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ON THE SOLVABILITY OF THE TRANSFER EQUATION COUPLED WITH BOLTZMANN EQUATION FOR TWO-LEVEL ATOMS

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ABSTRACT. We consider the nonlinear transfer equation coupled with the Boltzmann equation for two-level atoms in plane-parallel semi-infinite medium. It is assumed that the velocity distributions of atoms in the ground and excited states are different from the Maxwell distribution. This assumption is justified by the selective nature of the excitation of atoms by photons of different frequencies. In the linear approximation the problem is reduced to the Wiener-Hopf equation with a dissipative kernel. We prove the existence of a positive, bounded and integrable solution in the quasilinear approximation. We also obtain two-sided estimates for the source function.

Keywords: Boltzmann equation, nonlinear integral equation, two-level atom, Wiener-Hopf equation.

1. INTRODUCTION

It is well known that the resonance radiation field depends on the density of excited atoms and thereby on the velocity distribution of atoms. On the other hand, the velocity distribution of excited atoms is influenced by the radiation field (cf. [1],[2]). Therefore a rigorous study of the resonance scattering problem requires solving the transfer equation (Boltzmann equation for photons) together with the corresponding Boltzmann kinetic equation for atoms. It was shown in [1] for the model of two-level atoms that if the total velocity distribution of atoms is Maxwell

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the same can not be true for the distributions of atoms in the ground and, especially, excited states. This is due to the fact that the excitations of atoms by the quanta of various frequencies are of different nature.In the linear approximation it is natural to assume that the velocity distribution in the ground state is Maxwell in contrast with the velocity distribution of excited atoms. This assumption is based on the fact that the lifetime of excited atoms $t_{at} \backsim \frac{1}{A_{21}}$ (where A_{21} is the Einstein coefficient of spontaneous transition) is much less than the time interval between two successive collisions. In this paper we study and solve the nonlinear transfer equation coupled with the Boltzmann kinetic equation for the model case of two-level atoms in a plane-parallel semi-infinite medium. Mathematically, the problem is described by the system of nonlinear integro-differential equations. In the linear approximation it is reduced to the Wiener-Hopf equation with a dissipative kernel which is not difficult to solve. In the quasilinear approximation we can still prove the existence of a positive, bounded and integrable solution as well as two-sided estimates for the source function.

2. The statement of the problem. Derivation of the basic equation

Suppose that we have a plane-parallel semi-infinite medium uniformly filled with two-level atoms and free electrons. Assume that the medium from the exterior side of the boundary $z = 0$ is exposed to the radiation $I_0(\eta)$ where $\eta \in [-1, 1]$ is the cosine of the incidence angle of the quantum (with respect to the normal of the boundary). Suppose that the densities of atoms n_0 and electrons n_e are given and the temperatures of atoms and electrons are equal: $T_{at} = T_e = T$.

Denote by

$$F_k(z, \vec{s}) = n_k(z) f_k(\vec{s}), \quad k = 1, 2,$$

the distribution functions of atoms in the ground and excited states respectively. Here, $n_1(z)$ and $n_2(z)$ denote respectively the densities of atoms in the ground and excited states and $\vec{s} = (s_1, s_2, s_3)$ is the velocity of the atom.

We shall assume that the total distribution function of atoms is Maxwell which means that

$$F_1(z, \vec{s}) + F_2(z, \vec{s}) = F_0^M(\vec{s}) \tag{2.1}$$

where

$$F_0^M(\vec{s}) = n_0 f_0(\vec{s}) = \frac{n_0 e^{-\frac{s^2}{T}}}{(\pi T)^{\frac{3}{2}}}$$

is the Maxwell distribution function. Obviously

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_k(z, \vec{s}) d^3 s = n_k(z), \quad k = 1, 2. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$n_1(z) + n_2(z) = n_0. \tag{2.3}$$

The radiative transfer equation or the Boltzmann equation for photons in the case of the monochrome scattering can be written as (cf. [3]):

$$\eta \frac{\partial J(z, \eta)}{\partial z} + (n_1(z) B_{12} - n_2(z) B_{21})\, J(z, \eta) = n_2(z) A_{21} \tag{2.4}$$

where A_{12}, B_{12}, B_{21} are the Einstein transition coefficients and J is the unknown intensity.

In the one-dimensional approximation the Boltzmann kinetic equation for excited atoms has the form (cf.[1],[2]):

$$s_1 \frac{\partial F_2(z,\vec{s})}{\partial z} = \mathfrak{F}(F_2) \tag{2.5}$$

where $\mathfrak{F}(F)$ is the collision integral.

The collision integral contains the terms describing the inelastic collisions with electrons as well as the terms responsible for the interaction with the field:

$$\mathfrak{F}(F_2) = (a_{12} + B_{12}S(z))\, F_1(z,\vec{s}) - (a_{21} + A_{21} + B_{21}S(z))\, F_2(z,\vec{s}). \tag{2.6}$$

Here, a_{12} and a_{21} are the electron collision coefficients and

$$S(z) = \frac{1}{2}\int_{-1}^{1} J(z,\eta)d\eta \tag{2.7}$$

is the source function or the averaged (over the angles) radiation intensity.

Integrating both sides of (2.5) with respect to the velocity and taking into account the formulas (2.6), (2.1), (2.3), we obtain

$$æ\frac{dn_2(z)}{dz} + (a_{21}+a_{12}+A_{21})n_2(z) + (B_{12}+B_{21})n_2(z)S(z) = n_0a_{12} + B_{12}n_0S(z) \tag{2.8}$$

where

$$æ = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s_1 f(\vec{s})\, d^3s.$$

The boundary conditions for the equations (2.4) and (2.8) have the form

$$J(0,\eta)|_{\eta\in(0,1]} = I_0(\eta), \quad J(z,\eta)|_{\eta\in[-1,0)} = o(e^{\frac{z}{\eta}}) \text{ for } z \to +\infty, \quad n_2(0) = \alpha_0. \tag{2.9}$$

We introduce a new variable x in the equations (2.4), (2.8) by setting

$$dx = (n_1(z)B_{12} - n_2(z)B_{21})\, dz = (n_0B_{12} - (B_{12}+B_{21})n_2(z))\, dz \tag{2.10}$$

so that

$$x(z) = n_0B_{12}z - (B_{12}+B_{21})\int_0^z n_2(z')dz'. \tag{2.11}$$

Note that the new argument x depends on the solution of the original problem. After solving this problem we can return to the variable z by using the formula

$$z(x) = \int_0^x \frac{dx'}{n_0B_{12} - (B_{12}+B_{21})n_2(x')}\,. \tag{2.12}$$

The idea of introducing the new variable x, being the real optical depth, belongs to the academician V.A. Ambartsumyan and is known in the literature as the method of self-consistent optical depths (MSOD) (cf. [4, 5]).

In the paper [5] it was proposed a mathematical method, close to MSOD, for the solution of the nonlinear transfer problem. Using this method, it is possible in a number of cases to linearize or essentially simplify the problem as well as reconstruct the geometrical depth z by the formula (2.12).

Introduce the notations:

$$\varphi(x) = n_2(z(x)), \quad I(x,\eta) = J(z(x),\eta), \quad f(x) = S(z(x)). \tag{2.13}$$

Then the system (2.4) and (2.8) is reduced to the following nonlinear system of integro-differential equations

$$\eta\frac{\partial I(x,\eta)}{\partial x}+I(x,\eta)=\frac{\lambda_0\varphi(x)}{1-\lambda_1\varphi(x)}, \tag{2.14}$$

$$æ\frac{d\varphi}{dx}+\lambda_2\varphi(x)+\lambda_1\varphi(x)\left(f(x)-æ\frac{d\varphi}{dx}\right)=\lambda_3+f(x) \tag{2.15}$$

with boundary conditions

$$I(0,\eta)|_{\eta\in(0,1]}=I_0(\eta),\quad I(x,\eta)|_{\eta\in[-1,0)}=o(e^{\frac{x}{\eta}})\text{ for } x\to+\infty, \tag{2.16}$$

$$\varphi(0)=\alpha_0. \tag{2.17}$$

Here

$$\lambda_0=\frac{A_{21}}{n_0B_{12}};\quad \lambda_1=\frac{B_{12}+B_{21}}{n_0B_{12}};\quad \lambda_2=\frac{A_{21}+a_{12}+a_{21}}{n_0B_{12}};\quad \lambda_3=\frac{a_{12}}{B_{12}}. \tag{2.18}$$

It is easy to see that the unknown function $\varphi(x)$, representing the number of excited atoms on the optical depth x, satisfies the inequalities

$$0<\varphi(x)<\frac{1}{\lambda_1}<n_0. \tag{2.19}$$

The boundary value problem (2.14),(2.16) can be reduced by the standard argument to the following nonlinear integral equation on the source function $f(x)$:

$$f(x)=g(x)+\lambda_0\int_0^\infty E_1(x-t)\frac{\varphi(t)}{1-\lambda_1\varphi(t)}dt \tag{2.20}$$

where $\varphi(x)$ is determined by (2.15), and

$$g(x)=\frac{1}{2}\int_0^1 e^{-\frac{x}{\eta}}I_0(\eta)d\eta,\quad E_1(x)=\frac{1}{2}\int_0^1 e^{-\frac{|x|}{\eta}}\frac{d\eta}{\eta}. \tag{2.21}$$

Thus the initial problem is reduced to the solution of the nonlinear system (2.20) and (2.15) under condition (2.17).

It should be noted that the parameter æ characterizes the measure of deviation of the distribution of excited atoms from the Maxwell one. In particular, if the distribution of excited atoms $f_2(\vec{s})$ is Maxwell then

$$æ=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}s_1f_2(\vec{s})d^3s=0 \tag{2.22}$$

In the case æ $=0$ we have from (2.15):

$$\varphi(x)=\frac{\lambda_3+f(x)}{\lambda_2+\lambda_1f(x)}\,. \tag{2.23}$$

Substituting (2.23) in (2.20), we obtain the following well known linear integral equation (cf. [3],[6])

$$S(x)=S_0(x)+\lambda_4\int_0^\infty E_1(x-t)S(t)dt \tag{2.24}$$

where the free term is given by

$$S_0(x) = \frac{1}{2}\int_1^{\infty} e^{-xs}\left(I_0\left(\frac{1}{s}\right) - \frac{\lambda_4 a_{12}}{B_{12}}\right)\frac{ds}{s^2} + \frac{\lambda_4 a_{12}}{B_{12}}. \tag{2.25}$$

Here

$$\lambda_4 = \frac{A_{21}}{A_{21} + a_{21} - \dfrac{a_{12}B_{21}}{B_{12}}}\,.$$

Thus in the case æ $= 0$ the nonlinear boundary value problem (2.4)-(2.9) can be exactly linearized and reduced to the linear integral equation (2.24).

In the following sections we shall study the solvability of the nonlinear system (2.20), (2.15), (2.18) in the case when æ $\neq 0$.

3. Linear approximation. Exact solution.

When the medium interacts with a strong radiative field the local optical properties of the medium became dependent on this field. In spectral lines the strong radiative field leads to a large excitation of atoms thereby reducing the absorption capacity of the medium so that the stimulated emission becomes considerable. In the linear approximation the stimulated emission may be neglected because the number of excited atoms is much less than the number of atoms in the ground state:

$$n_2 \ll n_1 \approx n_0.$$

It implies that the velocity distribution of atoms in ground state is Maxwell.

Under these assumptions the equations (2.20), (2.15) and (2.17) are reduced to the equations

$$f(x) = g(x) + \lambda_0 \int_0^{\infty} E_1(x-t)\varphi(t)dt, \tag{3.1}$$

$$æ\frac{d\varphi}{dx} + \lambda_5\varphi(x) = \lambda_3 + f(x), \tag{3.2}$$

$$\varphi(0) = \alpha_0, \tag{3.3}$$

where

$$\lambda_5 = \frac{A_{21} + a_{21}}{n_0 B_{12}}\,.$$

The linear system (3.1), (3.2), (3.3) can be deduced directly from the nonlinear system (2.15) and (2.20) by assuming $\lambda_1 = 0$ and replacing λ_2 by λ_5.

It should be noted that in the linear approximation the optical depth x is related to the real depth z by the simple equation

$$x = n_0 B_{12} z. \tag{3.4}$$

Taking into consideration (3.3), we deduce from (3.2) the equation

$$\varphi(x) = \varphi_0(x) + \int_0^x e^{-(x-t)\frac{\lambda_5}{æ}} f(t)\frac{dt}{æ}, \tag{3.5}$$

where

$$\varphi_0(x) = \left(\alpha_0 - \frac{\lambda_3}{\lambda_5}\right)e^{-\frac{\lambda_5 x}{æ}} + \frac{\lambda_3}{\lambda_5}. \tag{3.6}$$

Substituting (3.5) in (3.1), we arrive at the following linear integral equation on the source function

$$f(x) = h(x) + \frac{\lambda_0}{æ}\int_0^\infty E_1(x-t)\left(\int_0^t e^{-\frac{\lambda_5}{æ}(t-y)} f(y)dy\right)dt \tag{3.7}$$

where

$$h(x) = g(x) + g_0(x); \quad g_0(x) = \lambda_0 \int_0^\infty E_1(x-t)\varphi_0(t)dt. \tag{3.8}$$

It is obvious that $h(x)$ is a bounded function. Making the change of variable and integration order in equation (3.7), we obtain the following Wiener-Hopf integral equation

$$f(x) = h(x) + \int_0^\infty W(x-t)f(t)dt \tag{3.9}$$

where

$$W(x) = \frac{\lambda_0}{æ}\int_0^\infty E_1(x-t)e^{-\frac{\lambda_5 t}{æ}}dt. \tag{3.10}$$

We note that

$$\|W\|_{L_1} = \int_{-\infty}^{+\infty} W(x)dx = \frac{\lambda_0}{\lambda_5}\|E_1\|_{L_1} = \lambda = \frac{A_{21}}{A_{21}+a_{21}} < 1$$

since $\|E\|_{L_1} = 1$.

The quantity λ is called the probability of survival of a photon under the elementary scattering act (cf. [6]).

It is known that the Wiener-Hopf integral equation in the dissipative case ($\lambda < 1$) has a unique positive solution in the space of bounded functions (cf. [7] and references therein).

Since the analytical and numerical solution of the Wiener-Hopf equation is the subject of numerous works, we shall skip a detailed discussion of this subject here.

Note that in the limiting case $æ \to 0$ we obtain from (3.10) that

$$W(x) = \lambda E_1(x).$$

Taking into account that in the linear approximation $\lambda_4 = \lambda$ ($B_{21} = 0$), it is easy to verify that in the limiting case (when $æ \to 0$) we shall have: $h(x) \to S_0(x)$ and the equation (3.9) reduces exactly to the equation (2.24).

4. On the solvability of the quasilinear system (2.20), (3.2).

This section is devoted to the solution of the nonlinear integral equation (2.20) where $\varphi(x)$ is given in terms of the unknown function $f(x)$ by the formula (3.5). We substitute (3.5) in (2.2) and consider the following iteration process:

$$f_{n+1}(x) = g(x) + \lambda_0 \int_0^\infty E_1(x-t)G\left(\int_0^t e^{-\frac{\lambda_5(t-y)}{æ}} f_n(y)\frac{dy}{æ} + \varphi_0(t)\right)dt, \tag{4.1}$$

starting from

$$f_0(x) = 0, \quad n = 0, 1, 2, \ldots. \tag{4.2}$$

Here

$$G(z) = \frac{z}{1 - \lambda_1 z}. \tag{4.3}$$

We denote

$$\sup_{x \geq 0} g(x) = M = \frac{1}{2} \int_0^1 I_0(\zeta) d\zeta \tag{4.4}$$

$$\sup_{x \geq 0} \varphi_0(x) \equiv \alpha_0. \tag{4.5}$$

Note that the function $G(z)$ has the following properties:

1. $G(z) \geq 0,\ z \in \left[0, \frac{\varepsilon}{\lambda_1}\right]$ for each $\varepsilon \in (0, 1)$;
2. $G(z)$ is monotonously increasing in z in the interval $[0, \frac{\varepsilon}{\lambda_1}]$;
3. $G(z) \leq \frac{z}{1-\varepsilon}$ for $z \in [0, \frac{\varepsilon}{\lambda_1}]$.

Together with equation (2.20) we consider the following auxiliary linear Wiener-Hopf equation:

$$F(x) = \tilde{g}(x) + \frac{\lambda_0}{1 - \varepsilon} \int_0^\infty T(x - t) F(y) dy, \quad x \geq 0 \tag{4.6}$$

on the unknown function $F(x)$. Here

$$\tilde{g}(x) = g(x) + \frac{\lambda_0}{1 - \varepsilon} \int_0^\infty E_1(x - t) \varphi_0(t) dt, \quad x \geq 0, \tag{4.7}$$

and

$$T(x) = \frac{1}{2æ} \int_0^\infty e^{-\frac{z\lambda_5}{æ}} E_1(x - z) dz, \quad x \in \mathbb{R}. \tag{4.8}$$

Direct computation shows that

$$\|T\|_{L_1} = \int_{-\infty}^{+\infty} T(x) dx = \frac{1}{\lambda_5}. \tag{4.9}$$

We can assume that

$$\frac{\lambda_0}{\lambda_5} < 1 - \varepsilon, \tag{4.10}$$

since

$$\frac{\lambda_0}{\lambda_5} = \frac{A_{21}}{A_{21} + a_{21}} = \lambda < 1.$$

Then the corresponding Wiener-Hopf integral operator will be contracting in all Banach spaces $L_p(\mathbb{R}^+)$ with $1 \leq p \leq \infty$. Therefore if the condition (4.10) is satisfied then the equation (4.6) has a unique integrable and bounded solution $F(x)$. Moreover

$$\tilde{g}(x) \le F(x) \le \frac{\sup\limits_{x\ge 0} \tilde{g}(x)}{1-\frac{\lambda_0}{\lambda_5(1-\varepsilon)}} . \tag{4.11}$$

Taking into account (4.4), we obtain from (4.7) that

$$0 \le \tilde{g}(x) \le M + \frac{\alpha_0\lambda_0}{1-\varepsilon}, \ x \ge 0. \tag{4.12}$$

It implies that

$$\tilde{g}(x) \le F(x) \le \frac{(M(1-\varepsilon)+\alpha_0\lambda_0)\lambda_5}{\lambda_5(1-\varepsilon)-\lambda_0}, \ x \ge 0. \tag{4.13}$$

Hereinafter we shall assume that the following condition is fulfilled: there exists a number $\varepsilon \in (0,1)$ such that

$$\max\left\{\alpha_0, \frac{\lambda_5\left(M(1-\varepsilon)+\alpha_0\lambda_0\right)}{\lambda_5(1-\varepsilon)-\lambda_0}\right\} \le \frac{\varepsilon}{2\lambda_1} . \tag{4.14}$$

We are going to prove that the sequence of functions $\{f_n(x)\}$ has the following properties:

$$f_n(x) \text{ is monotonously increasing in } n; \tag{4.15}$$

$$f_n(x) \le F(x) \text{ for } n = 0,1,2,\ldots, \ x \ge 0. \tag{4.16}$$

Indeed, from (4.1) we can deduce, using the monotonicity of function G on the interval $[0, \frac{\varepsilon}{\lambda_1}]$ and nonnegativity of functions g and E_1, that

$$f_1(x) = g(x) + \lambda_0 \int\limits_0^\infty E_1(x-t)G(\varphi_0(t))dt = f_0(x) \ge 0. \tag{4.17}$$

Note that by our assumptions we have

$$\varphi_0(t) \le \alpha_0 \le \frac{\varepsilon}{2\lambda_1} \le \frac{\varepsilon}{\lambda_1}, \ t \ge 0. \tag{4.18}$$

We also notice that

$$f_1(x) \le F(x), \ x \ge 0.$$

Indeed, since $\varphi_0(t) \le \frac{\varepsilon}{\lambda_1}$ we get from (4.6) and (4.7) that

$$f_1(x) \le g(x) + \frac{\lambda_0}{1-\varepsilon} \int\limits_0^\infty E_1(x-t)\varphi_0(t)dt = \tilde{g}(x) \le F(x).$$

We choose a natural n such that

$$f_n(x) \ge f_{n-1}(x) \text{ and } f_n(x) \le F(x) \text{ for } x \ge 0.$$

Then, due to (4.12),(4.1), we shall have

$$f_{n+1}(x) \ge g(x)+\lambda_0 \int\limits_0^\infty E_1(x-t)G\left(\int_0^t e^{-\frac{\lambda_5(t-y)}{æ}} f_{n-1}(y)\frac{dy}{æ} + \varphi_0(t)\right)dt = f_n(x) \text{ for } x \ge 0,$$

and

$$f_{n+1}(x) \leq g(x) + \lambda_0 \int_0^\infty E_1(x-t) G\left(\int_0^t e^{-\frac{\lambda_5(t-y)}{æ}} F(y)\frac{dy}{æ} + \varphi_0(t)\right) dt$$

$$\leq \tilde{g}(x) + \frac{\lambda_0}{1-\varepsilon}\int_0^\infty T(x-t)F(y)dy = F(x).$$

Hence, there exists a limit $\lim_{n\to\infty} f_n(x) = f(x) \leq F(x)$. Since $F \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$ we have $f \in L_1(\mathbb{R}^+) \cap L_\infty(\mathbb{R}^+)$. Thus we have proved the following

Theorem 1. *Under the conditions* (4.10), (4.14) *the nonlinear equation* (2.20), (3.2) *has a positive, bounded and integrable solution $f(x)$ which satisfies the estimate*

$$\tilde{g}(x) \leq f(x) \leq F(x), \ x \in \mathbb{R}^+.$$

Remark 1. *The condition* (4.14) *is satisfied if at least one of the following conditions holds:*

a) *if* $\lambda \in [1-\varepsilon-\lambda_0; 1-\varepsilon)$ *then* $M < \frac{\varepsilon}{2\lambda_1} = \frac{\varepsilon n_0 B_{12}}{2(B_{12}+B_{21})}$;

b) *if* $\lambda \in (0, 1-\varepsilon-\lambda_0)$ *then*

$$\begin{cases} M \leq \alpha_0\left(1 - \frac{\lambda+\lambda_0}{1-\varepsilon}\right), \\ \alpha_0 < \frac{\varepsilon}{2\lambda_1}, \end{cases} \quad or \quad \begin{cases} M > \alpha_0\left(1 - \frac{\lambda+\lambda_0}{1-\varepsilon}\right), \\ M < \frac{\varepsilon}{2\lambda_1}. \end{cases}$$

Remark 2. *The obtained results can be extended to the problem of resonance scattering in spectral line in the case of general frequency redistribution laws. We plan to devote a separate paper on this problem.*

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