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A NUMERICAL SOLUTION OF THE MEMBRANE EIGENPROBLEM BY THE MODEL ORDER REDUCTION

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Abstract. In this paper the Model Order Reduction technique to solve the problem of free oscillations of a heterogeneous rectangular elastic membrane is applied. Instead of solving 2D problem for the membrane in the exact formulation, we substitute it by a special network of 1D elastic strings. We present the characteristic equations for the spectrum of free oscillations of this network and develop the numerilal algorithm to solve the problem. We investigate the behavior of eigenvalues of a rectangular network and show that the eigenvalues and eigenvectors of rectangular networks of elastic strings and the rectangular membrane are close. The problem solution for the network of elastic strings has significantly less computational cost compared with the solution of free oscillations of a heterogeneous rectangular elastic membrane.

Keywords: networks of elastic strings, eigenvalue, eigenvector, model order reduction, finite-difference method.

1. Introduction

The eigenvalue problem is a problem of considerable theoretical interest and wide-ranging application**.** For example, this problem is crucial in solving systems of differential equations, biological molecular systems, optical switching technologies, social network analysis, etc. The main efforts here are focused on numerical solution of the eigenproblem, because it is impossible to theoretically predict the behavior of a spectrum. But it is worth mentioning that the computer costs for solving the eigenproblem are sufficiently high, in particular, for a matrix of order $\boldsymbol{n}$ it is equal

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to $\boldsymbol{O(n^3)}$. Hence, it is extremely time consuming for large-scale applications. In this paper we propose the new approach based on Model Order Reduction (MOR) [1] to reduce the computational costs needed for solving a membrane eigenvalue problem to an acceptable level. The MOR is a technique designed to reduce the computational complexity of mathematical models in numerical simulations. This technique may be a reduction of the model dimension or degrees of freedom, or an approximation to the original model with a lower accuracy but in significantly less computational time. In this paper we apply the MOR technique to efficiently solve the problem of free oscillations of the heterogeneous rectangular elastic membrane. Instead of solving the problem in exact formulation, we substitute it by a special one-dimensional network of elastic strings [2], [3]. This network consists of a finite number of strings attached to each other as is shown in Figure 1.

The numerical simulation of this new one-dimensional formulation of the problem needs significantly less computational costs, but corresponding eigenvalues and eigenvectors under some assumptions about the distribution of mass, the tension of each string and the elasticity coefficients of the springs are close to the eigenvalues of a rectangular membrane. Moreover, the corresponding modes (eigenvectors) of oscillations of the network and the membrane are also close to each other when the size of the square cells enclosed by the strings of the network is small enough.

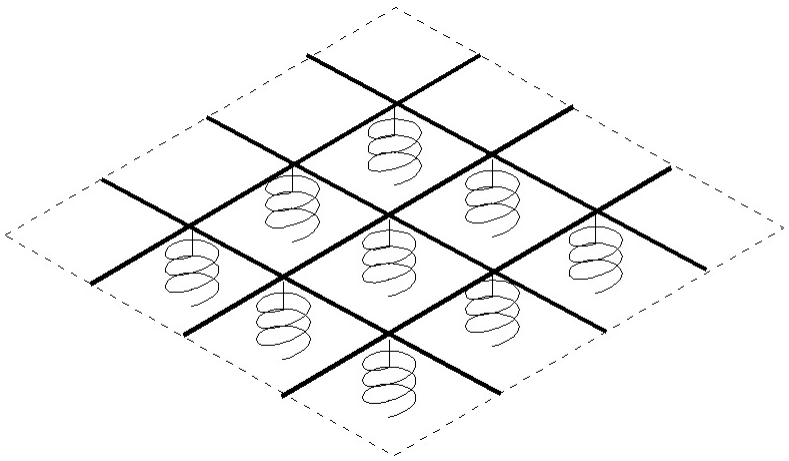


FIG. 1. A network of strings.

2. Mathematical model

Let us consider the network of a finite number of strings, attached to each other in the form of square cells. Each internal node is assumed to be attached to the spring. Next, suppose each node of the four adjacent strings can move only in the direction orthogonal to the coordinate plane O_{xy}. This plane gives a geometric representation of the network and contains an initial configuration of endpoints. The network, when moving, experiences the resistance of the springs. The nodes of the strings belonging to the endpoints are assumed to be fixed (Figure 1).

We introduce some notations for further considerations. The mass distribution along each string is given by the function:

$$\rho = \frac{h}{2}, \tag{1}$$

where h is the length of the string. If we assume the whole network stretched over the unit square, then the length h of an elementary cell equals *1/n*, where n is the number of equal strings in a horizontal or a vertical chain of strings joined together in one direction (between the dotted lines in Figure 1).

The network is stretched in such a way that the tension of each string equals:

$$T_h = h. \tag{2}$$

The elasticity coefficient k_h of each string equals:

$$k_h = h^2. \tag{3}$$

In order to reformulate the problem in mathematical terms, we interpret a network of strings as a graph G_h with the vertical and horizontal edges e (the line of joint strings) adjacent to each other in the vertices $v = (ih, jh),\ (0 \le i, j \le n)$. Each edge may be parameterized by the parameter $t \in [0; h]$. Here we use a natural parameter that is the arc length. Let the horizontal edges be directed from left to right, while the vertical edges - from bottom to top. We assume that the parameterization of each edge is agreed with such an orientation.

Under these re-assignments, the problem of free oscillations of the network can be easily reduced to the following analog of the Sturm - Liouville boundary value problem [4], [5]:

$$u_e'' + \frac{\lambda}{2} u_e = 0,\ \ e \in E, \tag{4}$$

$$\sum_{e \succ v} u_e'(v) - h u(v) = 0,\ \ v \in V_0, \tag{5}$$

$$u_e(v) = u(v),\ \ for\ e \succ v,\ \ v \in V_0, \tag{6}$$

$$u(v) = 0, \quad v \in \partial G_h. \tag{7}$$

Here u_e means the restriction of the function $u : G_h \to R$ on the edge e. In equation (4), the differentiation relates to the natural parameter. In this case, we need to clarify that the orientation of an edge does not assume sign of $u_e'(v)$ in (5). This expression defines the derivative in the internal direction of the edge e, i.e. from the vertex v into the interior of the edge e. And, finally, the notation V_0 indicates to the set of internal nodes.

Under the above notations and assumptions, we can consider eigenvalue problem (4)-(7) of the mechanical one-dimensional network of elastic strings as the MOR for the eigenvalue problem of rectangular membrane, where natural oscillations are described by the following boundary value problem:

$$\Delta u - u + \lambda u = 0, \tag{8}$$

$$u\,|_{\partial Q} = 0, \tag{9}$$

where $Q = [0,\ 1] \times [0,\ 1]$.

In fact, one can easily see from our assumption (1) that the total mass of the network of strings approximately equals 1 for sufficiently small h, while the mass of the membrane, described in (8) and (9), exactly equals 1. One can also see that assumption (2) related to the tensions of strings makes the network similar to the elastic membrane in the sense of similarity between elasticities of these two

mechanical systems. As a matter of fact, the domain "covered" with a string grid may be tasseled into squares with the side lengths h, centered at the nodes of the graph G_h. Each side of each square is intersected by the unique edge of the graph. If we distribute the tension of a corresponding string, which is orthogonal to the side of square and equals h along the side we will come to a set of squares stretched as the cells of membrane (8),(9). An additional argument follows from the mechanical assumption (3) about the elastic resistance of the springs to be applied to the network at its vertices. In fact, this assumption is equivalent to the exterior elastic resistance, applied to the membrane (the term u in the equation (8)).

All these heuristic arguments allow us to expect the following theorem be true.

Theorem 1. *$\Lambda^h \to \Lambda$ as $h \to 0$. Here $\Lambda^h = \{\lambda_1^h, \lambda_2^h, \lambda_3^h \ldots, \lambda_n^h, \ldots\}$ is the spectrum of problem (4)-(7) (one can prove that it produces a sequence of positive real numbers $\lambda_n^h \to +\infty$ $(n \to \infty)$, see for example [4]), Λ is a spectrum of the problem (8), (9). As for the notation $\Lambda^h \to \Lambda$ $(h \to 0)$, it means that for each $\varepsilon > 0$ and $N \in \mathrm{N}$ there exists a positive $\delta > 0$, such that the inequality $0 < h < \delta$ implies $\left|\lambda_i^h - \lambda_i\right| < \varepsilon$ for all $i = 1, 2, \ldots, N$.*

The proof of this theorem was presented in [6].

3. The Finite-difference approximation

In order to numerically resolve problem (4)-(7), we apply the finite difference method that is one of the simplest and, at the same time, one of the useful methods to solve differential equations in different application areas (see, for example [7],[8]).The approach to be applied consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients.

Let us agree that there are k vertical and k horizontal lines of the network so that each line contains *k+1* chained strings. In these notations problem (4)-(7) can be rewritten as follows:

$$(10)\quad \begin{cases} u''_{i,j}(x) + \frac{\lambda}{2} u_{i,j}(x) = 0, \;\; x \in G_h, \;\; i = \overline{1, k+1}, j = \overline{1, k}, \textit{horizontal edges}, \\ \\ w''_{i,j}(y) + \frac{\lambda}{2} w_{i,j}(y) = 0, \;\; y \in G_h, \;\; i = \overline{1, k}, j = \overline{1, k+1}, \textit{vertical edges}, \end{cases}$$

$$(11)\quad u'_{i,j}(v) + u'_{i+1,j}(v) + w'_{i,j}(v) + w'_{i,j+1}(v) - hu(v) = 0, \;\; v \in V_0, \;\; i, j = \overline{1, k},$$

$$(12)\quad u_{i,j}(v) = u_{i+1,j}(v) = w_{i,j}(v) = w_{i,j+1}(v), \;\; v \in V_0, \;\; i, j = \overline{1, k},$$

$$(13)\quad u_{1,i}(v) = 0, \;\; u_{k+1,i}(v) = 0, \;\; w_{i,1}(v) = 0, \;\; w_{i,k+1}(v) = 0, \;\; v \in \partial G_h, \;\; i = \overline{1, k},$$

where $u_{i,j}(x)$,$w_{i,j}(y)$ are the functions defined on the horizontal and vertical edges, respectively.

In order to construct a numerical scheme, we divide each string, belonging to one of horizontal/vertical edges into n parts in order to obtain the finite difference approximation of governing equations (10) and renumber all these elements to a finite one-dimensional array (Figure 2).

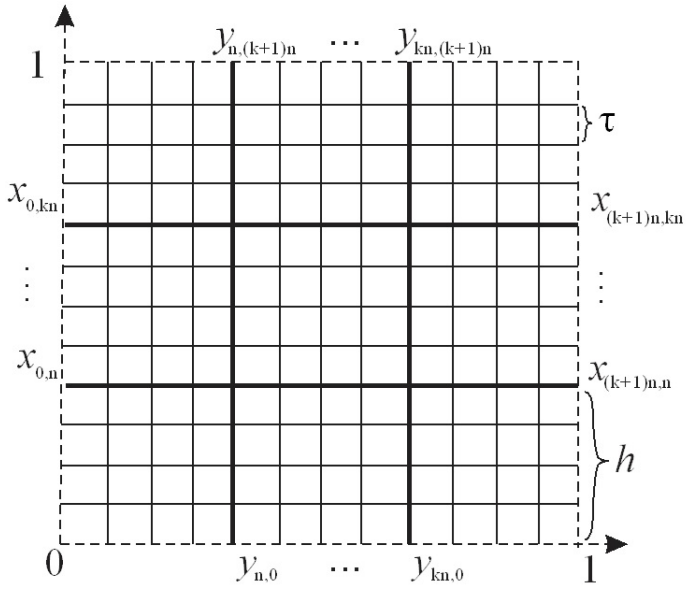


FIG. 2. The new grid $\tilde{G}_h$.

All new elements of $\tilde{G}_h$ satisfy the following conditions:

$$\tilde{G}_h = \left\{ \begin{array}{l} x_{i-1,jn} = ((i-1)\tau,\ jn\tau)\,,\ y_{jn,i-1} = (jn\tau,\ (i-1)\tau)\,, \\ \\ i = \overline{1, n(k+1)+1},\ j = \overline{1,k} \end{array} \right\},$$

(a) on the horizontal edges:

$$x_{0,n} = (0,\ n\tau),\ x_{1,n} = (\tau,\ n\tau),\ ...,\ x_{(k+1)n,n} = ((k+1)n\tau,\ n\tau)\,,$$
$$...$$
$$x_{0,kn} = (0,\ kn\tau),\ x_{1,n} = (\tau, k\,n\tau),\ ...,\ x_{(k+1)n,n} = ((k+1)n\tau,\ \ k\,n\tau)\,,$$

(b) on the vertical edges:

$$y_{n,0} = (\,n\tau,\ 0),\ y_{n,1} = (\,n\tau,\ \ \tau),\ ...\,,\ y_{n,(k+1)n} = (n\tau,\ (k+1)n\tau\ )\,,$$
$$...$$
$$y_{kn,0} = (kn\tau,\ 0),\ y_{kn,1} = (k\,n\tau,\ \tau),\ ...,\ y_{kn,(k+1)n} = (k\,n\tau,\ (k+1)n\tau,)$$

where $\tau = \frac{1}{(k+1)n}$ denotes the step of finite difference approximation and is equal to the step of the new grid. The points $x_{i,j}$, $y_{i,j}$ are nodes of the grid$\tilde{G}_h$. The functions $u_{i.j}(x)$, $w_{i.j}(y)$ are now defined for the points $x_{i,j}$, $y_{i,j}$, $\in \tilde{G}_h$ and further will be marked by $\tilde{u}_{i,j}$, $\tilde{w}_{i,j}$.

To construct a finite difference scheme, we use only interior nodes of the grid$\tilde{G}_h$ excluding all interior vertices from our explicit consideration (Figure 3). The reason is that condition (11) at the interior vertices $v_{i,j} = x_{in,jn}\ \big(i,j = \overline{1,k}\big)$ breaks the symmetry of the resulting matrix obtained after the straightforward finite difference discretization (10)-(13). We can easily overcome this difficulty by defining the values at all interior vertices $v_{i,j} = x_{in,jn}\ (i,j = \overline{1,k})$ of the grid $\tilde{G}_h$through the four nearest neighboring interior nodes $x_{in-1,jn}$, $x_{in+1,jn}$, $x_{in,jn-1}$, $x_{in,jn+1}$ by the formula:

$$u_{in,jn} = \frac{w_{in,jn-1} + w_{in,jn+1} + u_{in-1,jn} + u_{in+1,jn}}{4\tau h}, \quad i,j = \overline{1,k}. \tag{14}$$

Thus, the nodes of the network involved into the finite difference scheme approximating partial differential equations (10)-(13) can be graphically represented as follows:

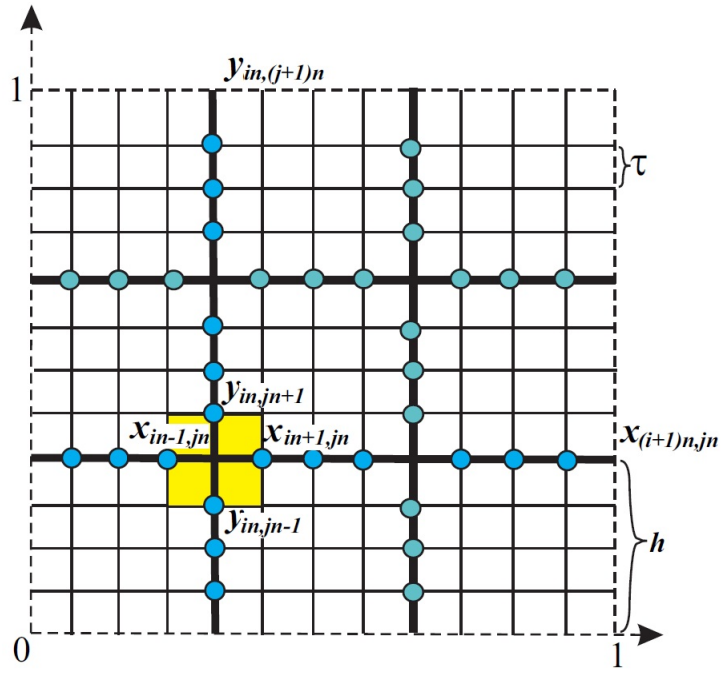


FIG. 3. The selected nodes of the network involved into the finite difference scheme.

We approximate the derivatives of system (10) using a second order central difference scheme with the second order accuracy and take into account conditions (11)-(13). As a result, we formulate the finite difference approximation of the system under study. At the first and at the last nodes of each edge e the finite difference scheme looks like:

$$2\frac{u_{2,jn} - 2u_{1,jn}}{\tau^2} + \lambda u_{1,jn} = 0, \quad 2\frac{u_{(k+1)n-2,jn} - 2u_{(k+1)n-1,jn}}{\tau^2} + \lambda u_{(k+1)n-1,jn} = 0,$$

$$2\frac{w_{jn,2} - 2w_{jn,1}}{\tau^2} + \lambda u_{jn,1} = 0, \quad 2\frac{w_{jn,(k+1)n-2} - 2w_{jn,(k+1)n-1}}{\tau^2} + \lambda u_{jn,(k+1)n-1} = 0,$$

where $j = \overline{1,k}$.
The finite difference approximation of (12),(13) looks like:

$$u_{in,jn} = w_{in,jn}, \quad i = \overline{1,k}\,, \quad j = \overline{1,k},$$

$$u_{0,in} = 0, \quad u_{(k+1)n,in} = 0, \quad w_{in,0} = 0, \quad w_{in,(k+1)n} = 0, \qquad i = \overline{1,k}.$$

The finite difference approximation of (10) for the interior nodes of each edge is as follows:

$$2\frac{u_{i+1,jn}-2u_{i,jn}+u_{i-1,jn}}{\tau^2}+\lambda u_{i,jn}=0,\ \ 2\frac{w_{jn,i+1}-2w_{jn,i}+w_{jn,i-1}}{\tau^2}+\lambda w_{jn,i}=0,$$

$$j=\overline{1,k},\qquad i=\overline{(\hat{i}-1)n+2,\hat{i}n-2},\qquad \hat{i}=\overline{1,k}.$$

Finally, following [9] and using condition (14), we write down the finite difference approximation of (10) for the nodes connecting with the interior vertices $x_{in,jn}$:

$$2\frac{u_{in-2,jn}-2u_{in-1,jn}}{\tau^2}+2\frac{u_{in-1,jn}+u_{in+1,jn}+w_{in,jn+1}+w_{in,jn-1}}{(4+\tau h)\tau^2}+\lambda u_{in-1,jn}=0,$$

$$2\frac{w_{in,jn-2}-2w_{in,jn-1}}{\tau^2}+2\frac{w_{in,jn-1}+w_{in,jn+1}+u_{in+1,jn}+u_{in-1,jn}}{(4+\tau h)\tau^2}+\lambda u_{in-1,jn}=0,$$

where $i,j=\overline{1,k}$.

As a result we come to the linear eigenvalue problem for a square symmetric matrix of order $p=2k(k+1)n\approx 2k^2n$. As was mentioned above, the computational costs of this eigenproblem is equal to $O(p^3)$. For comparison, if we solve an eigenvalue problem of a rectangular membrane with the same accuracy, we will solve it with computational costs equal to $\mathrm{O}(\mathrm{q}^3)$, where $q=(k+1)^2n^2\approx(kn)^2$. Thus, it is easy to estimate that using this approach we reduce the number of computations more than $(n/2)^3$ times.

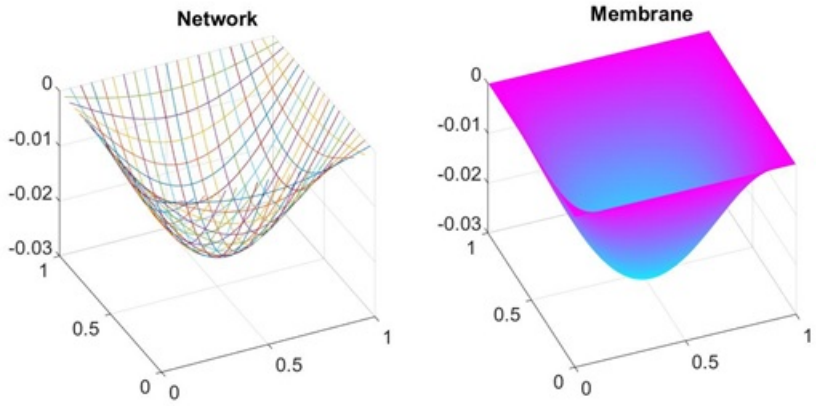
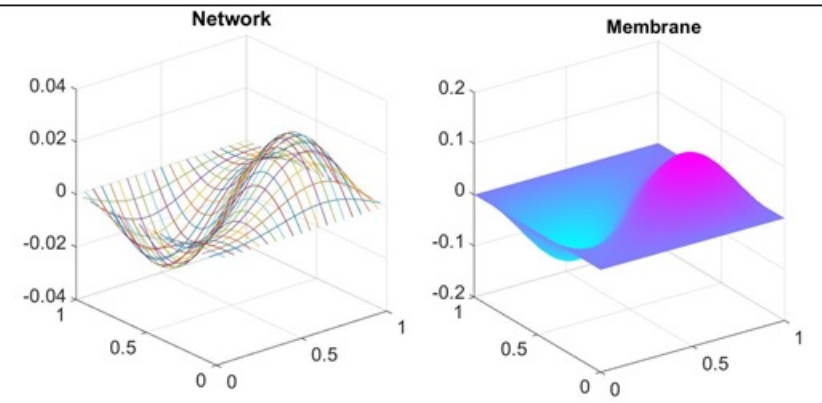
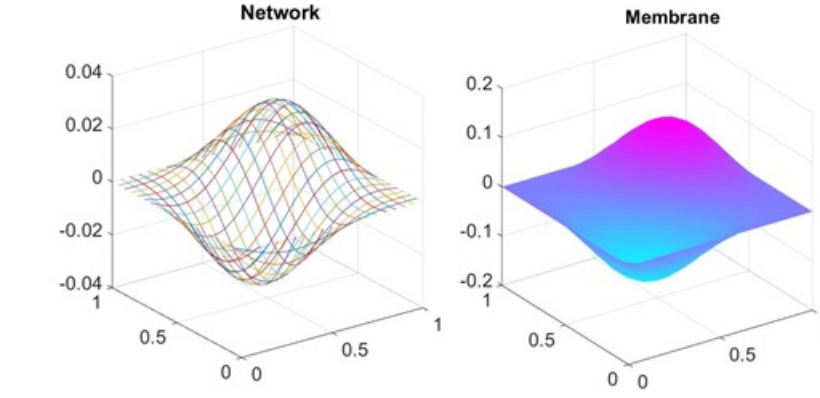
4. Numerical tests

In this Section we present some results of the numerical simulation of problems (4)-(7) and (8)-(9). The simulation of the two problems was implemented on $[0,\,1]\times[0,\,1]$ square for different input parameters and for different discretization. To compute the eigenvalues and the eigenvectors of the membrane we use the free package Pde Toolbox from Matlab [10].

First we carry out simulations to investigate eigenvalues and eigenvectors of the network of string with the number of vertical (horizontal) lines equal to $k=10$ and $n=20$. Thus, the approximation step τ of the grid $\tilde{G}_h$ is $\tau=0.0045$ and the length of each one $h=0.09$. In the second test we calculate eigenvalues of the network, where $k=20, n=20, h=0.05, \tau=0.0024$.

Table 1 presents the eigenvectors of this network of strings ($k=10, n=20$) in comparison with the membrane. As can be seen, even for rather a rough discretization of (4)-(7), the eigenvectors of the network of strings and of the membrane are similar.

Figures 4 and 5 present the graphic comparison and relative error of the eigenvalues of the network and the membrane for different numbers of vertical (horizontal) lines. These numerical results have revealed that this parameter substantially affects the accuracy of approximation $\lambda_i^h\approx\ \lambda_i$.

The index of Eigen-functions	Eigenfunctions
f_1	
f_2	
f_3	

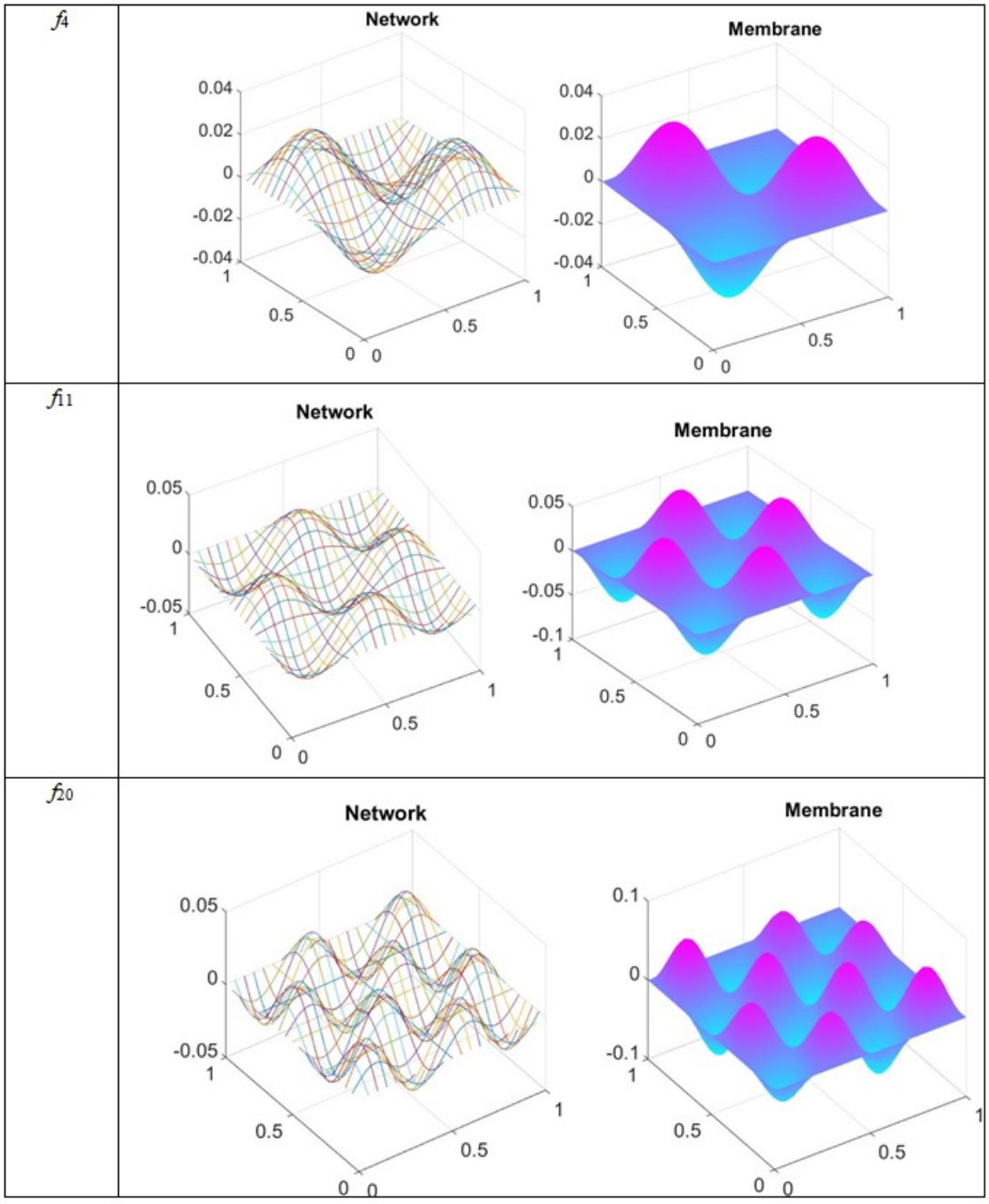
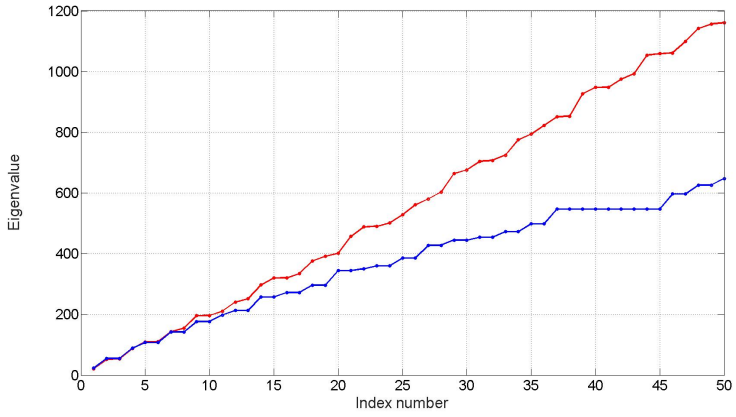
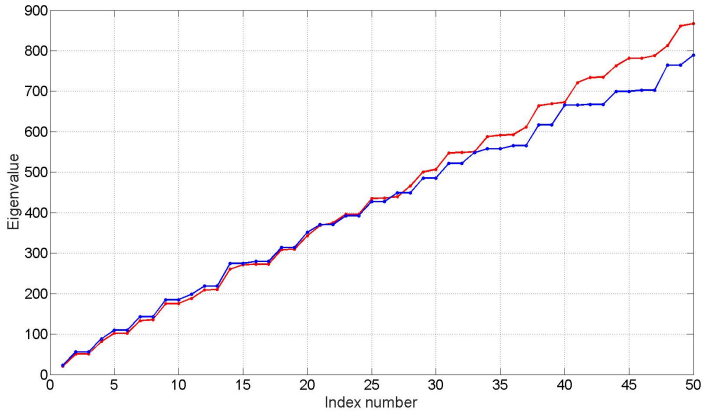


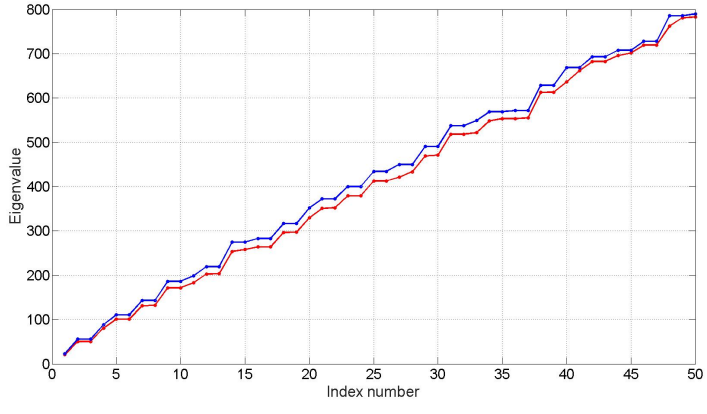
Table 1. *The eigenfunctions of network and membrane.*



a)



b)



c)

FIG. 4. Eigenvalues of the network (**blue**) and the membrane (**red**): a) the number of vertical (horizontal) lines $k = 9$, the length of the string $h = 0.1$ and the number of equal strings in a horizontal or a vertical chain of strings joined together in one direction $n = 10$; b) $k = 19, h = 0.05, n = 10$; c) $k = 29, h = 0.033, n = 10$.

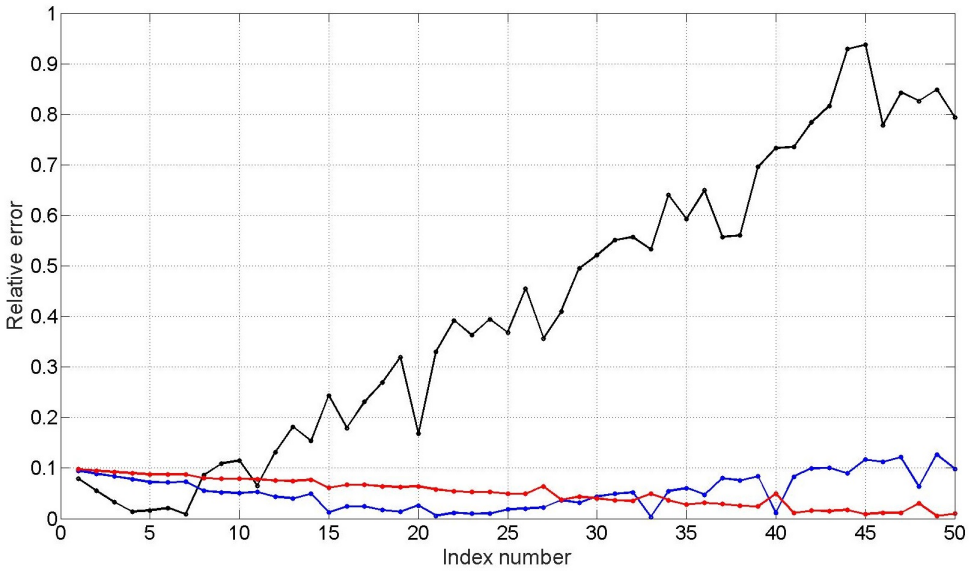


FIG. 5. Relative error of the eigenvalues of the network and the membrane for different numbers of vertical (horizontal) lines: a) **black** - the number of vertical (horizontal) lines $k = 9$, the length of the string $h = 0.1$ and the number of equal strings in a horizontal or a vertical chain of strings joined together in one direction $n = 10$; b) **blue** - $k = 19, h = 0.05, n = 10$; c) **red** - $k = 29, h = 0.033, n = 10$.

5. Conclusion

In this paper, we implement the Model Order Reduction to solve the problem of free oscillations of the rectangular elastic membrane. Instead of solving the 2D problem for the membrane, we substitute it by a special network of 1D elastic strings. By the numerical simulation of this new reduced problem we prove that the eigenvalues and eigenvectors of the network of strings under some reasonable assumptions are close to the eigenvalues and eigenvectors of the rectangular membrane. The advantage of the approach proposed is that we solve the original problem with a desired accuracy and with less computational costs.

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