C^2(\mathbb{R})$, $a(0) = 0$, $\boldsymbol{\varphi}(z) = (\varphi_1(z), \dots, \varphi_d(z))$, $z \in \mathbb{R}$, $\varphi_j \in C^2(\mathbb{R})$, $j = 1, \dots, d$, $\boldsymbol{\varphi}(0) = 0$. Function $a'(z)$ is not equal to zero identically on intervals of positive measure. In section 4 we impose additional condition* (57) *on $\boldsymbol{\varphi}(z)$.*

Remark 1. *In this paper we assume that function $a(z)$ in non-monotonic. Therefore, equation* (1a) *is a forward-backward parabolic equation. But we can also deal with the case when $a(z)$ is a monotonic function.*

We formally define boundary conditions on $\partial G_T = \Gamma_l \cup \Gamma_0 \cup \Gamma_T$:

$$u|_{\Gamma_l} = 0, \quad u|_{\Gamma_0} = u_0, \quad u|_{\Gamma_T} = u_T, \tag{1b}$$

where $\Gamma_l = [0, T] \times \partial\Omega$, $\Gamma_0 = \{0\} \times \Omega$, $\Gamma_T = \{T\} \times \Omega$, u_0, $u_T \in L^\infty(\Omega)$. With boundary condition $u|_{\Gamma_l} = 0$ being correct, it is important to note that this problem is ill-posed because even if function $a(z)$ is monotonic, only one of the conditions $u|_{\Gamma_0} = u_0$ and, respectively, $u|_{\Gamma_T} = u_T$ can be admissible. When $a(z)$ in non-monotonic, both initial and final conditions can be incorrect. Also, see the example ($d = 1$, $a(z) = z^2/2$, $\varphi(z) \equiv 0$, $\Omega = (0, 1)$) given in the very beginning of this paper.

We deal with a weak solution u_ε to elliptic regularization

$$\partial_t a(u_\varepsilon) + \mathrm{div}_x \boldsymbol{\varphi}(u_\varepsilon) = \Delta_x u_\varepsilon + \varepsilon \partial_t^2 u_\varepsilon \tag{2a}$$

in the domain $(t, \boldsymbol{x}) \in G_T$ with boundary conditions

$$u_\varepsilon|_{\Gamma_l} = 0, \quad u_\varepsilon|_{\Gamma_0} = u_0, \quad u_\varepsilon|_{\Gamma_T} = u_T. \tag{2b}$$

We recall that a weak solution $u_\varepsilon \in L^\infty(G_T) \cap W^{2,2}(G_T)$ satisfies equation (2a) in the sense of distributions. Let $\hat{u} \in L^\infty(G_T) \cap W^{2,2}(G_T)$ be an arbitrary extension of functions u_0 and u_T into G_T and $\hat{u}|_{\Gamma_l} = 0$. Therefore, boundary conditions (2b) can be satisfied in the following way:

$$(u_\varepsilon - \hat{u}) \in L^\infty(G_T) \cap \overset{\circ}{W}{}^{2,2}(G_T).$$

In comparison with (1a)–(1b) this boundary value problem is well-posed.

Theorem 1. *Suppose u_0, $u_T \in L^\infty(\Omega) \cap \overset{\circ}{W}{}^{2,2}(\Omega)$. For $\varepsilon > 0$ the weak solution $u_\varepsilon \in L^\infty(G_T) \cap W^{2,2}(G_T)$ exists and satisfies the maximum principle and integral estimates:*

$$\|u_\varepsilon\|_{L^\infty(G_T)} \le M, \tag{3}$$

$$\int_{G_T} \big(|\nabla_x u_\varepsilon|^2 + \varepsilon\, |\partial_t u_\varepsilon|^2 \big)\, dt d\boldsymbol{x} \le C_{(32)}, \tag{4}$$

and $\lim\limits_{\varepsilon \to 0+} u_\varepsilon$ corresponds to entropy solution of the boundary value problem (1a)–(1b), $M = \max\{\|u_0\|_{L^\infty(\Omega)}, \|u_T\|_{L^\infty(\Omega)}\}$, $C_{(32)}$ does not depend on ε and equals to the right hand of inequality (32).

This Theorem is proved in section 3.

To formulate entropy solution of the boundary value problem (1a)–(1b) we define an entropy-entropy flux pair $(\eta(z), \vec{q}(z))$ by the rule:

$$(5) \quad q_0'(z) = a'(z)\eta'(z), \quad q_j'(z) = \varphi_j'(z)\eta'(z), \quad j = 1, \dots, d,$$
$$\vec{q}(z) = (q_0(z), q_1(z), \dots, q_d(z)) = (q_0(z), \boldsymbol{q}(z)), \quad \eta''(z) \ge 0, \quad z \in \mathbb{R}.$$

Definition 1. *A function $u \in L^\infty(G_T) \cap L^2(0, T; \overset{\circ}{W}{}^{1,2}(\Omega))$ is called an entropy solution of the boundary value problem (1a)–(1b) if it satisfies the inequality (in the sense of distributions)*

$$\partial_t q_0(u) - \Delta_x \eta(u) + \mathrm{div}_x \boldsymbol{q}(u) \le -\eta''(u)|\nabla_x u|^2 \tag{6a}$$

and the entropy boundary conditions

$$q_0(u_0^\tau(\boldsymbol{x})) - q_0(u_0(\boldsymbol{x})) - \eta'(u_0(\boldsymbol{x}))(a(u_0^\tau(\boldsymbol{x})) - a(u_0(\boldsymbol{x}))) \le 0, \tag{6b}$$

$$q_0(u_T^\tau(\boldsymbol{x})) - q_0(u_T(\boldsymbol{x})) - \eta'(u_T(\boldsymbol{x}))(a(u_T^\tau(\boldsymbol{x})) - a(u_T(\boldsymbol{x}))) \ge 0 \tag{6c}$$

for any entropy-entropy flux pair $(\eta(z), \vec{q}(z))$, for a.e. $\boldsymbol{x} \in \Omega$.

Remark 2. *Instead of smooth entropy-entropy flux pairs we can use continuous boundary entropy-entropy flux pairs $(H^0(z,k), \vec{Q}^0(z,k))$:*

$$H^0(z,k) = |z-k|, \quad Q_0^0(z,k) = \mathrm{sgn}(z-k)\big(a(z) - a(k)\big),$$
$$\boldsymbol{Q}^0(z,k) = \mathrm{sgn}(z-k)\big(\boldsymbol{\varphi}(z) - \boldsymbol{\varphi}(k)\big).$$

In [22] entropy boundary conditions (6b)–(6c) are formulated in an equivalent way:

$$Q_0^0(u_0^\tau(\boldsymbol{x}), k) + Q_0^0(u_0^\tau(\boldsymbol{x}), u_0(\boldsymbol{x})) - Q_0^0(k, u_0(\boldsymbol{x})) \le 0, \tag{7a}$$

$$Q_0^0(u_T^\tau(\boldsymbol{x}), k) + Q_0^0(u_T^\tau(\boldsymbol{x}), u_T(\boldsymbol{x})) - Q_0^0(k, u_T(\boldsymbol{x})) \ge 0. \tag{7b}$$

When $a(z)$ is strictly increasing function, it follows that $u_0^\tau(\boldsymbol{x}) = u_0(\boldsymbol{x})$ and inequality (7b) is the triangle inequality.

Remark 3. *It is important to say that in Theorem 1 we assume that* u_0, $u_T \in L^\infty(\Omega) \cap W_0^{2,2}(\Omega)$, *but in Definition 1 it is only claimed that* u_0, $u_T \in L^\infty(\Omega)$. *We can resolve this contradiction with the help of Theorem 4 and, namely, estimate* (10). *Also, see Remark 6.*

In Definition 1 we have used Theorem 2 proved in [21] that guarantees the existence of traces u_0^τ and u_T^τ. It is important to note that boundary conditions (6b) and (6c) are similar to entropy conditions deduced in [6].

Theorem 2. *Let function* $u \in L^\infty(G_T) \cap L^2(0,T;W_0^{1,2}(\Omega))$ *be an entropy solution to the boundary value problem* (1a)–(1b). *Then there exist traces* u_0^τ, $u_T^\tau \in L^\infty(\Omega)$ *such that*

$$\operatorname*{esslim}_{t\to 0+} \int_\Omega |u(t,\boldsymbol{x}) - u_0^\tau(\boldsymbol{x})|\, d\boldsymbol{x} = 0, \quad \operatorname*{esslim}_{t\to T-} \int_\Omega |u(t,\boldsymbol{x}) - u_T^\tau(\boldsymbol{x})|\, d\boldsymbol{x} = 0. \tag{8}$$

Consider the function χ which is defined in the following way

$$\chi(\lambda;\upsilon) = \begin{cases} +1, & \text{for } 0 < \lambda < \upsilon, \\ -1, & \text{for } \upsilon < \lambda < 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition 2. *Let* N *be an positive integer,* $\mathcal{O}$ *be an open set of* $\mathbb{R}^N$ *and the microscopic function* $h \in L^\infty(\mathcal{O} \times (-L,L))$ *satisfying* $0 \le h(\boldsymbol{z},\lambda)\operatorname{sgn}(\lambda) \le 1$ *for almost every* $(\boldsymbol{z},\lambda) \in \mathbb{R}^{N+1}$. *It is said that* h *is a* χ*-function if there exists a function* $\upsilon \in L^\infty(\mathcal{O})$ *such that for a.e.* $\boldsymbol{z} \in \mathcal{O}$ *the following equality holds*

$$h(\boldsymbol{z},\lambda) = \chi(\lambda;\upsilon(\boldsymbol{z})).$$

Note that $\upsilon(\boldsymbol{z}) = \int_{-L}^{L} h(\boldsymbol{z},\lambda)\, d\lambda = \int_{-L}^{L} \chi(\lambda;\upsilon(\boldsymbol{z}))\, d\lambda$.

The following lemma formulated and proved in [8] guarantees the link between sequences of χ-functions and its limits.

Lemma 1. *Let* $\mathcal{O}$ *be an open set of* $\mathbb{R}^N$ *and* $h_n \in L^\infty(\mathcal{O} \times (-L,L))$ *be a sequence of* χ*-functions converging weakly to* $h \in L^\infty(\mathcal{O} \times (-L,L))$. *We define* $\upsilon_n(\cdot) = \int_{-L}^{L} h_n(\cdot,\lambda)\, d\lambda$ *and* $\upsilon(\cdot) = \int_{-L}^{L} h(\cdot,\lambda)\, d\lambda$. *Then the three following propositions are equivalent:*

- h_n *converges strongly to* h *in* $L^1_{\mathrm{loc}}(\mathcal{O} \times (-L,L))$,
- υ_n *converges strongly to* υ *in* $L^1_{\mathrm{loc}}(\mathcal{O})$,
- h *is a* χ*-function.*

We give kinetic formulation to the boundary value problem (1a)–(1b).

Theorem 3. *Function* $u = u(t,\boldsymbol{x})$ *is an entropy solution to the boundary value problem* (1a)–(1b) *if and only if there exist non-negative measures* n, $m \in \mathcal{M}^+(G_T \times (-M,M))$, $n = |\nabla_x u|^2 \delta_{(\lambda=u)}$, $\delta_{(\lambda=\cdot)}$ *is Dirac delta function, and* μ_0, $\mu_T \in \mathcal{M}^+(\Omega \times (-M,M))$, *such that related* χ*-function* f *defined by*

$$f(t,\boldsymbol{x},\lambda) = \chi(\lambda;u(t,\boldsymbol{x}))$$

for almost all $(t,\boldsymbol{x},\lambda) \in G_T \times (-M,M)$ *satisfies the following equalities*

$$a'(\lambda)\partial_t f(t,\boldsymbol{x},\lambda) + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f(t,\boldsymbol{x},\lambda) - \Delta_x f(t,\boldsymbol{x},\lambda) = \frac{\partial}{\partial\lambda}(m+n)(t,\boldsymbol{x},\lambda), \tag{9a}$$

(9b) $$a'(\lambda)(f_0^\tau(\boldsymbol{x},\lambda)-\chi(\lambda;u_0(\boldsymbol{x})))-\delta_{(\lambda=u_0(\boldsymbol{x}))}(a(u_0^\tau(\boldsymbol{x}))-a(u_0(\boldsymbol{x})))\\ =\partial_\lambda\mu_0(\boldsymbol{x},\lambda),$$

(9c) $$a'(\lambda)(f_T^\tau(\boldsymbol{x},\lambda)-\chi(\lambda;u_T(\boldsymbol{x})))-\delta_{(\lambda=u_T(\boldsymbol{x}))}(a(u_T^\tau(\boldsymbol{x}))-a(u_T(\boldsymbol{x})))\\ =-\partial_\lambda\mu_T(\boldsymbol{x},\lambda),$$

where $f_0^\tau(\boldsymbol{x},\lambda)$ and $f_T^\tau(\boldsymbol{x},\lambda)$ are χ-functions and are traces of χ-function f:

$$f_0^\tau(\boldsymbol{x},\lambda)=\chi(\lambda;u_0^\tau(\boldsymbol{x})),\quad f_T^\tau(\boldsymbol{x},\lambda)=\chi(\lambda;u_T^\tau(\boldsymbol{x})),$$

$$\operatorname*{esslim}_{t\to0+}\int_\Omega\int_{-M}^M|f(t,\boldsymbol{x},\lambda)-f_0^\tau(\boldsymbol{x},\lambda)|\,d\boldsymbol{x}d\lambda=0,$$

$$\operatorname*{esslim}_{t\to T-}\int_\Omega\int_{-M}^M|f(t,\boldsymbol{x},\lambda)-f_T^\tau(\boldsymbol{x},\lambda)|\,d\boldsymbol{x}d\lambda=0.$$

The proof of this theorem is trivial and it is omitted. We can only add that equivalence between (6a) and (9a) follows from the equation

$$\int_{-M}^M\eta'(\lambda)\big(a'(\lambda)\partial_t f(t,\boldsymbol{x},\lambda)+\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f(t,\boldsymbol{x},\lambda)-\Delta_x f(t,\boldsymbol{x},\lambda)\big)\,d\lambda\\ =\int_{-M}^M\eta'(\lambda)\partial_\lambda\big(n(t,\boldsymbol{x},\lambda)+m(t,\boldsymbol{x},\lambda)\big)\,d\lambda$$

which holds in the sense of distributions where η' is an arbitrary smooth function. The equivalence between (6b), (6c) and, respectively, (9b), (9c) is established in a similar way. Also we use here Theorem 2 and Lemma 1.

Due to the existence of traces u_0^τ and u_T^τ , see [21], boundary conditions (6b) and (6c) are analogues to conditions deduced in [6]. Furthermore, boundary conditions (9b) and (9c) are analogues to conditions deduced in [15], [16].

With the help of Theorems 1–3 and Theorem 5 formulated in section 2 we can prove the following theorem.

Theorem 4. *Suppose that u_0, $u_T\in L^\infty(\Omega)$. The boundary value problem* (1a)–(1b) *has the unique entropy solution $u\in L^\infty(G_T)\cap L^2(0,T;W_0^{1,2}(\Omega))$. Moreover, the following estimate holds*

(10) $$\int_{G_T}|u(t,\boldsymbol{x})-v(t,\boldsymbol{x})|\,dtd\boldsymbol{x}\\ \le C_{(57)}\int_\Omega\left(|u_0(\boldsymbol{x})-v_0(\boldsymbol{x})|+|u_T(\boldsymbol{x})-v_T(\boldsymbol{x})|\right)\,d\boldsymbol{x},$$

where two entropy solutions u and v correspond to boundary data (u_0,u_T) and, respectively, (v_0,v_T).

Remark 4. *This theorem was proved in* [22] *with the help of the double-variable method proposed in* [4] *and modified in* [7]*. The main innovation of the present paper is that we use here only kinetic formulation of entropy solution, see Theorem 3, and do not use boundary entropy-entropy flux pairs invented in* [7].

2. Relative compactness of a set of kinetic solutions $\{f_n\}_{n\in\mathbb{N}}$.

In this section the auxiliary result about relative compactness of kinetic solutions is given. We use this result in section 3 to establish relative compactness of a set $\{u_\varepsilon\}_{\varepsilon>0}$.

In the domain $(z_0, \boldsymbol{z}, \lambda) \in \mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda$ we deal with equation

$$(11)\quad a'(\lambda)\partial_{z_0} f_n(z_0, \boldsymbol{z}, \lambda) + \boldsymbol{\varphi}'(\lambda) \cdot \nabla_z f_n(z_0, \boldsymbol{z}, \lambda) - \Delta_z f_n(z_0, \boldsymbol{z}, \lambda) = \partial_\lambda \nu_n + \partial_t(g_{0,0,n} + \partial_\lambda g_{1,0,n}) + \sum_{k=1}^{d} \partial_{x_k}(g_{0,k,n} + \partial_\lambda g_{1,k,n}),$$

where $\nu_n \in \mathcal{M}^+(\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda)$, $\operatorname{supp} \nu_n \subset \mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times (-M, M)$, and function $f_n \in L^\infty(\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda)$ is χ-function with support in $\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times (-M, M)$. We assume that the sequence f_n converges *-weak in $L^\infty(\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda)$ to function f, and subsequence ν_n converges *weak in $\mathcal{M}^+(\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda)$ to measure ν, sequences $g_{j,k,n} \to 0$ in $L^2(\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda)$, $j = 0, 1$, $k = 0, 1, \dots, d$.

Condition 2. *We assume that the following non-degeneracy condition holds*

$$(12)\qquad 2\pi i a'(\lambda)\eta_0 - 4\pi^2|\boldsymbol{\eta}|^2 \neq 0 \quad (\text{for a.e. } \lambda \in \mathbb{R})$$

for every $(\eta_0, \boldsymbol{\eta}) \in P_2 = \{(\eta_0, \boldsymbol{\eta}) \in \mathbb{R} \times \mathbb{R}^d \;:\; |\eta_0|^2 + |\boldsymbol{\eta}|^4 = 1\}$.

This condition is similar to condition (11) formulated in [23].

Remark 5. *It is clear that when $\boldsymbol{\eta} = 0$, Condition 2 is equivalent to Condition 1 on function $a(z)$. It is known that similar conditions (nonlinear-diffusivity condition) were deduced in* [19], [24], *but in* [23] *the technique of H_{P}-measures enables to exclude terms with components of $\boldsymbol{\varphi}'(\lambda)$ in* (12).

Theorem 5. *Set $\{f_n\}_{n\in\mathbb{N}}$ is relatively compact in the space $L^1_{loc}(\mathbb{R}^+_{z_0} \times \mathbb{R}^d_{\boldsymbol{z}} \times \mathbb{R}_\lambda)$.*

This compactness theorem was proved in [23] when $g_{j,i,n} = 0$, $j = 0, 1$, $i = 0, \dots, d$. In (11) we take the similar right-hand side as in [25].

3. Proof of Theorem 1

The existence and uniqueness of weak solution to the boundary value problem (2a)–(2b) can be proved with help of well-known results on nonlinear elliptic equations [26], [28].

Remark 6. *We need here to impose additional condition on initial and final data u_0, u_T such that its extension into domain G_T should be smooth. Namely, $\hat{u} \in C^2(\overline{G_T})$, where $\widehat{u}(t, \boldsymbol{x}) = \frac{(T-t)}{T} u_0(\boldsymbol{x}) + \frac{t}{T} u_T(\boldsymbol{x})$. This enables to affirm that there exist traces on the boundaries Γ_0 and Γ_T of $\partial_t u_\varepsilon$: $\partial_t u_\varepsilon(0, \boldsymbol{x})$, $\partial_t u_\varepsilon(T, \boldsymbol{x}) \in L^\infty(\Omega)$. This result is significant in Lemma 2. Also, see Remark 3.*

3.1. **Maximum principle** (3).

Let us introduce the function u^M_ε:

$$(13)\qquad u^M_\varepsilon = \max(u_\varepsilon - M, 0) = \begin{cases} u_\varepsilon - M & \text{if } u_\varepsilon > M \\ 0 & \text{if } u_\varepsilon \leq M \end{cases}, \quad u^M_\varepsilon\big|_{\partial G_T} = 0.$$

Therefore, $\nabla_x u_\varepsilon^M$ and $\partial_t u_\varepsilon^M$ are defined in the following way:

(14) $$\nabla_x u_\varepsilon^M = \begin{cases} \nabla_x u_\varepsilon, & u_\varepsilon > M, \\ 0, & u_\varepsilon \le M, \end{cases} \qquad \partial_t u_\varepsilon^M = \begin{cases} \partial_t u_\varepsilon, & u_\varepsilon > M, \\ 0, & u_\varepsilon \le M. \end{cases}.$$

Multiplying equation (2a) by u_ε^M and integrating by parts give

(15) $$\int_{G_T} \left(\left|\nabla_x u_\varepsilon^M\right|^2 + \varepsilon \left|\partial_t u_\varepsilon^M\right|^2 \right) dtd\boldsymbol{x} = I_1 + I_2,$$

where

(16) $$I_1 := -\int_{G_T} \partial_t a(u_\varepsilon) u_\varepsilon^M \, dtd\boldsymbol{x}, \; I_2 := -\int_{G_T} \boldsymbol{\varphi}'(u_\varepsilon) \cdot \nabla_x u_\varepsilon^M u_\varepsilon^M \, dtd\boldsymbol{x}.$$

Taking into account the properties of the functions u_ε^M, $\nabla_x u_\varepsilon^M$, $\partial_t u_\varepsilon^M$, we have

(17)
$$I_1 = -\int_{G_T} a'(u_\varepsilon^M + M) u_\varepsilon^M \partial_t u_\varepsilon^M \, dtd\boldsymbol{x} = -\int_{G_T} \partial_t \left(\int_0^{u_\varepsilon^M} a'(\xi + M)\xi \; d\xi \right) dtd\boldsymbol{x}$$
$$= -\int_\Omega \left(\int_0^{u_\varepsilon^M} a'(\xi + M)\xi \; d\xi \right) \, d\boldsymbol{x} \Bigg|_{t=0}^{t=T} = 0,$$

(18)
$$I_2 = -\int_{G_T} \boldsymbol{\varphi}'(u_\varepsilon^M + M) \cdot \nabla_x u_\varepsilon^M u_\varepsilon^M \, dtd\boldsymbol{x} = -\int_{G_T} \operatorname{div}_x \left(\int_0^{u_\varepsilon^M} \boldsymbol{\varphi}'(\xi + M)\xi d\xi \right) dtd\boldsymbol{x} = 0.$$

Hence, according to (15), (16), (17), (18), we get

(19) $$\int_{G_T} \left(\left|\nabla_x u_\varepsilon^M\right|^2 + \varepsilon \left|\partial_t u_\varepsilon^M\right|^2 \right) dtd\boldsymbol{x} = 0$$

and

(20) $$u_\varepsilon^M = 0 \Longrightarrow u_\varepsilon \le M.$$

Analogously, we obtain that

(21) $$-u_\varepsilon \le M \text{ and } |u_\varepsilon| \le M.$$

3.2. **Estimates** (4)**.**

Assume that the functions u_0, u_T permit extension $\widehat{u}$ on G_T such that

(22) $$\widehat{u} \in W^{1,p}(G_T), \; \widehat{u}|_{\Gamma_l} = 0, \quad \widehat{u}|_{\Gamma_0} = u_0, \; \widehat{u}|_{\Gamma_T} = u_T,$$

and

(23) $$\|\widehat{u}\|_{L^\infty(G_T)} \le M = \max\Big(\|u_0\|_{L^\infty(\Omega)}, \|u_T\|_{L^\infty(\Omega)} \Big) < \infty.$$

Multiplying equation (2a) by $u_\varepsilon - \widehat{u}$ and integrating by parts, we obtain

(24) $$-\int_{G_T} \partial_t (u_\varepsilon - \hat{u}) a(u_\varepsilon) \, dtd\boldsymbol{x} = -\int_{G_T} \nabla_x (u_\varepsilon - \hat{u}) \cdot \nabla_x u_\varepsilon \, dtd\boldsymbol{x}$$
$$+ \int_{G_T} \nabla_x (u_\varepsilon - \hat{u}) \cdot \boldsymbol{\varphi}(u_\varepsilon) \, dtd\boldsymbol{x} - \varepsilon \int_{G_T} \partial_t (u_\varepsilon - \hat{u}) \partial_t u_\varepsilon \, dtd\boldsymbol{x}.$$

This reads in the following way

(25) $$\int_{G_T} \left(|\nabla_x u_\varepsilon|^2 + \varepsilon \, |\partial_t u_\varepsilon|^2 \right) dtd\boldsymbol{x} = J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 := \int_{G_T} (\nabla_x u_\varepsilon \cdot \nabla_x \widehat{u} + \varepsilon \partial_t u_\varepsilon \partial_t \widehat{u}) \, dtd\boldsymbol{x}, \quad J_2 := \int_{G_T} a(u_\varepsilon) \partial_t u_\varepsilon \, dtd\boldsymbol{x}, \tag{26}$$

$$J_3 := -\int_{G_T} a(u_\varepsilon) \partial_t \widehat{u} \, dtd\boldsymbol{x}, \quad J_4 := \int_{G_T} (\nabla_x u_\varepsilon - \nabla_x \widehat{u}) \cdot \boldsymbol{\varphi}(u_\varepsilon) \, dtd\boldsymbol{x}. \tag{27}$$

By the help of the Young inequality

$$ab \leq \frac{1}{2}(\delta a)^2 + \frac{1}{2}\left(\frac{b}{\delta}\right)^2, \quad a, b \geq 0, \ \delta \in (0,1),$$

we estimate the terms $|J_1|$, $|J_3|$, $|J_4|$:

$$|J_1| \leq \int_{G_T} \left(\frac{\delta^2}{2} |\nabla_x u_\varepsilon|^2 + \frac{1}{2}\delta^{-2} |\nabla_x \widehat{u}|^2 + \frac{\varepsilon \delta_1^2}{2} |\partial_t u_\varepsilon|^2 + \frac{\varepsilon \delta_1^{-2}}{2} |\partial_t \widehat{u}|^2 \right) dtd\boldsymbol{x}, \tag{28}$$

$$|J_3| \ \leq \int_{G_T} \left(\frac{1}{2} |\partial_t \widehat{u}|^2 + \frac{1}{2} |a(u_\varepsilon)|^2 \right) dtd\boldsymbol{x}, \tag{29}$$

$$|J_4| \ \leq \int_{G_T} \left(\frac{1}{2}\delta_2^2 |\nabla_x u_\varepsilon|^2 + \frac{1}{2}\delta_2^{-2} |\boldsymbol{\varphi}(u_\varepsilon)|^2 + \frac{1}{2} |\nabla_x \widehat{u}|^2 + \frac{1}{2} |\boldsymbol{\varphi}(u_\varepsilon)|^2 \right) dtd\boldsymbol{x}. \tag{30}$$

We evaluate the term J_2 in the following way

$$|J_2| = \left| \int_\Omega \left(\int_0^{u_\varepsilon(t,\boldsymbol{x})} a(\xi) \ d\xi \right) \ d\boldsymbol{x} \Big|_{t=0}^{t=T} \right| \leq 2 \sup_{|z| \leq M} \left| \int_0^z a(\xi) \ d\xi \right| |\Omega| . \tag{31}$$

Gathering last estimates and choosing δ_2, δ_1, δ appropriately small and taking into account (21), we find that

$$\begin{aligned} \int_{G_T} \left(|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_t u_\varepsilon|^2 \right) dtd\boldsymbol{x} \leq \\ C \int_{G_T} \left(|\partial_t \widehat{u}|^2 + |\nabla_x \widehat{u}|^2 + |a(u_\varepsilon)|^2 + |\boldsymbol{\varphi}(u_\varepsilon)|^2 \right) dtd\boldsymbol{x} \\ + 2 \sup_{|z| \leq M} \left| \int_0^z a(\xi) \ d\xi \right| |\Omega| \leq C \int_{G_T} \left(|\partial_t \widehat{u}|^2 + |\nabla_x \widehat{u}|^2 \right) dtd\boldsymbol{x} \\ + C(|\Omega|, T) \left(\sup_{|z| \leq M} |a(z)|^2 + \sup_{|z| \leq M} |\boldsymbol{\varphi}(z)|^2 + \sup_{|z| \leq M} \left| \int_0^z a(\xi) \ d\xi \right| \right) =: C_{(32)}. \end{aligned} \tag{32}$$

□

3.3. **Relative compactness of $\{u_\varepsilon\}$.**

Recall that we going to prove the existence of an entropy solution to the boundary value problem (1a)–(1b). We multiply equation (2a) by $\eta'(u_\varepsilon)$:

$$\partial_t q_0(u_\varepsilon) + \operatorname{div}_x \boldsymbol{q}(u_\varepsilon) - \Delta_x \eta(u_\varepsilon) = \varepsilon \partial_t^2 \eta(u_\varepsilon) - \eta''(u_\varepsilon)\big(|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_t u_\varepsilon|^2\big). \tag{33}$$

This equation can be expressed in the following way

$$\int_{-M}^{M}\eta'(\lambda)\big(a'(\lambda)\partial_t f_\varepsilon+\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f_\varepsilon-\Delta_x f_\varepsilon\big)\,d\lambda=\int_{-M}^{M}\big(\varepsilon\eta'(\lambda)\partial_t(\partial_t u_\varepsilon\delta_{(\lambda=u_\varepsilon)})$$
$$-\eta''(\lambda)\big((|\nabla_x u_\varepsilon|^2+\varepsilon|\partial_t u_\varepsilon|^2)\delta_{(\lambda=u_\varepsilon)}\big)\big)\,d\lambda=\int_{-M}^{M}\eta'(\lambda)\big(\varepsilon\partial_t(\partial_t u_\varepsilon\delta_{(\lambda=u_\varepsilon)})$$
$$+\partial_\lambda\big((|\nabla_x u_\varepsilon|^2+\varepsilon|\partial_t u_\varepsilon|^2)\delta_{(\lambda=u_\varepsilon)}\big)\big)\,d\lambda.$$

Therefore, we get a kinetic equation

$$a'(\lambda)\partial_t f_\varepsilon+\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f_\varepsilon-\Delta_x f_\varepsilon=\varepsilon\partial_t(\partial_t u_\varepsilon\delta_{(\lambda=u_\varepsilon)})+\partial_\lambda\big((|\nabla_{\boldsymbol{x}} u_\varepsilon|^2$$
$$+\varepsilon|\partial_t u_\varepsilon|^2)\delta_{(\lambda=u_\varepsilon)}\big)=\partial_t g_{0,\varepsilon}+\partial_\lambda\nu_\varepsilon,$$

where the function $g_{0,\varepsilon}:=\varepsilon\partial_t u_\varepsilon\delta_{(\lambda=u_\varepsilon)}\to 0$ in $L^2(G_T\times(-M,M))$;

$$\nu_\varepsilon:=\big(|\nabla_{\boldsymbol{x}} u_\varepsilon|^2+\varepsilon|\partial_t u_\varepsilon|^2\big)\delta_{(\lambda=u_\varepsilon)}\to n+m$$

*weak in $\mathcal{M}^+(G_T\times(-M,M))$. Therefore, we can apply Theorem 5 and Lemma 1.

Remark 7. *We can prove relative compactness for* $\{u_\varepsilon\}_{\varepsilon>0}$ *with help of estimates* (3), (4) *and two results:*

(1) *The set* $\{a(u_\varepsilon)-\varepsilon\partial_t u_\varepsilon\}_{\varepsilon>0}$ *is uniformly bounded in* $L^2(0,T;W^{-1,2}(\Omega))$.
(2) $\lim\limits_{h\to 0+}\|g(u_\varepsilon(t+h,\cdot))-g(u_\varepsilon(t,\cdot))\|_{W^{-s,2}(\Omega)}=0$, *where* $g(z)=\int_0^z(a'(\tau))^2\,d\tau$.

Therefore, we can apply the famous Aubin–Lions–Simon lemma to $\{g(u_\varepsilon)\}_{\varepsilon>0}$, *see* [29], [30]. *Since* $g(z)$ *is monotonic function, it follows that* $\{u_\varepsilon\}_{\varepsilon>0}$ *is a relatively compact set.*

Remark 8. *With the help of the relative compactness result in the limits as* $\varepsilon\to+0$ *inequality* (6a) *follows from equality* (33) *and estimates* (4).

3.4. **Entropy boundary conditions** (6b) **and** (6c).

The main idea is that we use compactness of u_ε and existence of the functions u_0^τ and u_T^τ defined by (8) in order to prove the following Lemma which is similar to Lemma 1.1 in [6].

Lemma 2. *For any test function* $\theta\in C_0^1(\Omega)$ *it follows that*

$$(34)\qquad -\lim_{\varepsilon\to 0+}\int_\Omega\theta(\boldsymbol{x})\varepsilon\partial_t u_\varepsilon(0,\boldsymbol{x})\,d\boldsymbol{x}=\int_\Omega\theta(\boldsymbol{x})\big(a(u_0^\tau(\boldsymbol{x}))-a(u_0(\boldsymbol{x}))\big)\,d\boldsymbol{x},$$

$$(35)\qquad -\lim_{\varepsilon\to 0+}\int_\Omega\theta(\boldsymbol{x})\varepsilon\partial_t u_\varepsilon(T,\boldsymbol{x})\,d\boldsymbol{x}=\int_\Omega\theta(\boldsymbol{x})\big(a(u_T^\tau(\boldsymbol{x}))-a(u_T(\boldsymbol{x}))\big)\,d\boldsymbol{x}.$$

Доказательство. We integrate by parts the following left hand side:

$$(36)\quad \varepsilon\int_\Omega\theta(\boldsymbol{x})\int_0^\delta\partial_t^2 u_\varepsilon\rho_\delta(t)\,dtd\boldsymbol{x}=-\varepsilon\int_\Omega\theta(\boldsymbol{x})\int_0^\delta\partial_t u_\varepsilon(t,\boldsymbol{x})\rho_\delta'(t)\,dtd\boldsymbol{x}$$
$$-\varepsilon\int_\Omega\theta(\boldsymbol{x})\partial_t u_\varepsilon(0,\boldsymbol{x})\;d\boldsymbol{x},$$

where θ is a nonnegative test function and $\rho_\delta(t)\in C^2(\mathbb{R}^+)$:

$$\rho_\delta(t)=0,\quad t>\delta,\quad \rho_\delta(0)=1,\quad |\rho_\delta'(t)|\le\frac{c}{\delta},\quad \lim_{\delta\to 0+}\int_0^\delta\Phi(t)\rho_\delta'(t)\,dt=-\Phi(0).$$

From (4) it follows that

$$\lim_{\varepsilon\to 0+} \sqrt{\varepsilon}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\sqrt{\varepsilon}\partial_t u_\varepsilon(t,\boldsymbol{x})\rho'_\delta(t)\, dtd\boldsymbol{x} = 0 \quad \text{for all} \quad \delta > 0.$$

In the limit as $\varepsilon \to 0+$ equation (36) reads

$$\lim_{\varepsilon\to 0+} \varepsilon\int_\Omega \theta(\boldsymbol{x})\partial_t u_\varepsilon(0,\boldsymbol{x})\ d\boldsymbol{x} = -\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\varepsilon\partial_t^2 u_\varepsilon(t,\boldsymbol{x})\rho_\delta(t)\, dtd\boldsymbol{x} =: A(\delta). \tag{37}$$

Function $A(\delta)$ can be rewritten in the following way

$$\begin{aligned}
A(\delta) = &-\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\varepsilon\partial_t^2 u_\varepsilon(t,\boldsymbol{x})\rho_\delta(t)\, dtd\boldsymbol{x}\\
&= -\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\partial_t a(u_\varepsilon(t,\boldsymbol{x}))\rho_\delta(t)\, dtd\boldsymbol{x}\\
&\quad -\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\varphi}(u_\varepsilon(t,\boldsymbol{x}))\rho_\delta(t)\, dtd\boldsymbol{x}\\
&\quad -\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \nabla_x\theta(\boldsymbol{x})\cdot\nabla_x u_\varepsilon(t,\boldsymbol{x})\rho_\delta(t)\, dtd\boldsymbol{x}\\
= &\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})a(u_\varepsilon(t,\boldsymbol{x}))\rho'_\delta(t)\, dtd\boldsymbol{x} + \int_\Omega \theta(\boldsymbol{x})a(u_0(\boldsymbol{x}))\ d\boldsymbol{x}\\
&\quad -\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\varphi}(u_\varepsilon(t,\boldsymbol{x}))\rho_\delta(t)\, dtd\boldsymbol{x}\\
&\quad -\lim_{\varepsilon\to 0+}\int_0^\delta \int_\Omega \nabla_x\theta(\boldsymbol{x})\cdot\nabla_x u_\varepsilon(t,\boldsymbol{x})\rho_\delta(t)\, dtd\boldsymbol{x}\\
= &\int_0^\delta \int_\Omega \theta(\boldsymbol{x})a(u(t,\boldsymbol{x}))\rho'_\delta(t)\, dtd\boldsymbol{x} + \int_\Omega \theta(\boldsymbol{x})a(u_0(\boldsymbol{x}))\ d\boldsymbol{x}\\
&-\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\varphi}(u(t,\boldsymbol{x}))\rho_\delta(t)\, dtd\boldsymbol{x} - \int_0^\delta \int_\Omega \nabla_x\theta(\boldsymbol{x})\cdot\nabla_x u(t,\boldsymbol{x})\rho_\delta(t)\, dtd\boldsymbol{x}.
\end{aligned}$$

With the help of

$$\lim_{\delta\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})\mathrm{div}_x\boldsymbol{\varphi}(u(t,\boldsymbol{x}))\rho_\delta(t)\, dtd\boldsymbol{x} = 0,$$

$$\lim_{\delta\to 0+}\int_0^\delta \int_\Omega \nabla_x\theta(\boldsymbol{x})\cdot\nabla_x u(t,\boldsymbol{x})\rho_\delta(t)\, dtd\boldsymbol{x} = 0$$

and

$$\lim_{\delta\to 0+}\int_0^\delta \int_\Omega \theta(\boldsymbol{x})a(u(t,\boldsymbol{x}))\rho'_\delta(t)\, dtd\boldsymbol{x} = -\int_\Omega \theta(\boldsymbol{x})a(u_0^\tau(\boldsymbol{x}))\ d\boldsymbol{x},$$

we obtain

$$\lim_{\varepsilon\to 0+} \varepsilon\int_\Omega \theta(\boldsymbol{x})\partial_t u_\varepsilon(0,\boldsymbol{x})\ d\boldsymbol{x} = \lim_{\delta\to 0+} A(\delta) = \int_\Omega \theta(\boldsymbol{x})\big(a(u_0(\boldsymbol{x})) - a(u_0^\tau(\boldsymbol{x}))\big)\ d\boldsymbol{x}.$$

It is obvious that formula (35) can be deduced in a similar way. □

Let $\vartheta(\boldsymbol{x})$ be nonnegative test function. We multiply (33) by $\vartheta(\boldsymbol{x})\rho_\delta(t)$ and integrate over G_T:

$$\int_\Omega\int_0^\delta \partial_t q_0(u_\varepsilon(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta(t)\,dtd\boldsymbol{x} = \int_\Omega\int_0^\delta \big(\Delta_x\eta(u_\varepsilon)+\varepsilon\partial_t^2\eta(u_\varepsilon) \tag{38}$$
$$-\operatorname{div}_x\boldsymbol{q}(u_\varepsilon)\big)\vartheta(\boldsymbol{x})\rho_\delta(t)\,dtd\boldsymbol{x} - \int_\Omega\int_0^\delta \eta''(u_\varepsilon)\big(|\nabla_x u_\varepsilon|^2+\varepsilon|\partial_t u_\varepsilon|^2\big)\vartheta(\boldsymbol{x})\rho_\delta(t)\,dtd\boldsymbol{x}.$$

In (38) we integrate by parts the left hand side and the right hand side:

$$-\int_\Omega\int_0^\delta q_0(u_\varepsilon(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta'(t)\,dtd\boldsymbol{x} - \int_\Omega q_0(u_0(\boldsymbol{x}))\vartheta(\boldsymbol{x})\;d\boldsymbol{x}$$
$$\le \int_\Omega\int_0^\delta \big(-\nabla_x\eta(u_\varepsilon)\cdot\nabla_x\vartheta(\boldsymbol{x}) - \operatorname{div}_x\boldsymbol{q}(u_\varepsilon)\vartheta(\boldsymbol{x})\big)\rho_\delta(t)\,dtd\boldsymbol{x}$$
$$-\sqrt{\varepsilon}\int_\Omega\int_0^\delta \sqrt{\varepsilon}\partial_t\eta(u_\varepsilon(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta'(t)\,dtd\boldsymbol{x} - \varepsilon\int_\Omega \eta'(u_0(\boldsymbol{x}))\partial_t u_\varepsilon(0,\boldsymbol{x})\vartheta(\boldsymbol{x})\,dtd\boldsymbol{x}.$$

In the limit as $\varepsilon\to 0+$ with the help of (34) we get:

$$-\int_\Omega\int_0^\delta q_0(u(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta'(t)\,dtd\boldsymbol{x} - \int_\Omega q_0(u_0(\boldsymbol{x}))\vartheta(\boldsymbol{x})\;d\boldsymbol{x} \tag{39}$$
$$\le \int_\Omega\int_0^\delta \big(-\nabla_x\eta(u)\cdot\nabla_x\vartheta(\boldsymbol{x}) - \operatorname{div}_x\boldsymbol{q}(u)\vartheta(\boldsymbol{x})\big)\rho_\delta(t)\,dtd\boldsymbol{x}$$
$$+\int_\Omega \eta'(u_0(\boldsymbol{x}))\big(a(u_0^\tau(\boldsymbol{x}))-a(u_0(\boldsymbol{x}))\big)\vartheta(\boldsymbol{x})\;d\boldsymbol{x}.$$

The boundary condition (6b) follows from (39) as $\delta\to 0+$.

Furthermore, the boundary condition (6c) is deduced in a similar way. We multiply (33) by $-\vartheta(\boldsymbol{x})\rho_\delta(T-t)$ and integrate by parts over G_T:

$$-\int_{T-\delta}^T\int_\Omega q_0(u_\varepsilon(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta'(T-t)\,dtd\boldsymbol{x} - \int_\Omega q_0(u_T(\boldsymbol{x}))\vartheta(\boldsymbol{x})\;d\boldsymbol{x} \tag{40}$$
$$\ge \int_{T-\delta}^T\int_\Omega \big(\nabla_x\eta(u_\varepsilon(t,\boldsymbol{x}))\cdot\nabla_x\vartheta(\boldsymbol{x}) + \operatorname{div}_x\boldsymbol{q}(u_\varepsilon(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\big)\rho_\delta(t)\,dtd\boldsymbol{x}$$
$$-\sqrt{\varepsilon}\int_{T-\delta}^T\int_\Omega \sqrt{\varepsilon}\partial_t\eta(u_\varepsilon(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta'(T-t)\,dtd\boldsymbol{x} - \varepsilon\int_\Omega \eta'(u_T(\boldsymbol{x}))\partial_t u_\varepsilon(T,\boldsymbol{x})\vartheta(\boldsymbol{x})\,d\boldsymbol{x}.$$

With the help of (35) we pass to the limit in (40) as $\varepsilon\to 0+$:

$$-\int_{T-\delta}^T\int_\Omega q_0(u(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\rho_\delta'(T-t)\,dtd\boldsymbol{x} - \int_\Omega q_0(u_T(\boldsymbol{x}))\vartheta(\boldsymbol{x})\;d\boldsymbol{x} \tag{41}$$
$$\ge \int_{T-\delta}^T\int_\Omega \big(\nabla_x\eta(u(t,\boldsymbol{x}))\cdot\nabla_x\vartheta(\boldsymbol{x}) + \operatorname{div}_x\boldsymbol{q}(u(t,\boldsymbol{x}))\vartheta(\boldsymbol{x})\big)\rho_\delta(T-t)\,dtd\boldsymbol{x}$$
$$+\int_\Omega \eta'(u_T(\boldsymbol{x}))(a(u_T^\tau(\boldsymbol{x}))-a(u_T(\boldsymbol{x})))\vartheta(\boldsymbol{x})\;d\boldsymbol{x}.$$

Finally, the boundary condition (6c) follows from (41) as $\delta\to 0+$.

4. Proof of Theorem 4

Let f and g be χ-functions having traces and satisfying the following equations taken from Theorem 3:

$$a'(\lambda)\partial_t f(t,\boldsymbol{x},\lambda) + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f(t,\boldsymbol{x},\lambda) - \Delta_x f(t,\boldsymbol{x},\lambda) = \partial_\lambda(m^1(t,\boldsymbol{x},\lambda) + n^1(t,\boldsymbol{x},\lambda)), \tag{42a}$$

$$a'(\lambda)(f_0^\tau(\boldsymbol{x},\lambda) - \chi(\lambda;u_0(\boldsymbol{x}))) - \delta_{(\lambda=u_0(\boldsymbol{x}))}(a(u_0^\tau(\boldsymbol{x})) - a(u_0(\boldsymbol{x}))) = \partial_\lambda \mu_{0,f}(\boldsymbol{x},\lambda), \tag{42b}$$

$$a'(\lambda)(f_T^\tau(\boldsymbol{x},\lambda) - \chi(\lambda;u_T(\boldsymbol{x}))) - \delta_{(\lambda=u_T(\boldsymbol{x}))}(a(u_T^\tau(\boldsymbol{x})) - a(u_T(\boldsymbol{x}))) = -\partial_\lambda \mu_{T,f}(\boldsymbol{x},\lambda), \tag{42c}$$

$$a'(\lambda)\partial_t g(t,\boldsymbol{x},\lambda) + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x g(t,\boldsymbol{x},\lambda) - \Delta_x g(t,\boldsymbol{x},\lambda) = \partial_\lambda(m^2 + n^2), \tag{43a}$$

$$a'(\lambda)(g_0^\tau(\boldsymbol{x},\lambda) - \chi(\lambda;v_0(\boldsymbol{x}))) - \delta_{(\lambda=v_0(\boldsymbol{x}))}(a(v_T^\tau(\boldsymbol{x})) - a(v_0(\boldsymbol{x}))) = \partial_\lambda \mu_{0,g}(\boldsymbol{x},\lambda), \tag{43b}$$

$$a'(\lambda)(g_T^\tau(\boldsymbol{x},\lambda) - \chi(\lambda;v_T(\boldsymbol{x}))) - \delta_{(\lambda=v_T(\boldsymbol{x}))}(a(v_T^\tau(\boldsymbol{x})) - a(v_T(\boldsymbol{x}))) = -\partial_\lambda \mu_{T,g}(\boldsymbol{x},\lambda), \tag{43c}$$

where $\mu_{0,f},\ \mu_{0,g},\ \mu_{T,f},\ \mu_{T,g} \in \mathcal{M}^+(\Omega\times(-M,M))$, $n^1,\ n^2,\ m^1,\ m^2 \in \mathcal{M}^+(G_T\times(-M,M))$, $f_0^\tau(\boldsymbol{x},\lambda) = \chi(\lambda;u_0^\tau(\boldsymbol{x}))$, $f_T^\tau(\boldsymbol{x},\lambda) = \chi(\lambda;u_T^\tau(\boldsymbol{x}))$, $g_0^\tau(\boldsymbol{x},\lambda) = \chi(\lambda;v_T^\tau(\boldsymbol{x}))$, $g_T^\tau(\boldsymbol{x},\lambda) = \chi(\lambda;v_T^\tau(\boldsymbol{x}))$.

Lemma 3. *For an arbitrary nonnegative function $\xi \in C^2(\Omega)\cap C_0^1(\overline{\Omega})$ the following inequality holds*

$$\int_{G_T}\int_{-M}^{M}\Big(-\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x\xi(\boldsymbol{x})|f(t,\boldsymbol{x},\lambda) - g(t,\boldsymbol{x},\lambda)|^2 - \Delta_x\xi(\boldsymbol{x})|f(t,\boldsymbol{x},\lambda) - g(t,\boldsymbol{x},\lambda)|^2\Big)\,dtd\boldsymbol{x}d\lambda \le \int_\Omega\int_{-M}^{M} a'(\lambda)\Big(|f_0^\tau(\boldsymbol{x},\lambda) - g_0^\tau(\boldsymbol{x},\lambda)|^2 - |f_T^\tau(\boldsymbol{x},\lambda) - g_T^\tau(\boldsymbol{x},\lambda)|^2\Big)\xi(\boldsymbol{x})\ d\boldsymbol{x}d\lambda. \tag{44}$$

Доказательство. Here we use methods from [13], [14], [15], [16]. Set $\epsilon = (\epsilon_1,\epsilon_2)$. We define auxiliary function ϕ_ϵ by

$$\phi_\epsilon(t,\boldsymbol{x}) = \frac{1}{\epsilon_1}\phi_1\Big(\frac{t}{\epsilon_1}\Big)\frac{1}{\epsilon_2^d}\phi_2\Big(\frac{\boldsymbol{x}}{\epsilon_2}\Big),$$

where $\phi_1 \in C_c^\infty(\mathbb{R})$, $\phi_2 \in C_c^\infty(\mathbb{R}^d)$, $\phi_j \ge 0$, $\int_{\mathbb{R}}\phi_1 = \int_{\mathbb{R}^d}\phi_2 = 1$, $\mathrm{supp}\phi_1 \subset (-1,1)$, $\mathrm{supp}\phi_2 \subset (-1,1)^d$. We shall use the following notations:

$$f_\epsilon(t,\boldsymbol{x},\lambda) = f(\cdot,\cdot,\lambda) *_{(t,\boldsymbol{x})} \phi_\epsilon(t,\boldsymbol{x}),\quad g_\epsilon(t,\boldsymbol{x},\lambda) = g(\cdot,\cdot,\lambda) *_{(t,\boldsymbol{x})} \phi_\epsilon(t,\boldsymbol{x}),$$
$$m_\epsilon^1(t,\boldsymbol{x},\lambda) = m^1(\cdot,\cdot,\lambda) *_{(t,\boldsymbol{x})} \phi_\epsilon(t,\boldsymbol{x}),\quad m_\epsilon^2(t,\boldsymbol{x},\lambda) = m^2(\cdot,\cdot,\lambda) *_{(t,\boldsymbol{x})} \phi_\epsilon(t,\boldsymbol{x}),$$
$$n_\epsilon^1(t,\boldsymbol{x},\lambda) = n^1(\cdot,\cdot,\lambda) *_{(t,\boldsymbol{x})} \phi_\epsilon(t,\boldsymbol{x}),\quad n_\epsilon^2(t,\boldsymbol{x},\lambda) = n^2(\cdot,\cdot,\lambda) *_{(t,\boldsymbol{x})} \phi_\epsilon(t,\boldsymbol{x}).$$

Therefore, we multiply the following equation

$$a'(\lambda)\partial_t\big(f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)\big)+\boldsymbol{\varphi}'(\lambda)\cdot\nabla_{\boldsymbol{x}}\big(f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)\big)\\ -\Delta_{\boldsymbol{x}}\big(f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)\big)=\partial_\lambda(m^1_\epsilon-m^2_\epsilon)+\partial_\lambda(n^1_\epsilon-n^2_\epsilon)$$

by $2(f_\epsilon-g_\epsilon)$:

$$a'(\lambda)\partial_t|f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)|^2+\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x|f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)|^2\\ -\Delta_x\big|f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)\big|^2+2|\nabla_x\big(f_\epsilon(t,\boldsymbol{x},\lambda)-g_\epsilon(t,\boldsymbol{x},\lambda)\big)|^2\\ =\partial_\lambda(m^1_\epsilon-m^2_\epsilon)+\partial_\lambda(n^1_\epsilon-n^2_\epsilon).$$

We multiply the latter equation by arbitrary nonnegative function $\xi\in C^2(\Omega)\cap C^1_0(\overline{\Omega})$ and integrate over $G_T\times(-M,M)$:

$$\int_{G_T}\int_{-M}^{M}\Big(a'(\lambda)\partial_t|f_\epsilon-g_\epsilon|^2\xi(\boldsymbol{x})-\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x\xi(\boldsymbol{x})|f_\epsilon-g_\epsilon|^2\\ -\Delta_x\xi(\boldsymbol{x})|f_\epsilon-g_\epsilon|^2+2|\nabla_x(f_\epsilon-g_\epsilon)|^2\xi(\boldsymbol{x})\Big)\,dtd\boldsymbol{x}d\lambda\\ =2\int_{G_T}\int_{-M}^{M}\partial_\lambda(m^1_\epsilon-m^2_\epsilon)(f_\epsilon-g_\epsilon)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda\\ +2\int_{G_T}\int_{-M}^{M}\partial_\lambda(n^1_\epsilon-n^2_\epsilon)(f_\epsilon-g_\epsilon)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda. \tag{45}$$

It is important to construct several limits:

$$\lim_{\epsilon\to0+}\int_{-M}^{M}\big|f_\epsilon(\cdot,\cdot,\lambda)-g_\epsilon(\cdot,\cdot,\lambda)\big|^2\,d\lambda\\ =\int_{-M}^{M}\big|f(\cdot,\cdot,\lambda)-g(\cdot,\cdot,\lambda)\big|^2\,d\lambda\ \text{ in }\ L^1(G_T),$$

$$\lim_{\epsilon\to0+}\int_{-M}^{M}a'(\lambda)\big|f_\epsilon(\cdot,\cdot,\lambda)-g_\epsilon(\cdot,\cdot,\lambda)\big|^2\,d\lambda\\ =\int_{-M}^{M}a'(\lambda)\big|f(\cdot,\cdot,\lambda)-g(\cdot,\cdot,\lambda)\big|^2\,d\lambda\ \text{ in }\ L^1(G_T),$$

$$\lim_{\epsilon\to0+}\int_{-M}^{M}\varphi_i'(\lambda)|f_\epsilon(\cdot,\cdot,\lambda)-g_\epsilon(\cdot,\cdot,\lambda)|^2\,d\lambda\\ =\int_{-M}^{M}\varphi_i'(\lambda)|f(\cdot,\cdot,\lambda)-g(\cdot,\cdot,\lambda)|^2\,d\lambda\ \text{ in }\ L^1(G_T),\quad i=1,\dots,d.$$

With the help of the following assertion which is analogous to Proposition 2.2 in [13]:

$$\lim_{\epsilon\to0+}\int_{G_T}\int_{-M}^{M}\big(m^1_\epsilon(t,\boldsymbol{x},\lambda)\delta_{(\lambda=u(\cdot,\cdot))}*\phi_\epsilon(t,\boldsymbol{x})\\ +m^2_\epsilon(t,\boldsymbol{x},\lambda)\delta_{(\lambda=v(\cdot,\cdot))}*\phi_\epsilon(t,\boldsymbol{x})\big)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda=0, \tag{46}$$

we can prove that

$$2\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}\partial_\lambda(m^1_\epsilon - m^2_\epsilon)(f_\epsilon - g_\epsilon)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda$$
$$= -2\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}(m^1_\epsilon - m^2_\epsilon)\partial_\lambda(f_\epsilon - g_\epsilon)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda = \dots$$

we take into account $\partial_\lambda(f-g) = \delta_{(\lambda=0)} - \delta_{(\lambda=u)} - \delta_{(\lambda=0)} + \delta_{(\lambda=v)} = \delta_{(\lambda=v)} - \delta_{(\lambda=u)}$

$$\dots = 2\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}(m^1_\epsilon - m^2_\epsilon)\big(\delta_{(\lambda=u(\cdot,\cdot))} - \delta_{(\lambda=v(\cdot,\cdot))}\big) * \phi_\epsilon(t,\boldsymbol{x})\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda$$
$$= 2\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}\big(m^1_\epsilon(t,\boldsymbol{x},\lambda)\delta_{(\lambda=u(\cdot,\cdot))} * \phi_\epsilon(t,\boldsymbol{x})$$
$$+ m^2_\epsilon(t,\boldsymbol{x},\lambda)\delta_{(\lambda=v(\cdot,\cdot))} * \phi_\epsilon(t,\boldsymbol{x})\big)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda$$
$$- 2\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}\big(m^1_\epsilon(t,\boldsymbol{x},\lambda)\delta_{(\lambda=v(\cdot,\cdot))} * \phi_\epsilon(t,\boldsymbol{x})$$
$$+ m^2_\epsilon(t,\boldsymbol{x},\lambda)\delta_{(\lambda=u(\cdot,\cdot))} * \phi_\epsilon(t,\boldsymbol{x})\big)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda \le 0.$$

Analogously, we can prove that

$$\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}\partial_\lambda(n^1_\epsilon - n^2_\epsilon)(f_\epsilon - g_\epsilon)\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda \le 0.$$

The existence of traces established in [21] implies the following result

$$(47)\quad -\lim_{\epsilon\to 0+}\int_{G_T}\int_{-M}^{M}a'(\lambda)\partial_t|f_\epsilon(t,\boldsymbol{x},\lambda) - g_\epsilon(t,\boldsymbol{x},\lambda)|^2\xi(\boldsymbol{x})\,dtd\boldsymbol{x}d\lambda$$
$$= \int_\Omega\int_{-M}^{M}a'(\lambda)\Big(|f^\tau_0(\boldsymbol{x},\lambda) - g^\tau_0(\boldsymbol{x},\lambda)|^2 - |f^\tau_T(\boldsymbol{x},\lambda) - g^\tau_T(\boldsymbol{x},\lambda)|^2\Big)\xi(\boldsymbol{x})\ d\boldsymbol{x}d\lambda.$$

If we apply above-mentioned results to inequality (45) as $\varepsilon \to 0+$, we would get inequality (44). □

We need to estimate the second term in the right hand side of (44).

Lemma 4. *For any nonnegative test function $\xi \in C_0(\overline{\Omega})$*

$$(48)\quad \int_\Omega\int_{-M}^{M}a'(\lambda)|f^\tau_0(\boldsymbol{x},\lambda) - g^\tau_0(\boldsymbol{x},\lambda)|^2\xi(\boldsymbol{x})\ d\boldsymbol{x}d\lambda$$
$$\le \int_\Omega\int_{-M}^{M}a'(\lambda)|\chi(\lambda;u_0(\boldsymbol{x})) - \chi(\lambda;v_0(\boldsymbol{x}))|^2\xi(\boldsymbol{x})\ d\boldsymbol{x}d\lambda,$$

$$(49)\quad -\int_\Omega\int_{-M}^{M}a'(\lambda)|f^\tau_T(\boldsymbol{x},\lambda) - g^\tau_T(\boldsymbol{x},\lambda)|^2\xi(\boldsymbol{x})\ d\boldsymbol{x}d\lambda$$
$$\le -\int_\Omega\int_{-M}^{M}a'(\lambda)|\chi(\lambda;u_T(\boldsymbol{x})) - \chi(\lambda;v_T(\boldsymbol{x}))|^2\xi(\boldsymbol{x})\ d\boldsymbol{x}d\lambda.$$

Доказательство. In order to prove Lemma 4 we need to represent the following function $|\chi(\lambda;u_0^\tau)-\chi(\lambda;v_0^\tau)|^2$:

$$
\begin{aligned}
|\chi(\lambda;u_0^\tau)-\chi(\lambda;v_0^\tau)|^2 &= |\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)-\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\\
&+\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)|^2=\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)^2+\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)^2\\
&+\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)^2-2\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\\
&-2\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)\\
&+2\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)=\dots
\end{aligned}
$$

here we use the following formulas

$$
\begin{aligned}
\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)^2&=\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\operatorname{sgn}(\lambda-u_0),\\
\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)^2&=\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\operatorname{sgn}(\lambda-v_0)
\end{aligned}
$$

$$
\begin{aligned}
\dots&=\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\operatorname{sgn}(\lambda-u_0)+\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\operatorname{sgn}(\lambda-v_0)\\
&+\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)^2-2\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\\
&-2\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)+2\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\big(\chi(\lambda;u_0)-\chi(\lambda;v_0)\big)\\
&=:\big(\chi(\lambda;u_0^\tau)-\chi(\lambda;u_0)\big)\alpha(\boldsymbol{x},\lambda)+\big(\chi(\lambda;v_0^\tau)-\chi(\lambda;v_0)\big)\beta(\boldsymbol{x},\lambda)+\big|\chi(\lambda;u_0)-\chi(\lambda;v_0)\big|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(\boldsymbol{x},\lambda)&=\operatorname{sgn}(\lambda-u_0(\boldsymbol{x}))-\chi(\lambda;v_0^\tau(\boldsymbol{x}))+2\chi(\lambda;u_0(\boldsymbol{x}))-\chi(\lambda;v_0(\boldsymbol{x})),\\
\beta(\boldsymbol{x},\lambda)&=\operatorname{sgn}(\lambda-v_0(\boldsymbol{x}))-\chi(\lambda;u_0^\tau(\boldsymbol{x}))-\chi(\lambda;u_0(\boldsymbol{x}))+2\chi(\lambda;v_0(\boldsymbol{x})).
\end{aligned}
$$

We can represent $|\chi(\lambda;u_T^\tau)-\chi(\lambda;v_T^\tau)|^2$ in a similar way.

Lemma 5. *Functions $\alpha(\boldsymbol{x},\lambda)$ and $\beta(\boldsymbol{x},\lambda)$ are equivalent to functions $\widetilde{\alpha}(\boldsymbol{x},\lambda)$ and $\widetilde{\beta}(\boldsymbol{x},\lambda)$ respectively:*

$$
\widetilde{\alpha}(\boldsymbol{x},\lambda)=\begin{cases}1 & \text{if} \quad \lambda>\max\{v_0(\boldsymbol{x}),v_0^\tau(\boldsymbol{x})\},\\ 0 & \text{if} \quad \lambda\in[\min\{v_0(\boldsymbol{x}),v_0^\tau(\boldsymbol{x})\},\max\{v_0(\boldsymbol{x}),v_0^\tau(\boldsymbol{x})\}],\\ -1 & \text{if} \quad \lambda<\min\{v_0(\boldsymbol{x}),v_0^\tau(\boldsymbol{x})\},\end{cases} \tag{50}
$$

$$
\widetilde{\beta}(\boldsymbol{x},\lambda)=\begin{cases}1 & \text{if} \quad \lambda>\max\{u_0(\boldsymbol{x}),u_0^\tau(\boldsymbol{x})\},\\ 0 & \text{if} \quad \lambda\in[\min\{u_0(\boldsymbol{x}),u_0^\tau(\boldsymbol{x})\},\max\{u_0(\boldsymbol{x}),u_0^\tau(\boldsymbol{x})\}],\\ -1 & \text{if} \quad \lambda<\min\{u_0(\boldsymbol{x}),u_0^\tau(\boldsymbol{x})\}.\end{cases} \tag{51}
$$

Доказательство. (1) If we assume that $0<u_0(\boldsymbol{x})<v_0^\tau(\boldsymbol{x})\le v_0(\boldsymbol{x})$, function $\alpha(\boldsymbol{x},\lambda)$ is represented in the following way:

$$
\alpha(\boldsymbol{x},\lambda)=\begin{cases}-1 & \text{if} \quad \lambda<u_0(\boldsymbol{x}),\\ -2 & \text{if} \quad \lambda=u_0(\boldsymbol{x}),\\ -1 & \text{if} \quad u_0(\boldsymbol{x})<\lambda<v_0^\tau(\boldsymbol{x}),\\ 0 & \text{if} \quad v_0^\tau(\boldsymbol{x})\le\lambda\le v_0(\boldsymbol{x}),\\ 1 & \text{if} \quad \lambda>v_0(\boldsymbol{x}).\end{cases}
$$

In the general case $u_0(\boldsymbol{x})\le v_0^\tau(\boldsymbol{x})\le v_0(\boldsymbol{x})$ function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function

$$
\widetilde{\alpha}(\boldsymbol{x},\lambda)=\begin{cases}-1 & \text{if} \quad \lambda<v_0^\tau(\boldsymbol{x}),\\ 0 & \text{if} \quad v_0^\tau(\boldsymbol{x})\le\lambda\le v_0(\boldsymbol{x}),\\ 1 & \text{if} \quad \lambda>v_0(\boldsymbol{x}).\end{cases}
$$

(2) When $v_0(\boldsymbol{x}) \le v_0^\tau(\boldsymbol{x}) \le u_0(\boldsymbol{x})$, function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x},\lambda) = \begin{cases} -1 & \text{if} \quad \lambda < v_0(\boldsymbol{x}), \\ 0 & \text{if} \quad v_0(\boldsymbol{x}) \le \lambda \le v_0^\tau(\boldsymbol{x}), \\ 1 & \text{if} \quad \lambda > v_0^\tau(\boldsymbol{x}). \end{cases}$$

(3) When $v_0^\tau(\boldsymbol{x}) \le u_0(\boldsymbol{x}) \le v_0(\boldsymbol{x})$, function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x},\lambda) = \begin{cases} -1 & \text{if} \quad \lambda < v_0^\tau(\boldsymbol{x}), \\ 0 & \text{if} \quad v_0^\tau(\boldsymbol{x}) \le \lambda \le v_0(\boldsymbol{x}), \\ 1 & \text{if} \quad v_0(\boldsymbol{x}) < \lambda. \end{cases}$$

(4) When $v_0^\tau(\boldsymbol{x}) \le v_0(\boldsymbol{x}) \le u_0(\boldsymbol{x})$, function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x},\lambda) = \begin{cases} -1 & \text{if} \quad \lambda < v_0^\tau(\boldsymbol{x}), \\ 0 & \text{if} \quad v_0^\tau(\boldsymbol{x}) \le \lambda \le v_0(\boldsymbol{x}), \\ 1 & \text{if} \quad v_0(\boldsymbol{x}) < \lambda. \end{cases}$$

(5) When $u_0(\boldsymbol{x}) \le v_0(\boldsymbol{x}) \le v_0^\tau(\boldsymbol{x})$, function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x},\lambda) = \begin{cases} -1 & \text{if} \quad \lambda < v_0(\boldsymbol{x}), \\ 0 & \text{if} \quad v_0(\boldsymbol{x}) \le \lambda \le v_0^\tau(\boldsymbol{x}), \\ 1 & \text{if} \quad v_0^\tau(\boldsymbol{x}) < \lambda. \end{cases}$$

(6) When $v_0(\boldsymbol{x}) \le u_0(\boldsymbol{x}) \le v_0^\tau(\boldsymbol{x})$, function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x},\lambda) = \begin{cases} -1 & \text{if} \quad \lambda < v_0(\boldsymbol{x}), \\ 0 & \text{if} \quad v_0(\boldsymbol{x}) \le \lambda \le v_0^\tau(\boldsymbol{x}), \\ 1 & \text{if} \quad v_0^\tau(\boldsymbol{x}) < \lambda. \end{cases}$$

From these representations it follows that function $\alpha(\boldsymbol{x},\lambda)$ is equivalent to function $\tilde{\alpha}(\boldsymbol{x},\lambda)$. The same analysis can be provided for $\beta(\boldsymbol{x},\lambda)$. □

With the help of Lemma 5 we get

$$\begin{aligned} (52) \quad & \int_\Omega \int_{-M}^{M} a'(\lambda) \big| f_0^\tau(\boldsymbol{x},\lambda) - g_0^\tau(\boldsymbol{x},\lambda)\big|^2 \xi(\boldsymbol{x})\, d\boldsymbol{x} d\lambda \\ &= \lim_{\epsilon\to 0+} \int_\Omega \Big(\int_{-M}^{u_0-\epsilon} + \int_{u_0+\epsilon}^{M} \Big) a'(\lambda) \big(f_0^\tau(\boldsymbol{x},\lambda) - \chi(\lambda; u_0(\boldsymbol{x}))\big) \alpha(\boldsymbol{x},\lambda)\xi(\boldsymbol{x})\, d\boldsymbol{x} d\lambda \\ &+ \lim_{\epsilon\to 0+} \int_\Omega \Big(\int_{-M}^{v_0-\epsilon} + \int_{v_0+\epsilon}^{M} \Big) a'(\lambda) \big(g_0^\tau(\boldsymbol{x},\lambda) - \chi(\lambda; v_0(\boldsymbol{x}))\big) \beta(\boldsymbol{x},\lambda)\xi(\boldsymbol{x})\, d\boldsymbol{x} d\lambda \\ &+ \int_\Omega \int_{-M}^{M} a'(\lambda) \big(\chi(\lambda; u_0(\boldsymbol{x})) - \chi(\lambda; v_0(\boldsymbol{x}))\big)^2 \xi(\boldsymbol{x})\, d\boldsymbol{x} d\lambda. \end{aligned}$$

Here the first and the second terms in the right-hand side can be estimated in such a way:

$$\begin{aligned} (53) \quad & \lim_{\epsilon\to 0+} \int_\Omega \Big(\int_{-M}^{u_0-\epsilon} + \int_{u_0+\epsilon}^{M} \Big) a'(\lambda) \big(f_0^\tau(\boldsymbol{x},\lambda) \\ & \qquad - \chi(\lambda; u_0(\boldsymbol{x}))\big) \alpha(\boldsymbol{x},\lambda)\xi(\boldsymbol{x})\ d\boldsymbol{x} d\lambda \\ &= \lim_{\epsilon\to 0+} \int_\Omega \Big(\int_{-M}^{u_0-\epsilon} + \int_{u_0+\epsilon}^{M} \Big) \partial_\lambda \mu_{0,f}(\boldsymbol{x},\lambda)\alpha(\boldsymbol{x},\lambda)\xi(\boldsymbol{x})\, d\boldsymbol{x} d\lambda \\ &= - \int_\Omega (\mu_{0,f}(\boldsymbol{x}, v_0(\boldsymbol{x})) + \mu_{0,f}(\boldsymbol{x}, v_0^\tau(\boldsymbol{x})))\xi(\boldsymbol{x})\ d\boldsymbol{x} \le 0, \end{aligned}$$

$$
\begin{aligned}
(54)\quad &\lim_{\epsilon\to 0+}\int_\Omega\Big(\int_{-M}^{v_0-\epsilon}+\int_{v_0+\epsilon}^{M}\Big)a'(\lambda)\big(g_0^\tau(\boldsymbol{x},\lambda)\\
&\qquad\qquad -\chi(\lambda;v_0(\boldsymbol{x}))\big)\beta(\boldsymbol{x},\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}d\lambda\\
&=\lim_{\epsilon\to 0+}\int_\Omega\Big(\int_{-M}^{v_0-\epsilon}+\int_{v_0+\epsilon}^{M}\Big)\partial_\lambda\mu_{0,g}(\boldsymbol{x},\lambda)\beta(\boldsymbol{x},\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}d\lambda\\
&=-\int_\Omega(\mu_{0,f}(\boldsymbol{x},u_0(\boldsymbol{x}))+\mu_{0,f}(\boldsymbol{x},u_0^\tau(\boldsymbol{x})))\xi(\boldsymbol{x})\,d\boldsymbol{x}\le 0.
\end{aligned}
$$

Therefore, the following representation finishes the proof of Lemma 4:

$$
\begin{aligned}
&\int_\Omega\int_{-M}^{M}a'(\lambda)(\chi(\lambda;u_0(\boldsymbol{x}))-\chi(\lambda;v_0(\boldsymbol{x})))^2\xi(\boldsymbol{x})\,d\boldsymbol{x}d\lambda\\
&\qquad=\int_\Omega\big(a(v_0(\boldsymbol{x}))-a(u_0(\boldsymbol{x}))\big)\mathrm{sgn}(v_0(\boldsymbol{x})-u_0(\boldsymbol{x}))\xi(\boldsymbol{x})\,d\boldsymbol{x}.
\end{aligned}
$$

□

Remark 9. *It is important to note that*

$$
\begin{aligned}
\int_{-M}^{M}\big|f(t,\boldsymbol{x},\lambda)-g(t,\boldsymbol{x},\lambda)\big|^2\,d\lambda&=\int_{-M}^{M}\big|f(t,\boldsymbol{x},\lambda)-g(t,\boldsymbol{x},\lambda)\big|\,d\lambda\\
&=|u(t,\boldsymbol{x})-v(t,\boldsymbol{x})|,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-M}^{M}a'(\lambda)\big|f(t,\boldsymbol{x},\lambda)-g(t,\boldsymbol{x},\lambda)\big|^2\,d\lambda&=\int_{-M}^{M}a'(\lambda)\big|f(t,\boldsymbol{x},\lambda)-g(t,\boldsymbol{x},\lambda)\big|\,d\lambda\\
&=\mathrm{sgn}(u(t,\boldsymbol{x})-v(t,\boldsymbol{x}))(a(u(t,\boldsymbol{x}))-a(v(t,\boldsymbol{x}))),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-M}^{M}\varphi_i'(\lambda)|f(t,\boldsymbol{x},\lambda)-g(t,\boldsymbol{x},\lambda)|^2\,d\lambda=\int_{-M}^{M}\varphi_i'(\lambda)|f(t,\boldsymbol{x},\lambda)-g(t,\boldsymbol{x},\lambda)|\,d\lambda\\
&\quad=\mathrm{sgn}(\varphi_i(u(t,\boldsymbol{x}))-\varphi_i(v(t,\boldsymbol{x})))(\varphi_i(u(t,\boldsymbol{x}))-\varphi_i(v(t,\boldsymbol{x})))\quad i=1,\dots,d,
\end{aligned}
$$

for a.e. $(t,\boldsymbol{x})\in G_T$.

Corollary 1. *The following inequality holds*

(55)

$$
\begin{aligned}
&-\int_{G_T}\sum_{i=1}^{d}\partial_{x_i}\xi(\boldsymbol{x})\,\mathrm{sgn}(\varphi_i(u(t,\boldsymbol{x}))-\varphi_i(v(t,\boldsymbol{x})))\big(\varphi_i(u(t,\boldsymbol{x}))-\varphi_i(v(t,\boldsymbol{x}))\big)\,dtd\boldsymbol{x}\\
&-\int_{G_T}\Delta_x\xi(\boldsymbol{x})|u(t,\boldsymbol{x})-v(t,\boldsymbol{x})|\,dtd\boldsymbol{x}\le\mathcal{A}\int_\Omega(|u_0(\boldsymbol{x})-v_0(\boldsymbol{x})|+|u_T(\boldsymbol{x})-v_T(\boldsymbol{x})|)\xi(\boldsymbol{x})\,d\boldsymbol{x}
\end{aligned}
$$

for any nonnegative function $\xi\in C^2(\Omega)\cap C_0(\overline{\Omega})$, $\mathcal{A}=\sup\limits_{|z|\le M}|a'(z)|$.

Remark 10. *In Lemma 3 we assume that* $\xi\in C^2(\Omega)\cap C_0^1(\overline{\Omega})$ $(\nabla\xi|_{\partial\Omega}=0)$. *If we take* $\xi\in C^2(\Omega)\cap C_0^1(\overline{\Omega})$ *in Corollary 1, we can rewrite the second term in the left hand side in the following way*

$$
\int_{G_T}\nabla_x\xi(\boldsymbol{x})\cdot\nabla|u(t,\boldsymbol{x})-v(t,\boldsymbol{x})|\,dtd\boldsymbol{x},
$$

because by definition $u, v \in L^\infty(G_T) \cap L^2(0,T; W_0^{1,2}(\Omega))$. *Then, we can choose the sequence* $\{\xi_l\}_{l\in\mathbb{N}} \subset C^2(\Omega) \cap C_0^1(\overline{\Omega})$ *with the limit* $\xi \in C^2(\Omega) \cap C_0(\overline{\Omega})$. *Therefore, we can assume that inequality* (55) *is valid for any* $\xi \in C^2(\Omega) \cap C_0(\overline{\Omega})$.

Let $\xi(\boldsymbol{x}) = \xi_P(\boldsymbol{x})$ in (55), where ξ_P is the solution of Poisson's equation in the domain Ω:

$$\Delta_x \xi_P(\boldsymbol{x}) = -1, \quad \xi_P(\boldsymbol{x})|_{\partial\Omega} = 0.$$

Finally, only for the small constant

$$C_{\varphi,\Omega} = \|\boldsymbol{\varphi}'\|_{C((-M,M))} \|\xi_P\|_{C^1(\Omega)} < 1 \tag{56}$$

it follows from (55) that

$$(1 - C_{\varphi,\Omega}) \int_{G_T} |u(t,\boldsymbol{x}) - v(t,\boldsymbol{x})| \, dt d\boldsymbol{x} \le \mathcal{A} \|\xi_P\|_{C(\Omega)} \int_\Omega (|u_0(\boldsymbol{x}) - v_0(\boldsymbol{x})| + |u_T(\boldsymbol{x}) - v_T(\boldsymbol{x})|) \, d\boldsymbol{x}. \tag{57}$$

The estimate (57) finishes the proof of Theorem 4.

Conclusion

In the present paper we have deduced kinetic formulation of boundary value problem (1a)–(1b). Moreover, with its help we have proved the uniqueness of kinetic solutions. Here we have applied methods developed for hyperbolic and degenerate parabolic equations.

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