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AUTOMORPHISMS OF DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{117, 80, 18, 1; 1, 18, 80, 117\}$

M. M. KHAMGOKOVA, A. A. MAKHNEV, D. V. PADUCHIKH

Abstract. Distance-regular graph Γ with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$ is an $AT4$-graph. Antipodal quotient $\bar{\Gamma}$ has parameters $(378, 117, 36, 36)$. Both graphs have strongly regular neighbourhoods with parameters $(117, 36, 15, 9)$. In the work automorphisms of the said graphs are found. In particular, there exist graphs of rank 3 with parameters $(117, 36, 15, 9)$ and $(378, 117, 36, 36)$, and graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$ is not arc-transitive.

Keywords: strongly regular graph, eigenvalue, automorphism of graph.

1. Introduction

We consider undirected graphs with no loops and multiple edges. For a vertex a of graph Γ we'll denote $\Gamma_i(a)$ i-neighbourhood of vertex a, that is, subgraph induced by Γ on the set of all vertices at distance i from a. Let's put $[a] = \Gamma_1(a)$, $a^\perp = \{a\} \cup [a]$.

Number of vertices in the neighbourhood of a vertex is called its *degree*. Graph Γ is called *regular of degree* k, if degree of any vertex of Γ is equal to k. We'll call graph Γ *edge-regular* with parameters (v, k, λ), if it contains v vertices, is regular of degree k, and each its edge lies in exactly λ triangles. Graph Γ is called *amply regular* with parameters (v, k, λ, μ), if it is edge-regular with corresponding parameters, and $[a] \cap [b]$ contains exactly μ vertices for any two vertices a, b at distance 2 from each other in Γ. Amply regular graph of diameter 2 is called *strongly regular*.

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Connected graph Γ of diameter d is called *antipodal*, if binary relation on the vertex set "to be at distance 0 or d" is an equivalence relation. Equivalence classes of this relation are called *antipodal classes. Antipodal quotient* $\bar{\Gamma}$ is defined on the set of antipodal classes of Γ as its vertices, where classes $\bar{u}, \bar{w}$ are adjacent iff $\bar{u}$ contains vertex adjacent with vertex from $\bar{w}$. If each antipodal class contains exactly r vertices, then r is called *antipodality index*, and Γ is called *antipodal r-cover* of graph $\bar{\Gamma}$.

Let Γ be a connected graph of diameter d. If vertices u, w are at distance i from each other in Γ, then by $b_i(u, w)$ (by $c_i(u, w)$) we'll denote number of vertices in the intersection $\Gamma_{i+1}(u)$ (correspondingly $\Gamma_{i-1}(u)$) with $[w]$. Graph Γ is called *distance-regular* with *intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, if for any $i = 0, \ldots, d$ values of $b_i(u, w)$ and $c_i(u, w)$ do not depend on choice of vertices u, w at distance i from each other in Γ. Clearly, distance-regular graph Γ is amply regular with parameters $k = b_0$, $\lambda = k - b_1 - 1$, $\mu = c_2$. Let's put $a_i = k - b_i - c_i$.

Further, by $p^l_{ij}(x, y)$ we'll denote number of vertices in subgraph $\Gamma_i(x) \cap \Gamma_j(y)$ for vertices x, y at distance l from each other in Γ. In distance-regular graph the numbers $p^l_{ij}(x, y)$ do not depend on choice of vertices x, y; they are denoted by p^l_{ij} and are called *intersection numbers* of Γ.

Distance-regular graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$ is a $AT4(9, 3, 2)$-graph (see [2]). An existence of this graph is not known (however $AT4(9, 3, 3)$-graph exists). Its antipodal quotient $\bar{\Gamma}$ has parameters $(378, 117, 36, 36)$ and non-principal eigenvalues $9, -9$; neighbourhoods of vertices in $\bar{\Gamma}$ are strongly regular with parameters $(117, 36, 15, 9)$.

In the work [3] there have been found intersection arrays of distance-regular graphs in which neighbourhoods of vertices are pseudogeometric graphs for $pG_{s-3}(s, t)$. In particular, locally pseudo $pG_2(5, 2)$-graph is a strongly regular graph with parameters $(117, 36, 15, 9)$.

Proposition 1. *Let Γ be a distance-regular graph, in which neighbourhoods of vertices are pseudogeometric graphs for $pG_{s-3}(s, t)$. Then one of the claims holds:*

(1) $d(\Gamma) \geq 4$ *and Γ is a Johnson graph $J(10, 5)$ or unique graph with intersection array* $\{45, 32, 12, 1; 1, 6, 32, 45\}$*;*

(2) $s > 4$ *and Γ is a strongly regular graph with parameters* $(117, 36, 15, 9)$, $(232, 81, 30, 27)$, $(287, 126, 45, 63)$, *or $s = 6$ and Γ is a Taylor graph, or $s = 5$, $t = 1$ and Γ is a halved 7-cube;*

(3) $s = 4$ *and Γ is either strongly regular graph with parameters* $(190, 45, 12, 10)$, $(126, 45, 12, 18)$, $(246, 85, 20, 34)$, $(726, 125, 28, 20)$, $(870, 165, 36, 30)$, $(486, 165, 36, 66)$, *or graph with intersection array* $\{25, 16, 1; 1, 8, 25\}$, $\{125, 96, 1; 1, 48, 125\}$.

In this work there have been found possible automorphisms of strongly regular graphs with parameters $(117, 36, 15, 9)$, $(378, 117, 36, 36)$ and those of distance-regular graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$.

For a set X of automorphisms of graph Γ we'll denote by $\mathrm{Fix}(X)$ subgraph induced by Γ on the set of all vertices of Γ which are fixed by each automorphism from X. For an automorphism g of connected graph Γ we'll denote by $\alpha_j(g)$ the number of vertices u of Γ such that $d(u, u^g) = j$.

Theorem 1. *Let Γ be a strongly regular graph with parameters* $(117, 36, 15, 9)$, *$G = \mathrm{Aut}(\Gamma)$, g be an element of G of prime order p, and $\Omega = \mathrm{Fix}(g)$. Then $\pi(G) \subseteq \{2, 3, 5, 13\}$, and one of the following claims holds:*

(1) Ω *is empty graph,* $p = 13$, $\alpha_1(g) = 39$ *and* $\alpha_2(g) = 78$, *or* $p = 3$, $\alpha_1(g) = 36l - 9$ *and* $\alpha_2(g) = 126 - 36l$*;*

(2) Ω *is a n-clique, and either* $p = 5$, $n = 2, 7, 12$, $\alpha_1(g) = 60l + 75 - 15n$ *and* $\alpha_2(g) = 42 + 14n - 60l$*; or* $p = 2$, $n = 5, 7, 9, 11, 13$, $\alpha_1(g) = 24l + 39 - 3n$ *and* $\alpha_2(g) = 78 + 2n - 24l$*;*

(3) Ω *contains geodetic* 2*-path, and either*

(*i*) $p = 3$, $|\Omega| \leq 33$ *or* $|\Omega| = 45$, *or*

(*ii*) $p = 2$ *and* $|\Omega| \leq 63$.

Corollary 1. *Vertex-symmetric strongly regular graph with parameters* $(117, 36, 15, 9)$ *is a graph of rank* 3 *with group of automorphisms* $L_4(3).Z_2$ *and stabilizer of vertex* $Z_2 \times (O(5, 3).Z_2)$.

Theorem 2. *Let* Γ *be a strongly regular graph with parameters* $(378, 117, 36, 36)$, *in which neighbourhoods of vertices are strongly regular with parameters* $(117, 36, 15, 9)$, $G = \mathrm{Aut}(\Gamma)$, *g be an element from G of prime order p, and* $\Omega = \mathrm{Fix}(g)$. *Then* $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$, *and one of the following claims holds:*

(1) Ω *is empty graph, and either* $p = 7$, $\alpha_1(g) = 126l$ *and* $\alpha_2(g) = 126(3 - l)$*; or* $p = 3$, $\alpha_1(g) = 54l$ *and* $\alpha_2(g) = 378 - 54l$*; or* $p = 2$, $\alpha_1(g) = 36l + 18$ *and* $\alpha_2(g) = 360 - 36l$*;*

(2) Ω *is a n-clique, and either* $p = 13$, $n = 1$, $\alpha_1(g) = 117$ *and* $\alpha_2(g) = 260$ *or* $\alpha_1(g) = 351$ *and* $\alpha_2(g) = 26$*; or* $p = 5$, $n = 3, 8, 13$, $\alpha_1(g) = 45l$*; or* $p = 2$, $n = 6, 8, \ldots, 14$ *and* $\alpha_1(g) = 18l$*;*

(3) Ω *is a m-coclique,* $p = 3$, $m = 3, 6, \ldots, 27$ *and* $\alpha_1(g) = 9l$*;*

(4) Ω *contains geodetic* 2*-path,* $|\Omega| \leq 126$, *and either*

(*i*) $p = 3$, $|\Omega(a)| \leq 30$ *or* $|\Omega(a)| = 45$, $|\Omega| = 3t$ *and* $\alpha_1(g) = 54m - 27t$, *or*

(*ii*) $p = 2$, *degree of vertex in* Ω *is not greater than* 63, $|\Omega| = 2t$ *and* $\alpha_1(g) = 36m + 18 - 18t$.

Corollary 2. *Let strongly regular graph* Γ *with parameters* $(378, 117, 36, 36)$ *be vertex-symmetric,* $G = \mathrm{Aut}(\Gamma)$ *contain an element of order* 13, *T be a socle of group G. Then* $T \cong \Omega_7(3)$, T_a *is an extension* $L_4(3)$ *by the group of order* 2, *and* Γ *is a graph of rank* 3.

Theorem 3. *Let* Γ *be a distance-regular graph with intersection array* $\{117, 80, 18, 1; 1, 18, 80, 117\}$, $G = \mathrm{Aut}(\Gamma)$, *g be an element of G of prime order p, and* $\Omega = \mathrm{Fix}(g)$. *Then* $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$ *and one of the following claims holds:*

(1) *g induces trivial automorphism of antipodal quotient* $\bar{\Gamma}$, $p = 2$ *and* $\alpha_4(g) = 756$*;*

(2) Ω *is empty graph,* $\alpha_4(g) = 0$, *and either* $p = 7$, $\alpha_1(g) = 294n + 336 - 126m$, $\alpha_2(g) = 252m$, $\alpha_3(g) = -294n + 420 - 126m$*; or* $p = 3$, $\alpha_1(g) = 126n + 378 - 54m$, $\alpha_2(g) = 108m$, $\alpha_3(g) = -126n + 378 - 54m$*; or* $p = 2$, $\alpha_1(g) = 84n + 336 - 36m$, $\alpha_2(g) = 72m$, $\alpha_3(g) = 420 - 84n - 36m$*;*

(3) Ω *is antipodal class,* $p = 13$, $\alpha_1(g) = 104$, $\alpha_2(g) = 520$, $\alpha_3(g) = 130$*;*

(4) $\bar{\Omega}$ *is a n-clique, and either* $p = 5$, $n = 3$, $\alpha_1(g) = 210$, $\alpha_2(g) = 480$, $\alpha_3(g) = 60$*; or* $p = 2$, $n = 6$, $\alpha_1(g) = 84m + 150$, $\alpha_2(g) = 456$, $\alpha_3(g) = 138 - 84m$, *or* $n = 12$, $\alpha_1(g) = 84m + 132$, $\alpha_2(g) = 408$, $\alpha_3(g) = 192 - 84m$*;*

(5) Ω *is a* 42*-coclique,* $p = 3$, $\alpha_1(g) = 126n + 18$, $\alpha_2(g) = 108l - 420 = 552$, $\alpha_3(g) = 144 - 126n$*;*

(6) $\bar{\Omega}$ *contains geodetic* 2*-path, and either*

(i) $p = 3$, $|\Omega| = 54$, $\alpha_1(g) = 126n + 540 - 54l$, $\alpha_2(g) = 108l - 540$ *and* $\alpha_3(g) = 702 - 126n - 54l$, *or*

(ii) $p = 2$, $|\Omega| = 36s$, $s \leq 7$, $\alpha_1(g) = 84n + 420 - 36l + 144s$, $\alpha_2(g) = 72l - 360s$ $\alpha_3(g) = 336 - 84n - 36l + 180s$.

Corollary 3. *Let* Γ *be a distance-regular graph with intersection array* $\{117, 80, 18, 1; 1, 18, 80, 117\}$. *If* $G = \mathrm{Aut}(\Gamma)$ *acts transitively on set of vertices, then* $|G|$ *is not divisible by* 13. *In particular, graph* Γ *is not arc-transitive.*

Proofs of the theorems employ the method by G. Higman [4, § 3.7] of computing automorphisms of a distance-regular graph which we'll describe in brief now.

Let Γ be a distance-regular graph of diameter d on v vertices. Permutation representation of group $G = \mathrm{Aut}(\Gamma)$ on vertices of graph Γ gives in a usual way a matrix representation ψ of G in $GL(v, \mathbb{C})$. Vector space $\mathbb{C}^v$ is orthogonal sum of G-invariant eigenspaces $W_0, \dots, W_d$ of adjacency matrix $A = A_1$ of graph Γ. For any $g \in G$ the matrix $\psi(g)$ commutes with A, so subspace W_i is $\psi(G)$-invariant. Let χ_i be a character of representation ψ_{W_i}. Then for $g \in G$ we'll get

$$\chi_i(g) = v^{-1} \sum_{j=0}^{d} Q_{ij} \alpha_j(g),$$

where coefficients Q_{ij} depend only on intersection numbers of Γ. If all Q_{ij} are rational numbers, then $\chi_i(g)$ being algebraic integer is a whole number. This gives limitation on possible values of $\alpha_j(g)$ which helps to find automorphisms of Γ.

2. Automorphisms of graph with parameters $(117, 36, 15, 9)$

First we'll bring some auxiliary results.

Lemma 1. *Let* Γ *be a strongly regular graph with parameters* (v, k, λ, μ) *and non-principal eigenvalues* r, s, $s < 0$. *If* D *is an induced regular subgraph of* Γ *of degree* d *on* w *vertices, then*

$$s \leq d - \frac{w(k-d)}{v-w} \leq r,$$

and one of equalities is reached iff each vertex from $\Gamma - D$ *is adjacent with exactly* $w(k-d)/(v-w)$ *vertices from* D.

Proof. This statement is well known (see, for example, § 2 from [5]). □

Up to the end of the section we'll assume that Γ is a strongly regular graph with parameters $(117, 36, 15, 9)$ and spectrum $36^1, 9^{26}, -3^{90}$. Let $G = \mathrm{Aut}(\Gamma)$, g be an element of G of prime order p, and $\Omega = \mathrm{Fix}(g)$.

If Δ is an induced regular subgraph of Γ of degree d on w vertices, then $d - 9 \leq \frac{w(36-d)}{117-w} \leq d + 3$. Therefore number of vertices in a coclique is not greater than 9, and in a clique is not greater than 13.

Lemma 2. *Let* χ_1 *be a character of projection of the representation* ψ *on eigenspace of dimension* 26. *Then* $\alpha_i(g) = \alpha_i(g^l)$ *for any whole number* $l > 0$ *coprime with* $|g|$, $\chi_1(g) = (\alpha_0(g) + \alpha_1(g)/3 - 13)/4$. *If* $|g| = p$ *is a prime number, then* $\chi_1(g) - 26$ *is divisible by* p. *If* $|g| = p^2$, p *is a prime number, then* p^2 *divides* $\chi_1(g^p) - 26$.

Proof. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 26 & 13/2 & -13/4 \\ 90 & -15/2 & 9/4 \end{pmatrix}.$$

Therefore $\chi_1(g) = (8\alpha_0(g)+2\alpha_1(g)-\alpha_2(g))/36$. Substituting $\alpha_2(g) = 117-\alpha_0(g)-\alpha_1(g)$ we'll get $\chi_1(g) = (\alpha_0(g) + \alpha_1(g)/3 - 13)/4$.

Remaining claims of the lemma follow from the lemmas 1–2 [6]. □

Lemma 3. *Let A be a 3-vertex subgraph from Γ, y_i be a number of vertices from $\Gamma - A$ adjacent with exactly i vertices from A. If A is a coclique, then $y_0 = 33 - y_3$; if A is a union of isolated vertex and edge, then $y_0 = 40 - y_3$; if A is a geodetic path, then $y_0 = 47 - y_3$; and if A is a clique, then $y_0 = 54 - y_3$.*

Proof. Let A be a coclique. Then $y_1 = 54 + 3y_3$, $y_2 = 27 - 3y_3$ and $y_0 = 33 - 3y_3$. Let A be a clique. Then $y_1 = 18 + 3y_3$, $y_2 = 42 - 3y_3$ and $y_0 = 54 - 3y_3$. The remaining cases are dealt with similarly. □

Note that if a, b are two vertices from Ω, and $p > 13$, then $[a] \cap [b] \subset \Omega$.

Lemma 4. *The following claims hold:*

(1) *in Γ there are no proper strongly regular subgraphs with parameters $(v', k', 15, 9)$;*

(2) *if Ω is empty graph, then $p = 13$, $\alpha_1(g) = 39$ and $\alpha_2(g) = 78$, or $p = 3$, $\alpha_1(g) = 36l - 9$ and $\alpha_2(g) = 126 - 36l$;*

(3) *if Ω is a n-clique, then $n > 1$, and either $p = 5$, $n = 2, 7, 12$, $\alpha_1(g) = 60l + 75-15n$ and $\alpha_2(g) = 42+14n-60l$, or $p = 2$, $n = 5, 7, 9, 11, 13$, $\alpha_1(g) = 24l+39-3n$ and $\alpha_2(g) = 78 + 2n - 24l$;*

(4) *if Ω is not a clique or empty graph, then Ω contains geodetic 2-path and $p \leq 13$.*

Proof. Let Δ be a strongly regular graph with parameters $(v', k', 15, 9)$. Since $n^2 = 36+4(k'-9)$, we have $n = 2u, k' = u^2$, and Δ has eigenvalues $u+3, -(u-3)$. Multiplicity of $u + 3$ is equal to $(u - 4)u(u^2 + u - 3)/18$, therefore $u \geq 6$.

Let Ω be empty graph. Since $117 = 13 \cdot 9$, we have $p \in \{3, 13\}$. Let's put $\alpha_i(g) = pw_i$.

Let $p = 13$. Then $\chi_1(g) = 13(w_1/3-1)/4$, therefore $\alpha_1(g) = 39$ and $\alpha_2(g) = 78$.

Let $p = 3$. Then $\chi_1(g) = (w_1 - 13)/4$ is congruent to 2 modulo 3, therefore $\alpha_1(g) = 36l - 9$ and $\alpha_2(g) = 126 - 36l$.

Let Ω be a n-clique, a be a vertex from Ω. If $n = 1$, then p divides 36 and 80, therefore $p = 2$, a contradiction with the fact that for a vertex $u \in \Gamma - a^{\perp}$ subgraph $[u] \cap [u^g]$ intersects Ω.

If $n > 1$, then p divides 20, 50 and $17-n$, therefore either $p = 5$ and $n = 2, 7, 12$, or $p = 2$ and $n = 3, 5, \ldots, 13$. In any case $\chi_1(g) = (n + pw_1/3 - 13)/4$.

In the case $p = 5$ we have $\alpha_1(g) = 60l + 75 - 15n$ and $\alpha_2(g) = 42+14n-60l$. In the case $p = 2$ number $(n+2w_1/3-13)/4$ is even, therefore $\alpha_1(g) = 24l + 39 - 3n$ and $\alpha_2(g) = 78 + 2n - 24l$. If $n = 3$, then some vertex from $\Gamma - \Omega$ is not adjacent with vertices from Ω, a contradiction.

Let Ω is a m-clique, $m > 1$. Then p divides 9 and 26, a contradiction.

Let Ω contains an edge and is a union of $t \geq 2$ isolated cliques. Then p divides 9 and 20, a contradiction.

Let Ω contains geodetic 2-path. If $p>13$, then Ω is a strongly regular graph with $\lambda=15$ and $\mu=9$, a contradiction with claim (1). □

In lemmas 5–6 it is presumed that Ω contains geodetic path b,a,c.

Lemma 5. *The following claims hold:*

(1) *if Ω contains vertex a of degree 36, then either $p=3$ and $\alpha_0(g)=45$, or $p=2$, $37\le\alpha_0(g)\le 63$ and number $\alpha_0(g)+\alpha_1(g)/3-13$ is divisible by 8;*

(2) *p is not greater than 5.*

Proof. Let Ω contain vertex a of degree 36. In view of lemma 2 any $\langle g\rangle$-orbit of length p does not contain 3-cocliques. Since any vertex from $\Gamma-a^\perp$ is adjacent with 9 vertices from $[a]$, any $\langle g\rangle$-orbit of length p does not contain geodetic 2-paths. If $p>2$, then any $\langle g\rangle$-orbit of length p is a clique and $p\le 7$. In this case $\chi_1(g)=(\alpha_0(g)+(117-\alpha_0(g))/3-13)/4$, therefore $\alpha_0(g)=6l+3$ for $p>3$, and $\alpha_0(g)=18l+9$ for $p=3$. Hence $p=3$ and $\alpha_0(g)=45$.

If $p=2$, then number $\chi_1(g)=(\alpha_0(g)+\alpha_1(g)/3-13)/4$ is even. Claim (1) is proven.

Let $p=13$. Then $\lambda_\Omega=2,15$, $\mu_\Omega=9$, $|\Omega|=13,26,39,52$ and degrees of vertices in Ω are equal to $10,23$. If $|\Omega|=13$, then Ω is a strongly regular graph with parameters $(13,10,2,9)$, a contradiction.

Let $|\Omega|=26$. If Ω contains vertex a of degree 23, then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to 18, but not less than $23\cdot 7$, a contradiction. So Ω is a strongly regular graph with parameters $(26,10,2,9)$, a contradiction.

Let $|\Omega|=39$. If Ω contains vertex a of degree 10, then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to $28\cdot 9$, but not greater than $20\cdot 10$, a contradiction. So Ω is a regular graph of degree 23, a contradiction.

Let $|\Omega|=52$. If Ω contains vertex a of degree 10, then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to $41\cdot 9$, but not greater than $10\cdot 21$, a contradiction. So Ω is a regular graph of degree 23, and number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to $28\cdot 9=20y+7(23-y)$, therefore $y=13$. But if b,c are two vertices of degree 2 in graph $\Omega(a)$, then $[b]\cap[c]$ contains 13 vertices from $[a]-\Omega$ and at least 12 vertices from $\Omega_2(a)$, a contradiction.

Let $p=11$. Then $\lambda_\Omega=4,15$, $\mu_\Omega=9$, $|\Omega|=18,29,40,51$ and degrees of vertices in Ω are equal to $14,25$. If Ω contains vertex a of degree 14, then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is not less than $14\cdot 9$, is equal to $9(|\Omega|-15)$, but not greater than $14\cdot 20$, therefore $|\Omega|=40$ and the said number is equal to $9\cdot 25=20y+9(14-y)$. $y=9$. But if b,c are two vertices from $\Omega(a)$ adjacent with 20 vertices from $\Omega_2(a)$, then $[b]\cap[c]$ contains a and at least 15 of vertices from $\Omega_2(a)$, a contradiction.

So Ω is a regular graph of degree 25, $|\Omega|=40$, and number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to $14\cdot 9$, but not less than $9\cdot 25$, a contradiction.

Let $p=7$. Then $\lambda_\Omega=1,8,15$, $\mu_\Omega=2,9$, $|\Omega|=19,26,33,40,47,54$, and degrees of vertices in Ω are equal to $8,15,22,29$. If $|\Omega|>33$, then Any $\langle g\rangle$-orbit of length 7 does not contain 3-cocliques. Further, $\chi_1(g)=(\alpha_0(g)+\alpha_1(g)/3-13)/4$, and in the case $\alpha_0(g)=40$ we have $\alpha_1(g)=63$. In this case on $\Gamma-\Omega$ there are 5 clique $\langle g\rangle$-orbits and 6 orbits of degree 4. A contradiction with the fact that for an edge and a vertex from $\langle g\rangle$-orbit of degree 4 which are isolated from each other, subgraph, consisting of vertices adjacent with 0 or 3 vertices from this triplet, contains 40 vertices from Ω and 2 vertices from this $\langle g\rangle$-orbit, a contradiction. In the case $\alpha_0(g) = 47$ we have $\alpha_1(g) = 42$. In this case for geodetic 2-path from $\langle g\rangle$-orbit, subgraph, consisting of vertices adjacent with 0 or 3 vertices from this triplet, contains 47 of vertices from Ω and 2 vertices from this $\langle g\rangle$-orbit, a contradiction. In the case $\alpha_0(g) = 54$ we have $\alpha_1(g) = 63$ and $\chi_1(g) = (41 + 21)/4$, a contradiction.

So $|\Omega| \leq 33$. If Ω contains vertex a of degree 8, then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is not less than $8 \cdot 6$ and is equal to $2(|\Omega| - 9)$, therefore $|\Omega| = 33$, and Ω is a strongly regular graph with parameters $(33, 8, 1, 2)$. In this case Ω has eigenvalues 2, -3, and the multiplicity of 2 is equal to $2 \cdot 8 \cdot 11/10$, a contradiction.

If Ω contains vertex a of degree 22, then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is not less than $22 \cdot 6$ and not greater than $10 \cdot 9$, a contradiction. So Ω is a regular graph of degree 15, $|\Omega| = 26$, and number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to $15 \cdot 6 = 10 \cdot 9$. Hence Ω is a strongly regular graph with parameters $(26, 15, 8, 9)$, and $\chi_1(g) = (13 + \alpha_1(g)/3)/4$, therefore $\alpha_1(g) = 21$.

Number of edges between Ω and $\Gamma - \Omega$ is equal to $26 \cdot 21$, and vertex from $\Gamma - \Omega$ is adjacent in average with $26 \cdot 21/91 = 6$ vertices from Ω. Since $\mu_\Omega = 9$, a vertex from $\Gamma - \Omega$ is adjacent with clique from Ω. In view of the Hoffman bound [1, 1.3.2] the maximal order of clique in Ω is equal to 6, therefore each vertex from $\Gamma - \Omega$ is adjacent with 6-clique from Ω. For $a \in \Omega$ the subgraph $[a]$ contains three $\langle g\rangle$-orbits of length 7 and three 5-cliques Δ_i from Ω adjacent with these orbits. Vertex from Δ_1 is adjacent with two vertices from Δ_2. Suppose that Δ_1 is a coclique. Then each vertex from Δ_1 is adjacent with 10 vertices from $\Delta_2 \cup \Delta_3$. A contradiction with the fact that for two vertices $u, w \in \Delta_1$ the subgraph $[u] \cap [w]$ contains 6 vertices from Ω and at least 6 vertices from $\Delta_2 \cup \Delta_3$. Suppose that Δ_1 is a heptagon. Then each vertex from Δ_1 is adjacent with exactly 4 vertices from Δ_2 and from Δ_3 (for vertices u, w at distance 2 in Δ_1 the subgraph $[u] \cap [w]$ contains 6 vertices from Ω, a vertex from Δ_1, and one vertex from Δ_2 and from Δ_3). Suppose that Δ_1 is a complement of a heptagon. Then each vertex from Δ_1 is adjacent exactly with 3 vertices from Δ_2 and from Δ_3. So, for any vertex $a \in \Omega$ all $\langle g\rangle$-orbits of length 7 on $[a]$ are pairwise isomorphic. Since for $u \in \Gamma - \Omega$ a union of neighbourhoods of vertices from $[u] \cap \Omega$ contains $\Gamma - \Omega$, all $\langle g\rangle$-orbits of length 7 are pairwise isomorphic.

Note that number $\chi_1(g) = (13+\alpha_1(g)/3)/4$ is congruent to 5 modulo 7, therefore $\alpha_1(g) = 63$ and $\alpha_2(g) = 28$. Hence each $\langle g\rangle$-orbit of length 7 is a complement of a heptagon. Then there are four orbits in which vertex is not adjacent with its own image under the action of g. Similarly, there are four orbits in which vertex is not adjacent with its own image under the action of g^2, and there are four orbits in which vertex is not adjacent with its own image under the action of g^3. A contradiction with the fact that $|\Gamma - \Omega| = 91$. □

Lemma 6. *The following claims hold:*

(1) *number p is not equal to* 5*;*

(2) *if $p = 3$, then $|\Omega| \leq 33$ or $|\Omega| = 45$;*

(3) *if $p = 2$, then $|\Omega| \leq 63$.*

Proof. Let $p = 5$. Then $\lambda_\Omega = 0, 5, 10, 15$, $\mu_\Omega = 4, 9$, $|\Omega| = 12, 17, 22, 27, 32, 37, 42, 47, 52$, degrees of vertices in Ω are equal to $6, 11, 16, 21, 26, 31$, and $\chi_1(g) = (\alpha_0(g) + \alpha_1(g)/3 - 13)/4$.

Let Y be a set of vertices from $\Omega_2(a)$ adjacent with 9 vertices from $\Omega(a)$, and $y = |Y|$. Then number of edges between $\Omega(a)$ and $\Omega_2(a)$ is equal to $6|\Omega(a)| + 5x$. On the other hand, the said number of edges is equal to $9y + 4(|\Omega_2(a)| - y)$ and is divisible by 5, a contradiction.

Let $p = 3$. Then $\lambda_\Omega = 0, 3, \ldots, 15$, $\mu_\Omega = 0, 3, 6, 9$, $|\Omega| = 6, 9, \ldots, 54$, and degrees of vertices in Ω are equal to $0, 3, 6, \ldots, 36$, and $\chi_1(g) = (\alpha_0(g) + \alpha_1(g)/3 - 13)/4$, therefore $(\alpha_0(g) + \alpha_1(g)/3 - 13)/4$ is congruent to 2 modulo 3.

If $|\Omega| > 33$, then in view of lemma 2 any $\langle g\rangle$-orbit of length 3 is a clique, $\alpha_1(g) = 117 - \alpha_0(g)$, and $(2\alpha_0(g)/3 + 26)/4$ is congruent to 2 modulo 3. Therefore $\alpha_0(g) = 45$ and $\alpha_1(g) = 72$.

Let $p = 2$. Then $\lambda_\Omega = 1, 3, \ldots, 15$, $\mu_\Omega = 1, 3, \ldots, 9$, $|\Omega| = 5, 7, \ldots, 65$, and degrees of vertices in Ω are equal to $0, 2, 4, \ldots, 36$, and $\chi_1(g) = (\alpha_0(g) + \alpha_1(g)/3 - 13)/4$, therefore $(\alpha_0(g) + \alpha_1(g)/3 - 13)/4$ is even.

If $|\Omega| > 63$, then any $\langle g\rangle$-orbit of length 2 is a clique, $\alpha_1(g) = 117 - \alpha_0(g) = 52$, and $\chi_1(g) = (65 + 52/3 - 13)/4$, a contradiction. □

Now theorem 1 follows from lemmas 4–6.

Let's prove corollary 1. Up to the end of the section it is presumed that Γ is a strongly regular graph with parameters $(117, 36, 15, 9)$, and group $G = \mathrm{Aut}(\Gamma)$ acts transitively on the vertex set of graph Γ. Then $|G|$ is divisible by $9 \cdot 13$. By theorem 1 $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$.

Lemma 7. *Let f be an element of order 13 from G. Then*

(1) *G does not contain elements of order 52 and $13p$, where p is a prime number, $2 < p < 11$;*

(2) *if involution $g \in G$ centralizes f, then $\mathrm{Fix}(g)$ is a 13-clique and $\alpha_1(g) = 0$;*

(3) *$|C_G(f)|$ divides 26 and $S(G) = O_3(G)$.*

Proof. Let G contains subgroup $\langle h\rangle$ of order $13p$, where p is a prime number less than 11, $g = h^{13}$, $f = h^p$. In view of the theorem $\mathrm{Fix}(f)$ is empty graph, $p = 2$ and either $|\Omega| = 13$ and $\alpha_1(g) = 24l$ is divisible by 13, or $|\Omega| = 39$. In the latter case $\chi_1(g) = (\alpha_1(g)/3 + 26)/4$, number $(\alpha_1(g)/3 + 26)/4$ is even, and $\alpha_1(g) = 3(8l - 26)$ is divisible by 13, a contradiction. From action g on $U_i = \{u \in \Gamma \mid d(u, u^{f^i}) = 1\}$ it follows that $\Omega = \mathrm{Fix}(g)$ intersects U_i for any i not multiple of 13, therefore Ω is a 13-coclique.

Let V be a subgroup of order 4 from $C_G(f)$. Since $\chi_1(g) - 26$ is not divisible by 4, V is an elementary abelian group. From action of V on $U = \{\, u \in \Gamma \mid d(u, u^f) = 1 \,\}$ it follows that $\Omega = \mathrm{Fix}(g)$ is contained in U for any involution $g \in V$, a contradiction with action of V on $W = \{\, w \in \Gamma \mid d(w, w^f) = 2 \,\}$.

In view of claims (1–2) we have $S(G) = O_3(G)$. □

We'll finish proof of corollary 1. In view of lemma 7 we have $S(G) = O_3(G)$. Let $\bar{G} = G/O_3(G)$, $\bar{T}$ be a socle of the group $\bar{G}$. From action of subgroup of order 13 on minimal normal subgroup $\bar{N}$ of $\bar{G}$ it follows that $|\bar{N}|$ is divisible by 13. Hence $\bar{T}$ is a simple nonabelian group, and in view of [7, table 1] the group $\bar{T}$ is isomorphic to $L_3(3)$, $L_2(25)$, $U_3(4)$, $PSp_4(5)$, $L_4(3)$, ${}^2F_4(2)'$, $L_2(13)$, $L_2(27)$, $G_2(3)$, ${}^3D_4(2)$, $Sz(8)$, $L_2(64)$, $U_4(5)$, $L_3(9)$, $PSp_6(3)$, $P\Omega_7(3)$, $G_2(4)$, $PSp_4(8)$, $P\Omega_8^+(3)$. Among these groups only $L_3(3)$ and $L_4(3)$ contain maximal subgroup of index dividing 117. Computation in GAP [8] shows that Γ is a graph of rank 3 with group of automorphisms $L_4(3).Z_2$ and vertex stabilizer $Z_2 \times (O(5, 3).Z_2)$. Corollary 1 is proven.

3. Automorphisms of graph with parameters $(378, 117, 36, 36)$

In this section it is presumed that Γ is a strongly regular graph with parameters $(378, 117, 36, 36)$ and spectrum $117^1, 9^{182}, -9^{195}$ in which neighbourhoods of vertices are strongly regular with parameters $(117, 36, 15, 9)$. Let $G = \mathrm{Aut}(\Gamma)$, g be an element from G of prime order p, and $\Omega = \mathrm{Fix}(g)$.

If Δ is an induced regular subgraph of Γ of degree d on w vertices, then $d-9 \leq \frac{w(117-d)}{378-w} \leq d+9$. Therefore number of vertices in coclique is not greater than 27, and in clique not greater than 14.

Lemma 8. *Let χ_1 be the character of projection of representation ψ on eigenspace of dimension 182. Then $\alpha_i(g) = \alpha_i(g^l)$ for any whole number $l > 0$ coprime with $|g|$, $\chi_1(g) = (9\alpha_0(g)+\alpha_1(g))/18-7$. If $|g| = p$ is a prime number, then $\chi_1(g)-182$ is divisible by p. If $|g| = p^2$, p is a prime number, then p^2 divides $\chi_1(g^p) - 182$.*

Proof. We have
$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 182 & 14 & -7 \\ 195 & -15 & 6 \end{pmatrix}.$$
Therefore $\chi_1(g) = (26\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g))/54$. Substituting $\alpha_2(g) = 378 - \alpha_0(g) - \alpha_1(g)$ we'll get $\chi_1(g) = (9\alpha_0(g) + \alpha_1(g))/18 - 7$.

Remaining claims of the lemma follow from the lemmas 1–2 [6]. $\square$

Lemma 9. *The following claims hold:*

(1) *if Ω is empty graph, then either $p = 7$, $\alpha_1(g) = 126l$, and $\alpha_2(g) = 126(3-l)$; or $p = 3$, $\alpha_1(g) = 54l$, and $\alpha_2(g) = 378 - 54l$; or $p = 2$, $\alpha_1(g) = 36l + 18$, and $\alpha_2(g) = 360 - 36l$;*

(2) *if Ω contains $[a]$ for some vertex a, then $p = 2, 5, 13$, consequently, $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$;*

(3) *if Ω is a n-clique, then either $n = 1$, $p = 13$, $\alpha_1(g) = 117$, and $\alpha_2(g) = 260$, or $\alpha_1(g) = 351$ and $\alpha_2(g) = 26$; or $p = 5$, $n = 3, 8, 13$, $\alpha_1(g) = 45l$; or $p = 2$, $n = 6, 8, \ldots, 14$, and $\alpha_1(g) = 18l$;*

(4) *if Ω is a m-coclique, $m > 1$, then $p = 3$, $m = 3, 6, \ldots, 27$ and $\alpha_1(g) = 9l$;*

(5) *if Ω is not a clique or an empty graph, then Ω contains geodetic 2-path.*

Proof. Let Δ be a strongly regular subgraph with parameters $(v', k', 36, 36)$. Since $n^2 = 4(k'-36)$, we have $n = 2u, k' = u^2 + 36$, and Δ has eigenvalues $u, -u$. Multiplicity of u is equal to $(u-1)(u^2+36)(u^2+u+36)/(72u)$, therefore $u = 3, 6$ and Δ has parameters $(56, 45, 36, 36)$ or $(143, 72, 36, 36)$. On the other hand, between Δ and $\Gamma - \Delta$ there are $v'(117-k')$ edges. A contradiction with the fact that some vertex from $\Gamma - \Delta$ is adjacent at least with 2 vertices from Δ.

Let Ω be empty graph. Since $378 = 9 \cdot 42$, we have $p \in \{2, 3, 7\}$. Let's put $\alpha_i(g) = pw_i$.

Let $p = 7$. Then $\chi_1(g) = 7w_1/18 - 7$, therefore $\alpha_1(g) = 126l$ and $\alpha_2(g) = 126l(3-l)$.

Let $p = 3$. Then number $\chi_1(g) = w_1/6 - 7$ is congruent to 2 modulo 3, therefore $\alpha_1(g) = 54l$ and $\alpha_2(g) = 378 - 54l$.

Let $p = 2$. Then number $\chi_1(g) = w_1/9 - 7$ is even, therefore $\alpha_1(g) = 36l + 18$ and $\alpha_2(g) = 360 - 36l$.

Let Ω contain $[a]$ for some vertex a. Then $[u]$ contains 36 vertices from Ω for any vertex $u \in \Gamma - \Omega$. If $b \in \Omega - a^{\perp}$, then Ω contains $[b]$, a contradiction. So $|\Omega| = 118$,

$\alpha_2(g) = 378 - 118 = 260$, and $\chi_1(g) - 182 = 9\alpha_0(g)/18 - 189 = -130$ is divisible by p. Hence $p = 2, 5, 13$, and in view of theorem 1 we have $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$.

Let Ω be a n-clique, a be a vertex from Ω. If $n = 1$, then p divides 117 and 260, therefore $p = 13$, $\chi_1(g) = (9 + \alpha_1(g))/18 - 7$, and $\alpha_1(g) = 117, 351$.

If $n > 1$, then p divides 80, 180 and $38 - n$, therefore either $p = 5$, $n = 3, 8, 13$, $\alpha_1(g) = 45l$, number $\chi_1(g) = (n + 5l)/2 - 7$ is congruent to 2 modulo 5, and $n + 5l = 10m - 2$, or $p = 2$ and $n = 2, 4, \ldots, 14$. In the latter case $\alpha_1(g) = 18l$, and number $\chi_1(g) = n/2 + l - 7$ is odd. Since any vertex from $\Gamma - \Omega$ is adjacent with even number of vertices from Ω, we have $n \geq 6$.

Let Ω be a m-coclique, $m > 1$. Then p divides 81 and 36, therefore $p = 3$, $m = 3, 6, \ldots, 27$, $\alpha_1(g) = 9l$, number $\chi_1(g) = (m + l)/2 - 7$ is congruent to 2 modulo 3, therefore $m + l$ is divisible by 6.

Let Ω contain an edge and be a union of $t \geq 2$ isolated cliques. Then p divides 80 and 81, a contradiction. □

Lemma 10. *If Ω contains geodetic path b, a, c, then the following claims hold:*

(1) *p is not greater than 3, and if $p = 3$ then $|\Omega| \leq 30$ and $\alpha_1(g) = 54m - 27t$;*

(2) *if $p = 2$ then $|\Omega| \leq 126$, degree of vertex in Ω is not greater than 63, and $\alpha_1(g) = 36m + 18 - 18t$.*

Proof. Let $\alpha'_i(g)$ be a number of vertices in $\Omega(a)$ moved by g at a distance i. One can account that g acts exactly on $[a]$. Since $|\Omega| \leq v \cdot \max\{\lambda, \mu\}/(k - r) = 378 \cdot 36/108$, we have $|\Omega| \leq 126$.

In view of theorem 1 $\Omega(a)$ contains geodetic 2-path, and either

(*i*) $p = 3$, $|\Omega(a)| = 3t \leq 33$ or $|\Omega(a)| = 45$, or

(*iii*) $p = 2$, $|\Omega| \leq 126$, $|\Omega(a)| \leq 63$.

Let $p = 3$. If Ω is a regular graph of degree 45, then by lemma 1 we have $|\Omega| = 126$, and each vertex from $\Gamma - \Omega$ is adjacent exactly with 27 vertices from Ω. Now number $\chi_1(g) = 56 + \alpha_1(g)/18$ is congruent with 2 modulo 3, therefore $\alpha_1(g)$ is divisible by 54.

Let $p = 3$, $|\Omega| = 3t$. Then number $\chi_1(g) = (27t + \alpha_1(g))/18 - 7$ is congruent to 2 modulo 3. Therefore $\alpha_1(g) = 54m - 27t$.

Let $p = 2$, $|\Omega| = 2t$. Then number $\chi_1(g) = (18t + \alpha_1(g))/18 - 7$ is even. Therefore $\alpha_1(g) = 36m + 18 - 18t$. □

Now theorem 2 follows from lemmas 9–10.

4. Strongly regular graph with parameters $(378, 117, 36, 36)$, vertex-symmetric case

In this section corollary 2 is proven.

Let Γ be a strongly regular graph with parameters $(378, 117, 36, 36)$, group $G = \mathrm{Aut}(\Gamma)$ act transitively on vertex set of Γ, $|G|$ be divisible by 13, g be an element from G of prime order p, and $\Omega = \mathrm{Fix}(g)$. Then $|G : G_a| = 378$, and by theorem 2 $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$.

Lemma 11. *The following claims hold:*

(1) *if f is an element of G of order 13, g is an element of $C_G(f)$ of prime order $p < 13$, then either $p = 3$, Ω is a 27-coclique, $|C_G(f)|$ is not divisible by 81, and $C_G(f)$ does not contain elements of order 9, or $p = 2$, Ω is a 14-clique, and $|C_G(f)|$ is not divisible by 4;*

(2) *if f is an element of G of order 7, g is an element of $C_G(f)$ of order 3, then either Ω is empty graph, $\alpha_1(g) = 378$, and $|C_G(f)|$ is not divisible by 27, or $|\Omega| = 21s$, $\alpha_2(g) = 378 - 21s - 27l$, and $|C_G(f)|$ is not divisible by 27;*

(3) $S(G) = O_3(G)$;

(4) *if $\bar{T}$ is a socle of the group $\bar{G} = G/O_3(G)$, then either*

(*i*) $\bar{T} \cong L_2(13)$, *$\bar{T}_a$ is an extension of group of order 13 by group of order 2 or 6, or*

(*ii*) $T \cong L_2(27)$, *T_a is an extension of group of order 13 by group of order 2, or*

(*iii*) $T \cong \Omega_7(3)$, *T_a is an extension of $L_4(3)$ by group of order 2.*

Proof. Let f be an element of G of order 13, g be an element of $C_G(f)$ of prime order $p < 13$. Then $\mathrm{Fix}(f) = \{a\}$ is a 1-vertex graph, $\alpha_1(f) = 117$ and $\alpha_2(f) = 260$, or $\alpha_1(f) = 351$ and $\alpha_2(f) = 26$. From action f on Ω it follows that $|\Omega| - 1$ is divisible by 13. In view of lemma 7 either $p = 2$, Ω is a 14-clique, or $p = 3$, Ω is a 27-coclique.

If $|\Omega| = 27$ then from the proof of lemma 10 it follows that $\alpha_1(g) = 9l$, and $27 + l$ is divisible by 6. Hence $\alpha_1(g) = 351$, and $|C_G(f)|$ is not divisible by 81. Further, $\chi_1(g) = (9\alpha_0(g) + \alpha_1(g))/18 - 7 = 26$, $\chi_1(g) - 182$ is not divisible by 9, and by lemma 8 $C_G(f)$ does not contain elements of order 9. Let $U = \langle f, h\rangle$ be a subgroup of order 9 from $C_G(f)$, and $\Delta = \mathrm{Fix}(U)$ contain 3^t vertices. Since $378 - (3^t + 4(27 - 3^t))$ is divisible by 9, we have that 3^{t+1} is divisible by 9. If $u \in \Omega - \Delta$, then, as shown above, $d(u, u^h) = 1$, a contradiction with the fact that Ω is a 27-coclique. So $\Delta = \mathrm{Fix}(x)$ for any element x of order 3 from U.

If Ω is a 14-clique, then in view of lemma 10 we have $\alpha_1(g) = 234$ and $\alpha_2(g) = 130$, therefore $|C_G(f)|$ is not divisible by 4.

Let f be an element of G of order 7, g be an element of $C_G(f)$ of order 3. By theorem 2 $\mathrm{Fix}(f)$ is empty graph, $\alpha_1(f) = 126l$ and $\alpha_2(f) = 126(3 - l)$. If Ω is empty graph, then $\alpha_1(g) = 54l$ is divisible by 7, $\alpha_1(g) = 378$, and $|C_G(f)|$ is not divisible by 27. If $C_G(f)$ contains element h of order 9 such that $g = h^3$, then $\chi_1(g) = \alpha_1(g)/18 - 7 = 14$, $\chi_1(g) - 182$ is not divisible by 9, a contradiction with lemma 8.

If Ω is a nonempty graph, then $|\Omega| = 21s$, $s \leq 6$, $\alpha_1(g) = 27l$, $\chi_1(g) = (21s + 3l)/2 - 7$ and $\alpha_2(g) = 378 - 21s - 27l$, and $|C_G(f)|$ is not divisible by 27.

Since $v = 378$, we have that $S(G)$ is a $\{2,3,7\}$-group. In view of claim (1) $S(G)$ is a $\{2,3\}$-group. Let P be a Sylow 2-subgroup from $S(G)$. Then $|P : P_a|$ divides 2. By lemma 7 we have $S(G_a) = O_3(G_a)$, therefore $|P_a| = 1$, and $|P|$ divides 2. If $|P| = 2$, then element f of G of order 13 centralizes involution $g \in P$, and Ω is a 14-clique. A contradiction with the fact that subgraph Ω admits G.

In view of table 1 from [7] a socle $\bar{T}$ of group $\bar{G} = G/O_3(G)$ is isomorphic to $L_2(13)$, $L_2(27)$, $G_2(3)$, $Sz(8)$, $L_2(64)$, $U_4(5)$, $L_3(9)$, $PSp_6(3)$, $\Omega_7(3)$, $G_2(4)$, $P\Omega_8^+(3)$.

Since $\bar{T}$ contains proper subgroup of index divisible by 7 and dividing 378, we have that either $\bar{T} \cong L_2(13)$, $\bar{T}_a$ is an extension of group of order 13 by group of order 2 or 6, or $\bar{T} \cong L_2(27)$, $\bar{T}_a$ is an extension of group of order 13 by group of order 2, or $\bar{T} \cong \Omega_7(3)$, $\bar{T}_{\{F\}}$ is an extension of $L_4(3)$ by group of order 2. In the latter two cases we have $O_3(G) = 1$. □

Lemma 12. *We have $T \cong P\Omega_7(3)$, T_a is an extension of $L_4(3)$ by group of order 2, and Γ is a graph of rank 3.*

Proof. Let's put $V = O_3(G)$ and fix element f from G_a of order 13.

Let $\bar{T} \cong L_2(13)$, $\bar{T}_a$ be an extension of group of order 13 by group of order 2. Then $|\bar{T} : \bar{T}_a| = 42$ and $|V : V_a| = 9$. A contradiction with the fact that $C_V(f)$ is not contained in V_a.

Let $\bar{T} \cong L_2(13)$, $\bar{T}_a$ be an extension of group of order 13 by group of order 6. Then $|\bar{T} : \bar{T}_a| = 14$ and $|V : V_a| = 27$. If W is an irreducible $F_3\bar{T}_a$-submodule of V, then $|W| = 3^7, 3^{12}, 3^{13}$ and $|W : W_a| = 27$, otherwise we'll get a contradiction with the fact that $C_W(f)$ is not contained in W_a. Now $W = V$.

Group $3^{13} : L_2(13)$ is built using package AtlasRep in GAP [8], then we compute subgroup $\langle V, T_a \rangle$ and find all its normal subgroups lying in V. They have orders $1, 3, 3^6, 3^6, 3^7, 3^7, 3^12, 3^13$. A contradiction with the fact that $|V : V_a| \neq 27$.

Let $T \cong L_2(27)$, T_a be an extension of group of order 13 by group of order 2. Using computation in GAP [8] we'll get that strongly regular graph with parameters $(378, 117, 36, 36)$ does not arise.

Let $T \cong \Omega_7(3)$, T_a be an extension $L_4(3)$ by group of order 2. Using computation in GAP [8] we'll get that corresponding strongly regular graph with parameters $(378, 117, 36, 36)$ is a graph of rank 3. The lemma and corollary 2 are proven. □

5. Automorphisms of graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$

Let Γ be a distance-regular graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$ and spectrum $117^1, 39^{27}, 9^{182}, -3^{351}, -9^{195}$, $G = \mathrm{Aut}(\Gamma)$, g be an element of G of prime order p, and $\Omega = \mathrm{Fix}(g)$.

Lemma 13. *Let χ_1 be the character obtained by projecting $\psi(G)$ into eigenspace of dimension 27, χ_2 be the character obtained by projecting $\psi(G)$ into eigenspace of dimension 182, and χ_4 be the character obtained by projecting $\psi(G)$ into eigenspace of dimension 195. Then $\chi_1(g) = (3\alpha_0(g) + \alpha_1(g) - \alpha_3(g) - 3\alpha_4(g))/84$, $\chi_2(g) = (8\alpha_0(g) - \alpha_2(g) + 8\alpha_4(g))/36 + 14$, and $\chi_4(g) = (10\alpha_0(g) + \alpha_2(g) + 10\alpha_4(g))/36 - 15$. Furthermore, $\chi_1(g) - 27$, $\chi_2(g) - 182$, and $\chi_4(g) - 195$ are divisible by p.*

Proof. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 27 & 9 & 0 & -9 & -27 \\ 182 & 14 & -7 & 14 & 182 \\ 351 & -9 & 0 & 9 & -351 \\ 195 & -15 & 6 & -15 & 195 \end{pmatrix}.$$

Therefore $\chi_1(g) = (3\alpha_0(g) + \alpha_1(g) - \alpha_3(g) - 3\alpha_4(g))/84$. Further, $\chi_2(g) = (26\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) + 2\alpha_3(g) + 26\alpha_4(g))/108$. Having substituted $\alpha_1(g) + \alpha_3(g) = 756 - \alpha_0(g) - \alpha_2(g) - \alpha_4(g)$ we'll get $\chi_2(g) = (8\alpha_0(g) - \alpha_2(g) + 8\alpha_4(g))/36 + 14$.

Similarly, $\chi_4(g) = (65\alpha_0(g) - 5\alpha_1(g) + 2\alpha_2(g) - 5\alpha_3(g) + 65\alpha_4(g))/252$. Substituting in this formula value $\alpha_1(g) + \alpha_3(g) = 756 - \alpha_0(g) - \alpha_2(g) - \alpha_4(g)$ we'll get $\chi_4(g) = (10\alpha_0(g) + \alpha_2(g) + 10\alpha_4(g))/36 - 15$.

The last claim of the lemma follows from lemma 1 [6]. □

Lemma 14. *If g induces trivial automorphism of antipodal quotient $\bar{\Gamma}$, then $p = 2$ and $\alpha_4(g) = v$. Furthermore, order of the subgroup of G, inducing trivial automorphisms of $\bar{\Gamma}$, divides 2.*

Proof. By condition, $\alpha_i(g) \neq 0$ may be only for $i = 0, 4$. If $u = u^g$, then $[u]$ consists of vertices fixed by g. Therefore g leaves fixed each vertex of Γ, a contradiction. So $\alpha_4(g) = v$. Since $r = 2$, order of the subgroup of G inducing trivial automorphisms of $\bar{\Gamma}$ divides 2. □

Lemma 15. *If g induces nontrivial automorphism of graph $\bar{\Gamma}$, then one of the claims holds:*

(1) Ω *is empty graph,* $\alpha_4(g) = 0$, *and either* $p = 7$, $\alpha_1(g) = 294n + 336 - 126m$, $\alpha_2(g) = 252m$, $\alpha_3(g) = -294n + 420 - 126m$; *or* $p = 3$, $\alpha_1(g) = 126n + 378 - 54m$, $\alpha_2(g) = 108m$, $\alpha_3(g) = -126n + 378 - 54m$; *or* $p = 2$, $\alpha_1(g) = 84n + 336 - 36m$, $\alpha_2(g) = 72m$, $\alpha_3(g) = 420 - 84n - 36m$;

(2) Ω *is an antipodal class,* $p = 13$, $\alpha_1(g) = 104$, $\alpha_2(g) = 520$, $\alpha_3(g) = 130$;

(3) $\bar{\Omega}$ *is a n-clique, and either* $p = 5$, $n = 3$, $\alpha_1(g) = 210$, $\alpha_2(g) = 480$, $\alpha_3(g) = 60$; *or* $p = 2$, $n = 6$, $\alpha_1(g) = 84m + 150$, $\alpha_2(g) = 456$, $\alpha_3(g) = 138 - 84m$, *or* $n = 12$, $\alpha_1(g) = 84m + 132$, $\alpha_2(g) = 408$, $\alpha_3(g) = 192 - 84m$;

(4) Ω *is a 42-coclique,* $p = 3$, $\alpha_1(g) = 126n + 18$, $\alpha_2(g) = 108l - 420 = 552$, $\alpha_3(g) = 144 - 126n$;

(5) $\bar{\Omega}$ *contains geodetic 2-path, and either*

(*i*) $p = 3$, $|\Omega| = 6t \leq 252$, $\alpha_1(g) = 126n + 378 - 54l + 18t$, $\alpha_2(g) = 108l - 60t$, $\alpha_3(g) = 378 - 126n - 54l + 36t$, *or*

(*ii*) $p = 2$, $|\Omega| + \alpha_4(g) = 4t \leq 252$, $\alpha_1(g) + \alpha_3(g) = 756 - 72l + 36t$, $\alpha_2(g) = 72l - 40t$.

Proof. By theorem 2 one of the claims holds:

(1) $\bar{\Omega}$ is empty graph, and either $p = 7$, $\bar{\alpha}_1(g) = 126l$, $\bar{\alpha}_2(g) = 126(3 - l)$; or $p = 3$, $\bar{\alpha}_1(g) = 54l$, $\bar{\alpha}_2(g) = 378 - 54l)$; or $p = 2$, $\bar{\alpha}_1(g) = 36l + 18$, $\bar{\alpha}_2(g) = 360 - 36l$;

(2) $\bar{\Omega}$ is a n-clique, and either $n = 1$, $p = 13$, $\bar{\alpha}_1(g) = 117$, $\bar{\alpha}_2(g) = 260$; or $n = 3, 8, 13$, $p = 5$, $\bar{\alpha}_2(g) = 378 - n - 45l$; or $n = 6, 8, \ldots, 14$, $p = 2$, and $\bar{\alpha}_2(g) = 378 - n - 18l$;

(3) $\bar{\Omega}$ is a m-coclique, $p = 3$, $m = 3, 6, \ldots, 27$, $\bar{\alpha}_1(g) = 9l$, $\bar{\alpha}_2(g) = 378 - m - 9l$;

(4) $\bar{\Omega}$ contains geodetic 2-path, and either

(*i*) $p = 3$, $|\bar{\Omega}| = 3t \leq 126$ and $\alpha_2(g) = 378 - 54m + 24t$, or

(*ii*) $p = 2$, $|\bar{\Omega}| = 2t \leq 126$, degrees of vertices in $\bar{\Omega}$ are not greater than 63, and $\alpha_2(g) = 360 - 36m + 16t$.

If $\bar{\Omega}$ is empty graph, then Ω is empty graph, and $\alpha_4(g) = 0$. In the case $p = 7$ we have $\chi_4(g) = \alpha_2(g)/36 - 15$, therefore $\alpha_2(g) = 252m$. Further, $\alpha_1(g) + \alpha_3(g) = 756 - 252m$, number $\chi_1(g) = (\alpha_1(g) - 378 + 126m)/42$ is congruent with -1 modulo 7, therefore $\alpha_1(g) = 294n + 336 - 126m$, $\alpha_3(g) = -294n + 420 - 126m$.

In the case $p = 3$ number $\chi_4(g) = \alpha_2(g)/36 - 15$ is divisible by 3, therefore $\alpha_2(g) = 108m$. Further, $\alpha_1(g) + \alpha_3(g) = 756 - 108m$, number $\chi_1(g) = (\alpha_1(g) - 378 + 54m)/42$ is divisible by 3, therefore $\alpha_1(g) = 126n + 378 - 54m$, $\alpha_3(g) = -126n + 378 - 54m$.

In the case $p = 2$ number $\chi_4(g) = \alpha_2(g)/36 - 15$ is odd, therefore $\alpha_2(g) = 72m$. Further, $\alpha_1(g) + \alpha_3(g) = 756 - 72m$, number $\chi_1(g) = (\alpha_1(g) - 378 + 36m)/42$ is odd, therefore $\alpha_1(g) = 84n + 336 - 36m$, $\alpha_3(g) = 420 - 84n - 36m$.

Let $\bar{\Omega}$ is a n-clique. If $p = 13$, then $|\Omega| = 2$, $\alpha_4(g) = 0$, $\chi_4(g) = (20 + \alpha_2(g))/36 - 15$, therefore $\alpha_2(g) = 13(36l + 4)$. On the other hand, $\bar{\alpha}_2(g) = 260 = 13(18l + 2)$ and $l = 1$. Further, $\alpha_1(g) + \alpha_3(g) = 754 - 520 = 234$, therefore $\chi_1(g) = (\alpha_1(g) - 114)/42$, $\alpha_1(g) = 104$, $\alpha_3(g) = 130$.

If $p = 5$, then $|\Omega| = 2n - \alpha_4(g)$, $n = 3, 8, 13$. Note that $\alpha_4(g) = 0$. Now number $\chi_4(g) = (20n+\alpha_2(g))/36-15$ is divisible by 5, therefore $\alpha_2(g) = 180l-20n$. On the other hand, in view of theorem 2 we have $\alpha_2(g)/2 = 378-n-45l = 90l-10n$, and n is divisible by 3. Hence $n = l = 3$ and $\alpha_2(g) = 480$. Now $\alpha_1(g)+\alpha_3(g) = 810-180l$, number $\chi_1(g) = (\alpha_1(g) - 414 + 90l)/42$ is congruent with 2 modulo 5, therefore $\alpha_1(g) = 210m+210$, $\alpha_3(g) = 60-210m$. By theorem 1 in neighbourhood of a vertex from Ω we have $n = 2$ and $\alpha_1'(g) = 60l + 45$, therefore $\alpha_1(g) = 210$, $\alpha_3(g) = 60$.

If $p = 2$, then $|\Omega| = 2n - \alpha_4(g)$, $n = 6, 8, \ldots, 14$. Note that either $\alpha_4(g) = 0$, or $|\Omega| = 0$. Now number $\chi_4(g) = (20n + \alpha_2(g))/36 - 15$ is odd, therefore $\alpha_2(g) = 72l - 20n$. On the other hand, in view of theorem 2 we have $\alpha_2(g)/2 = 378 - n - 18l = 36l - 10n$, $378 + 9n$ is divisible by 27, and $n = 6, 12$. Now $\alpha_1(g)+\alpha_3(g) = 756-72l+18n$, number $\chi_1(g) = (\alpha_1(g)-378+36l-3n)/42$ is odd, therefore $\alpha_1(g) = 84m+420-36l+3n$, $\alpha_3(g) = 336-84m-36l+15n$. In the case $n = 6$ we have $l = 8$, $\alpha_1(g) = 84m + 150$, $\alpha_2(g) = 456$, $\alpha_3(g) = 138 - 84m$, and in the case $n = 12$ we have $l = 9$, $\alpha_1(g) = 84m+132$, $\alpha_2(g) = 408$, $\alpha_3(g) = 192-84m$.

If $\bar{\Omega}$ is a m-coclique, then Ω is a coclique, $p = 3$, $m = 3, 6, \ldots, 27$, and $\alpha_4(g) = 0$. Further, number $\chi_4(g) = (20m + \alpha_2(g))/36 - 15$ is divisible by 3, therefore $\alpha_2(g) = 108l - 20m$. On the other hand, in view of theorem 2 we have $54l - 10m = 378 - m - 9l$, and $7l - m$ is divisible by 42. Hence $m = 21$, number l is odd and is divisible by 3, $\alpha_2(g) = 108l - 420 = 552$. Further, $\alpha_1(g) + \alpha_3(g) = 162$, number $\chi_1(g) = (\alpha_1(g) - 18)/42$ is divisible by 3, therefore $\alpha_1(g) = 126n + 18$, $\alpha_3(g) = 144 - 126n$.

Let $\bar{\Omega}$ contain geodetic 2-path. If $p = 3$, then $|\Omega| = 6t \leq 252$, and $\alpha_4(g) = 0$. Further, number $\chi_4(g) = (60t + \alpha_2(g))/36 - 15$ is divisible by 3, therefore $\alpha_2(g) = 108l - 60t$. Hence $\alpha_1(g) + \alpha_3(g) = 756 - 108l + 54t$, number $\chi_1(g) = (\alpha_1(g)-378+54l-18t)/42$ is divisible by 3, therefore $\alpha_1(g) = 126n+378-54l+18t$, $\alpha_3(g) = 378 - 126n - 54l + 36t$.

If $p = 2$, then $|\Omega|+\alpha_4(g) = 4t \leq 252$. Further, number $\chi_4(g) = (40t+\alpha_2(g))/36-15$ is odd, therefore $\alpha_2(g) = 72l - 40t$. Hence $\alpha_1(g) + \alpha_3(g) = 756 - 72l + 36t$. □

Now theorem 3 follows from lemmas 14–15.

6. Distance-regular graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$, vertex-symmetric case

In this section the corollary 3 is proven. Let Γ be a distance-regular graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$, group $G = \mathrm{Aut}(\Gamma)$ act transitively on vertex set of graph Γ, $|G|$ be divisible by 13, K be a subgroup of G acting semiregularly on each antipodal class, and F be an antipodal class containing vertex a of Γ. Then $|G : G_{\{F\}}| = 378$, $|K|$ divides 2, and by theorem 3 $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$.

Let's fix element f of G of order 13. From lemma 11 and theorem 3 it follows that $C_G(f) = K\langle f\rangle$. In view of corollary 2 we have $O_3(G) = 1$, and either

(1) $K \neq 1$, T is a nonsplit extension of group of order 2 by $\Omega_7(3)$, $T_{\{F\}} \cong (Z_2 \times L_4(3)).Z_2$, and $T_a \cong L_4(3).Z_2$, or

(2) $K = 1$, $T \cong \Omega_7(3)$, $T_{\{F\}} \cong L_4(3).Z_2$, and $T_a \cong L_4(3)$.

Using computation in GAP [8] one can see that in the case (1) distance-regular graph does not arise. And in the case (2) distance-regular graph with intersection array $\{117, 80, 18, 1; 1, 18, 80, 117\}$ does not arise. (But there is bipartite double of strongly regular graph with parameters $(378, 117, 36, 36)$.) Corollary 3 is proven.

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Aleksandr Alekseevich Makhnev
Uralskii Federalnii Universitet,
Krasovskii Institute of Mathematics and Mechanics,
ul. S. Kovalevskoi, 16,
620990, Ekaterinburg, Russia
E-mail address: makhnev@imm.uran.ru

Dmitrii Viktorovich Paduchikh
Krasovskii Institute of Mathematics and Mechanics,
ul. S. Kovalevskoi, 16,
620990, Ekaterinburg, Russia
E-mail address: dpaduchikh@gmail.com

Madina Mukhatdinovna Khamgokova
Kabardino-Balkarskii University,
ul. Mira, 16,
360000, Nalchik, Russia
E-mail address: hamgokova.madina@yandex.ru