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ON SUBGROUPS OF FINITE GROUPS WITH A COVER AND AVOIDANCE PROPERTY

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Abstract. Subgroups of finite groups which cover the Frattini chief factors and avoid the supplemented ones are studied in the paper. Some relations with the profrattini and prefrattini subgroups are also exhibited.

Keywords: finite group, prefrattini subgroups, profrattini subgroups, cover and avoidance properties.

1. Introduction

All groups considered here are finite.

This note is an outgrowth of the investigation into subgroups which satisfy the cover and avoidance property. Let A/B a factor of a group G; i. e. A and B are subgroups of G and B is a normal subgroup of A. Let X be any subgroup of G. Then $B(A \cap X)$ is a subgroup of G between B and A. We say that X *covers* A/B if $A = B(A \cap X)$ and X *avoids* A/B if $B = B(A \cap X)$. We say that X has the *cover and avoidance* property in G, or X is a CAP-subgroup of G for short, if X either covers or avoids every chief factor of G.

The study of families of subgroups with the cover and avoidance property has been a consistent theme in the theory of groups, first in the context of soluble

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groups, and more recently as a way of characterising certain classes of groups (see, for example, [1] and [2]). One of the earliest examples of work in this area is the paper of Gaschütz [3] where an interesting class of conjugate subgroups called *prefrattini subgroups* are introduced in the soluble universe. Gaschütz showed that the prefrattini subgroups cover the Frattini chief factors and avoid the complemented ones, and asked whether this cover and avoidance property characterises these subgroups in every soluble group. Gillam shows in [4] that the answer is negative in general.

Prefrattini subgroups can be defined in the general finite universe ([5], [6, Chapter 4, Section 3]). Unfortunately, they are not neither conjugate not CAP-subgroups in general. In fact, conjugacy of prefrattini subgroups characterises solubility ([6, Theorem 4.3.15]).

The main goal of the present paper is to describe the subgroups of a group that cover the Frattini chief factors and avoid the supplemented ones. The profrattini normal subgroups introduced in [7] (see [6, Definitions 4.3.19 and 4.3.23]) will play an important role in our approach. These subgroups arise naturally with the study of prefrattini subgroups and totally non-saturated formations in the general finite case.

Definition 1. *Let G be a group. A normal subgroup $1 \neq N$ of G is said to be* profrattini *if every chief factor of G of the form N/K is a Frattini chief factor of G, i.e., $N/K \leq \Phi(G/K)$.*

According to [6, Theorem 4.3.21], the product of two profrattini subgroups of G is again profrattini. Therefore, each group G has a largest profrattini subgroup denoted by $Pro(G)$ and called the *profrattini subgroup* of G.

Definition 2. *Let G be a group.*

*1) An element $g \in G$ is said to be Q-*Frattini *if the normal closure $\langle g\rangle^G$ is a profrattini subgroup of G.*

*2) A subgroup A of G is called Φ-*isolator *if A covers the Frattini chief factors of G and avoids the supplemented ones.*

By the above remark, a normal subgroup N of a group G is profrattini provided that every element of N is Q-Frattini. The converse does not hold in general.

Example. Let p be an odd prime and let P be an extraspecial group of order p^3 and exponent p. Applying [8, Theorem B.10.3], P has an irreducible and faithful module V over $GF(q)$, the finite field of q-elements, for a prime $q \neq p$. Let $G = V \rtimes P$ be the corresponding semidirect product. Then $N = V\Phi(P)$ is a profrattini normal subgroup of G but if $1 \neq v \in V$, $V = \langle v\rangle^G$ is not profrattini.

It is also clear that every prefrattini subgroup of a soluble group G is a Φ-isolator of G.

Our first result characterises the profrattini normal subgroups of a group.

Theorem 1. *Let G be a group and let $\mathcal{X}$ be a complete set of representatives of the G-isomorphism classes of chief factors of G. Let $\mathfrak{F} = form(\mathcal{X})$ be the formation generated by $\mathcal{X}$. Then a normal subgroup N of G is profrattini if and only if $N/N^{\mathfrak{F}}$ is contained in $\Phi(G/N^{\mathfrak{F}})$.*

The description of the Φ-*isolator* subgroups is contained in the following:

Theorem 2. *Let G be a group. Then all the Φ-isolators of G have the same order. Futhermore, if A is a Φ-isolator of G, we have:*

1) each element of A is Q-Frattini;
2) if A is contained in B and every element of B is Q-Frattini, then $A = B$;
3) $Core_G(A) = \Phi(G)$;
4) if W is a prefrattini subgroup of G, then A is contained in $\langle W\rangle^G$.

The example due to Gillam mentioned above shows that not every Φ-isolator is a prefrattini subgroup. However, by [4, Theorem 3.2], we have:

Corollary 1. *Let A be a Φ-isolator of a soluble group G. Then A is a prefrattini subgroup of G if and only if A permutes with a Hall system of G.*

2. Proofs

We recall first the following lemma.

Lemma 1. *Let G be a group and let $\mathfrak{X}$ be a complete set of representatives of the G-isomorphism classes of chief factors of G. Let $\mathfrak{F} = form(\mathfrak{X})$ be the formation generated by $\mathfrak{X}$. Then every element of $\mathfrak{F}$ is a direct product of groups in $\mathfrak{X}$.*

Proof. Let $\mathfrak{H}$ be the class composed of all groups that are direct products of groups in $\mathfrak{X}$ together the trivial group. Clearly $\mathfrak{H}$ is contained in $\mathfrak{F}$. Assume that $\mathfrak{F}$ is not contained in $\mathfrak{H}$ and let $X \in \mathfrak{F} \setminus \mathfrak{H}$ be a group of minimal order. Suppose that N and L are two different minimal normal subgroups of X. Then $N \cap L = 1$, and X/N and X/L belong to $\mathfrak{H}$ by minimality of X. Hence $X/N = Y/N \times Z/N$, where Y/N is a direct product of cyclic groups of prime orders and Z/N is a direct product of non-abelian simple groups. In particular, there exists a normal subgroup T of X such that $X/N = LN/N \times T/N$. Hence $X = LT$ and $L \cap T = 1$. Hence $T \in \mathfrak{H}$. Since L is isomorphic to $LN/N \in \mathfrak{H}$, it follows that $X \in \mathfrak{H}$. This contradiction shows that $N = Soc(X)$ is a minimal normal subgroup of X. Since X is a direct product of groups in $\mathfrak{X}$, we have that $X \in \mathfrak{X} \subseteq \mathfrak{H}$, contrary to supposition. Therefore $\mathfrak{F} = \mathfrak{H}$. The lemma is proved.

Proof of Theorem 1. Assume that N is a profrattini normal subgroup of a group G. We may assume that $N \neq 1$. By Lemma 1, $N/N^{\mathfrak{F}}$ is a direct product of minimal normal subgroups of $N/N^{\mathfrak{F}}$. Suppose one of them, $A/N^{\mathfrak{F}}$ say, is not Frattini. Since every chief factor of N of the form N/K is Frattini, it follows that $A/N^{\mathfrak{F}}$ is abelian and central in $N/N^{\mathfrak{F}}$. Hence there exists a maximal subgroup M of G containing $N^{\mathfrak{F}}$ such that $G = AM$ and $A \cap M = N^{\mathfrak{F}}$. Let $R = C_G(A/N^{\mathfrak{F}})$ and $S = C_M(A/N^{\mathfrak{F}})$. It is clear that $S = Core_G(M)$. Therefore G/S is a primitive group such that $R/S = Soc(G/S)$ is a chief factor of G which is G-isomorphic to $A/N^{\mathfrak{F}}$. Since N is a subgroup of R which is not contained in S, it follows that $R = NS$. Thus $N/N \cap S$ is a chief factor of G which is G-isomorphic to R/S. Since $N \cap M$ is a normal subgroup of G containing $N \cap S$, we have that $N \cap M = N \cap S$. This means that $N/N \cap S$ is a complemented chief factor of G, which contradicts the profrattini character of N. Therefore $N/N^{\mathfrak{F}}$ is a direct product of minimal normal subgroups of $N/N^{\mathfrak{F}}$ contained in $\Phi(G/N^{\mathfrak{F}})$. This means that $N/N^{\mathfrak{F}}$ is a subgroup of $\Phi(G/N^{\mathfrak{F}})$.

Assume now that $N/N^{\mathfrak{F}}$ is a subgroup of $\Phi(G/N^{\mathfrak{F}})$. Let N/B a chief factor of G. Then $N/B \in \mathfrak{F}$ and so $N^{\mathfrak{F}}$ is contained in B. This implies that N/B is a Frattini chief factor of G. Consequently, N is a profrattini normal subgroup of G. The theorem is proved.

The following easy lemma is useful in induction arguments.

Lemma 2. *If A is a profrattini normal subgroup of a group G and N is normal in G, then AN/N is a profrattini subgroup of G/N.*

Proof. Let B/N be a normal subgroup of G/N such that $(AN/N)/(B/N)$ is a chief factor of G/N. Then $AN = AB$ and $AB/B = AN/B$ is a chief factor of G which is G-isomorphic to $A/A \cap B$. Suppose that AB/B is not Frattini and let M be a maximal subgroup of G such that $B \leq M$ and $G = AM$. Then M is a supplement of $A/A \cap B$ and so $A/A \cap B$ is not a Frattini chief factor of G. This contradicts the fact that A is profrattini in G. The lemma is proved.

Corollary 2. *If N is a normal subgroup of a group G and H is a subgroup of G such that every element of H is Q-Frattini in G, then every element of HN/N is Q-Frattini in G/N.*

The proof of the next lemma is straigthforward.

Lemma 3. *If A is a Φ-isolator of a group G and N is a normal subgroup of G, then AN/N is a Φ-isolator of G/N.*

Lemma 4. *Let H be a subgroup of a group G such that every element of H is Q-Frattini in G. Then H avoids every non-Frattini chief factor of G.*

Proof. Assume that every element of H is Q-Frattini, and let U/V be a non-Frattini chief factor of G. By Corollary 2, every element of HV/V is Q-Frattini. Using induction on the order of G, we may assume that $V = 1$. Suppose that $H \cap U \neq 1$ and let $1 \neq h \in H \cap U$. Since h is Q-Frattini, it follows that $\langle h \rangle^G = U$ is a Frattini chief factor of G, against supposition. Thus H avoids U. The lemma is proved.

Proof of Theorem 2. Let

$$1 = U_0 < U_1 < \cdots < U_n = G \quad (*)$$

be a chief series of G, and let $\{U_j/U_{j-1} : j \in J\}$ be the set of all Frattini chief factors of $(*)$. If H is a Φ-isolator of G, we can apply [8, Lemma A.1.7] to conclude that $|H| = \prod_{j \in J} |U_j/U_{j-1}|$. Therefore all the Φ-isolators of G have the same order.

Let A be an Φ-isolator of G and let $a \in A$. Set $L = \langle a \rangle^G$. If L/M is a chief factor of G, then $L \cap A = M \cap A$ or $L = M(L \cap A)$. Since $a \notin M$, we have that $L = M(L \cap A)$, that is, A covers L/M. This implies that L/M is a Frattini chief factor of G. Hence L is profrattini subgroup of G and a is Q-Frattini. This proves Statement (1).

Assume that B is a subgroup of G containing A such that every element of B is Q-Frattini. By Lemma 4, B avoids all non-Frattini chief factors of G. Since A covers all the Frattini ones, the same is true for B. Therefore $A = B$ and Statement (2) holds.

Write $C = Core_G(A)$ and let

$$1 = C_0 < C_1 < \cdots < C_t = C \quad (**)$$

be part of a chief series of G passing through C. Then every chief factor C_i/C_{i-1} of $(**)$ is covered by A. Hence C_i/C_{i-1} is a Frattini chief factor of G for all $i \in \{1, \ldots, t\}$. In particular, C is contained in $\Phi(G)$.

Now let

$$1 = F_0 < F_1 < \cdots < F_r = \Phi(G)$$

be part of a chief series of G passing through $\Phi(G)$. Then all the chief factors of this series is Frattini. Thus F_i is contained in AF_{i-1} for all $i \in \{1, \dots, r\}$. Consequently, $\Phi(G)$ is contained in A and so $C = \Phi(G)$. This proves Statement (3).

Let W be a prefrattini subgroup of G, and let $a \in A$. Then $\langle a \rangle^G$ is a profrattini normal subgroup of G by Statement (1). Applying [6, Corollary 4.3.31], A is contained in $\langle W \rangle^G$. The theorem is proved.

3. Final Remark

In [6, Chapter 4, Section 3], a common extension of all prefrattini subgroups appeared in the literature is analysed by using different types of solid sets of maximal subgroups. By arguments essentially identical to those given above, one can establish the corresponding extensions of our main theorems by means of such solid sets.

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