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ASYMPTOTICS OF GROWTH FOR NON-MONOTONE COMPLEXITY OF MULTI-VALUED LOGIC FUNCTION SYSTEMS

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Abstract. The problem of the complexity of multi-valued logic functions realization by circuits in a special basis is investigated. This kind of basis consists of elements of two types. The first type of elements are monotone functions with zero weight. The second type of elements are non-monotone elements with unit weight. The non-empty set of elements of this type is finite.

In the paper the minimum number of non-monotone elements for an arbitrary multi-valued logic function system F is established. It equals $\lceil \log_u(d(F)+1)\rceil - O(1)$. Here $d(F)$ is the maximum number of the value decrease over all increasing chains of tuples of variable values for at least one function from system F; u is the maximum (over all non-monotone basis functions and all increasing chains of tuples of variable values) length of subsequence such that the values of the function decrease over these subsequences.

Keywords: combinational machine (logic circuits), circuits complexity, bases with zero weight elements, k-valued logic functions, inversion complexity, Markov's theorem, Shannon function.

In this paper the problem of the non-monotone complexity of a k-valued logic function system realization is studied. Non-monotone complexity means that the considered basis contains all monotone functions (they have zero weight) and a finite number of non-monotone functions (they have positive weight). Markov [1, 2] obtained the exact answer for the problem for Boolean functions with the only non-monotone function negation. In [3, 4] the exact values of the non-monotone complexity of multi-valued logic function systems for two bases is obtained. These

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bases contain one non-monotone function (Post negation or Łukasiewicz negation) and all monotone functions.

In this paper the upper and lower bounds for the non-monotone complexity of multi-valued function systems in the general case are obtained. The upper and lower bounds are asymptotically the same.

Denote by E_k the set $\{0, 1, \dots, k-1\}$. A sequence

$$\tilde{\alpha}_1 = (\alpha_{11}, \dots, \alpha_{1n}),\ \tilde{\alpha}_2 = (\alpha_{21}, \dots, \alpha_{2n}),\ \dots,\ \tilde{\alpha}_r = (\alpha_{r1}, \dots, \alpha_{rn})$$

of the pairwise different tuples from E_k^n is called *increasing chain with respect to the order* $0 < 1 < \dots < k-1$ or *chain* if

$$\alpha_{ij} \le \alpha_{i+1,j}, \quad i = 1, \dots, r-1, \quad j = 1, \dots, n.$$

The tuples $\tilde{\alpha}_1$ and $\tilde{\alpha}_r$ are called *the initial tuple* and *the terminal tuple* of the chain respectively.

Let $f(x_1, \dots, x_n)$ be a k-valued logic function. An ordered pair of tuples $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \dots, \beta_n)$, $\tilde{\alpha}$, $\tilde{\beta} \in E_k^n$, is called *a jump* for the function f if

1) $\alpha_j \le \beta_j,\ j = 1, \dots, n$;

2) $f(\tilde{\alpha}) > f(\tilde{\beta})$.

A jump for a system of functions is a pair of tuples which is a jump for any function of the system.

Let $F = \{f_1, \dots, f_m\}$, $m \ge 1$, be a system of k-valued logic functions with arguments $x_1, \dots x_n$. Let C be a chain of the form

$$\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_r.$$

Decrease $d_C(F)$ of the system F over chain C is the number of jumps of the form $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$ for the system F.

Decrease $d(F)$ of the system F is the maximum $d_C(F)$ over all chains C.

Let P_k be the set of all k-valued logic functions, M be the class of all monotone functions from P_k with respect to the order $0 < 1 < \dots < k-1$.

The aim of this paper is to study the non-monotone complexity of multi-value logic functions. That is, k-valued logic function realization by circuits of funcitonal elements (logic circuits, combinational machine — see e.g. [5, 6]) over bases B of the form

$$B = M \cup \{\omega_1, \dots, \omega_p\}, \quad \omega_i \in P_k \setminus M,\ i = 1, \dots, p \ \ (p \ge 1). \qquad (*)$$

Here any function from M has zero weight, functions $\omega_1, \dots, \omega_p$ have positive weights $\rho_1, \dots, \rho_p$ respectively. Further, in case the oposite is not mentioned, consider $\rho_1 = \dots = \rho_p = 1$ to simplify the reasoning and to emphasize the essential idea.

Non-monotone complexity $I_B(S)$ of the circuit S over basis B is the sum of the weights of all non-monotone elements of the circuit S. Non-monotone functional elements correspond to non-monotone basis functions.

Non-monotone complexity of the multi-valued logic function system F over basis B is the minimum non-monotone complexity of the circuits that realize the function system F over basis B. Denote non-monotone complexity of the system F over basis B by $I_B(F)$.

The exact value of the non-monotone complexity for Boolean functions over bases $B_0 = M \cup \{\overline{x}\}$ (i.e. inversion complexity) has been obtained by Markov [1, 2]. For any Boolean function system F Markov proved that

$$I_{B_0}(F) = \lceil \log_2(d(F) + 1) \rceil .$$

Let us note that [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] also concern problem of Boolean functions computation by circuits that contain the minimum number of negations.

In papers [3, 4] the classical Markov result has been extended to the case of k-valued logic function systems, $k \geq 2$. It is shown that the minimum number of negations for arbitrary k-valued logic system F computation equals $\lceil \log_2(d(F)+1) \rceil$ in the case of Post negation ($x+1 \pmod{k}$) and equals $\lceil \log_k(d(F)+1) \rceil$ in the case of Łukasiewicz negation ($k-1-x$). If $k=2$ these formulas give the same result that coincides with Markov's theorem.

In addition let us note that papers [18, 19] deal with the problem of non-monotone complexity for systems of Boolean functions over an arbitrary basis of the form $(*)$. It is proved that for any system F there exists constant $c(B)$ such that

$$\lceil \log_2(d(F)+1) \rceil - c(B) \leq I_B(F) \leq \lceil \log_2(d(F)+1) \rceil.$$

Moreover, for any N there exists a Boolean basis B_N and a Boolean function g_N such that

$$\lceil \log_2(d(g_N)+1) \rceil - I_{B_N}(g_N) > N.$$

This paper proves a similar result for a multi-valued logic function system. We need to define extra notions for the exact statement of the main result.

Let f be a k-valued logic function, $C = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_r)$ be an arbitraty chain from E_k^n. By $u_C(f)$ denote the maximum length t of the subsequence $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_t$ of the sequence C such that $f(\tilde{\beta}_1) > f(\tilde{\beta}_2) > \ldots > f(\tilde{\beta}_t)$. Note that $u_C(f) = 1$ if f does not decrease over chain C.

Now we define *the inversion force $u(f)$ of the function f*. By definition put

$$u(f) = \max_{C \subset E_k^n} u_C(f).$$

Obviously, for any function f we get $1 \leq u(f) \leq d(f)+1$. Moreover, f is non-monotone iff $u(f) \geq 2$.

Finally, we define *the inversion force $u(B)$ of the basis B*. By definition put

$$u(B) = \max_{f \in B} u(f).$$

Theorem 1. *Suppose B is the complete basis of the form $(*)$. Then there exists the constant $c(B)$ such that for any finite k-valued logic function system F the inequalities*

$$\left\lceil \log_{u(B)}(d(F)+1) \right\rceil - c(B) \leq I_B(F) \leq \left\lceil \log_{u(B)}(d(F)+1) \right\rceil$$

hold.

The upper bound estimation follows directly from Theorem 2 from [4].

The lower bound estimation is based on the following four lemmas.

Lemma 1. *Let $A = (a_1, a_2, \ldots, a_r)$ be a sequence of elements from $\{0, 1 \ldots, k-1\}$. Suppose there exists $l \in \mathbb{N}$ such that for any subsequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_t})$ of A such that $a_{i_1} > a_{i_2} > \ldots > a_{i_t}$, the inequality $t \leq l$ holds. Then A can be partitioned into l disjoint non-decreasing subsequences.*

Proof. The proof is by induction over l.

If $l = 1$ then A does not decrease. Hence, sequence A itself is the partition into l subsequences.

Assume Lemma statement is valid for all l, $l \le l' - 1$. Let us prove it for $l = l'$.

If sequence A does not contain any subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_{l'}}$ with exact l' components such that $a_{i_1} > a_{i_2} > \ldots > a_{i_{l'}}$ then using the inductive assumption we obtain that Lemma statement for l' holds. Let A contain s subsequenses of the length l', $s > 0$. They are

$$a_{i(1,1)}, a_{i(2,1)}, \ldots, a_{i(l',1)};$$

$$a_{i(1,2)}, a_{i(2,2)}, \ldots, a_{i(l',2)};$$

$$\ldots$$

$$a_{i(1,s)}, a_{i(2,s)}, \ldots, a_{i(l',s)}.$$

Note that inequality $i(1, j_1) < i(1, j_2)$ implies the relation $a_{i(1,j_1)} \le a_{i(1,j_2)}$. Assume the converse, i.e. $a_{i(1,j_1)} > a_{i(1,j_2)}$. Then inequalities

$$a_{i(1,j_1)} > a_{i(1,j_2)} > a_{i(2,j_2)} > \ldots > a_{i(l',j_2)}$$

hold for the subsequence

$$a_{i(1,j_1)}, a_{i(1,j_2)}, a_{i(2,j_2)}, \ldots, a_{i(l',j_2)}$$

of the length $l' + 1$. That contradicts Lemma statement.

The sequence $\{a_{i(1,1)}, a_{i(1,2)}, \ldots, a_{i(1,s)}\}$ is non-decreasing subsequence of A. Consider sequence

$$A' = A \backslash \{a_{i(1,1)}, a_{i(1,2)}, \ldots, a_{i(1,s)}\}.$$

Then for any subsequence $a_{i(1)}, a_{i(2)}, \ldots, a_{i(t)}$ of the sequence A' such that $a_{i(1)} > a_{i(2)} > \ldots > a_{i(t)}$ inequality $t \le l' - 1$ holds. Using the inductive assumption we obtain that A' can be partitioned into $l - 1$ disjoint non-decreasing subsequences. Hence, sequence A can be partitioned into l disjoint non-decreasing subsequences. This complete the proof of Lemma 1. $\square$

Lemma 2. *Let*

$$h(x_1, \ldots, x_n) = \omega(m_1(x_1, \ldots, x_n), \ldots, m_s(x_1, \ldots, x_n)),$$

where $m_1, \ldots, m_s$ are monotone functions. Then for any chain of tuples $\tilde\alpha_1, \tilde\alpha_2, \ldots, \tilde\alpha_r$ from E_k^n sequence $h(\tilde\alpha_1), h(\tilde\alpha_2), \ldots, h(\tilde\alpha_r)$ changes values at most $k(d(\omega) + 1) - 1$ times.

Proof. Indeed, sequence $h(\tilde\alpha_1), h(\tilde\alpha_2), \ldots, h(\tilde\alpha_r)$ contains at most $d(\omega) + 1$ non-decreasing ranges. Each of these ranges contains at most $k - 1$ value changes. The sequence value decreases $d(\omega)$ times. Hence, sequence $h(\tilde\alpha_1), h(\tilde\alpha_2), \ldots, h(\tilde\alpha_r)$ changes values at most $(k - 1)(d(\omega) + 1) + d(\omega) = k(d(\omega) + 1) - 1$ times. $\square$

Let $B = M \cup \{\omega_1, \ldots, \omega_p\}$, where $\omega_i \in P_2 \setminus M$ $(i = 1, \ldots, p)$. Let us extend the notion of the decrease to an arbitrary basis of the form $(*)$. By definition put $d(B) = \max\{d(\omega_1), \ldots, d(\omega_p)\}$.

Lemma 3. *Let F be a k-valued logic function system. Then*

$$d(F) \le \frac{k(d(B) + 1) - 1}{u(B) - 1} \left((u(B))^{I_B(F)} - 1 \right).$$

Proof. Let $F = \{f_1, \dots, f_m\}$, $m \geq 1$, be a k-valued logic function system. Let $x_1, \dots, x_n$ be all variables in functions $f_1, \dots, f_m$. The proof is by induction over $I_B(F)$.

For $I_B(F) = 0$ all functions from F are monotone. Hence, $d(F) = 0$.

Assume Lemma statement is valid for all function systems G such that $I_B(G) \leq I_B(F) - 1$. Consider arbitrary circuit S with inputs $x_1, \dots, x_n$ that realizes function system F and contains exactly $I_B(F)$ nodes that correspond to functions from $\{\omega_1, \dots, \omega_p\}$. Let us select the first vertex (according to any correct numeration) corresponding to a function from $\{\omega_1, \dots, \omega_p\}$. Denote the corresponding gate by E. Denote by $h(x_1, \dots, x_n)$ the function that is calculated at the output E. Denote by S' a circuit with inputs $y, x_1, x_2, \dots, x_n$ which is obtained from the circuit S by replacing the selected gate with one more input by variable y. Denote by $G = \{g_1, \dots, g_m\}$ the system of the functions that is realized at the outputs of G. We stress that

$$f_i(x_1, \dots, x_n) = g_i\left(h(x_1, \dots, x_n), x_1, \dots, x_n\right), \quad i = 1, \dots, m.$$

Moreover $I_B(G) \leq I_B(F) - 1$.

Consider a chain

$$C = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_r)$$

such that $d(F) = d_C(F)$.

Denote by C' the sequence of the $(n+1)$-tuples

$$(h(\tilde{\alpha}_1), \tilde{\alpha}_1), \dots, (h(\tilde{\alpha}_r), \tilde{\alpha}_r).$$

Note that the sequence C' is not a chain. Let us decompose C' into chains in the following way. By Lemma 1 the sequence $h(\tilde{\alpha}_1), h(\tilde{\alpha}_2), \dots, h(\tilde{\alpha}_r)$ can be partitioned into $u(B)$ disjoint non-decreasing subsequences. Without loss of generality we can assume that equality $h(\tilde{\alpha}_i) = h(\tilde{\alpha}_{i+1})$ implies that $h(\tilde{\alpha}_i)$ and $h(\tilde{\alpha}_{i+1})$ belong to the same subsequence (if not we can move element $h(\tilde{\alpha_{i+1}})$ to the subsequence containing $h(\tilde{\alpha}_i)$). This partition of the sequence $h(\tilde{\alpha}_1), h(\tilde{\alpha}_2), \dots, h(\tilde{\alpha}_r)$ gives us a partition of the sequence C' into subsequences $C'_1, C'_2, \dots, C'_{u(B)}$. For all $j = 1, \dots, u(B)$, the sequence C'_j is the chain of $(n+1)$-tuples.

By inductive assumtion the inequalities

$$d_{C'_j}(G) \leq d(G) \leq \frac{k(d(B)+1)-1}{u(B)-1}\left((u(B))^{I_B(G)} - 1\right) \leq \frac{k(d(B)+1)-1}{u(B)-1}\left((u(B))^{I_B(F)-1} - 1\right)$$

are valid for $j = 1, \dots, u(B)$.

Note that by Lemma 2 the first positions in the tuples from C' change values at most $k(d(B)+1) - 1$ times. Now if we recall constructing the sequence C' partition we can note that the number of transfers from one subsequence to another while looking through elements of C' from initial to the terminal tuples is also at most $k(d(B)+1) - 1$.

Now using the equalities

$$f_i(x_1, \dots, x_n) = g_i\left(h(x_1, \dots, x_n), x_1, \dots, x_n\right), \quad i = 1, \dots, m,$$

we obtain

$$d(F) = d_C(F) \le d_{C'_1}(G) + d_{C'_2}(G) + \ldots + d_{C'_{u(B)}}(G) + k(d(B)+1) - 1$$
$$\le \frac{k(d(B)+1)-1}{u(B)-1}\left(u(B)((u(B))^{I_B(F)-1} - 1) + u(B) - 1\right)$$
$$= \frac{k(d(B)+1)-1}{u(B)-1}\left((u(B))^{I_B(F)} - 1\right).$$

These completes the proof of Lemma 3 □

Let
$$c(B) = \log_{u(B)} \frac{k(d(B)+1)-1}{u(B)-1} + 1.$$

Lemma 4. *Let F be a k-valued logic function system. Then*
$$I_B(F) \ge \lceil \log_{u(B)}(d(F)+1) \rceil - c(B).$$

Proof. Lemma 3 implies
$$(u(B))^{I_B(F)} - 1 \ge d(F)\frac{u(B)-1}{k(d(B)+1)-1}.$$

Hence,
$$I_B(F) \ge \log_{u(F)}\left(d(F)\frac{u(B)-1}{k(d(B)+1)-1} + 1\right)$$
$$\ge \log_{u(F)}\left(d(F)\frac{u(B)-1}{k(d(B)+1)-1} + \frac{u(B)-1}{k(d(B)+1)-1}\right)$$
$$\ge \log_{u(B)}(d(F)+1) - \log_{u(B)} \frac{k(d(B)+1)-1}{u(B)-1}$$
$$\ge \lceil \log_{u(B)}(d(F)+1) \rceil - \left(\log_{u(B)} \frac{k(d(B)+1)-1}{u(B)-1} + 1\right)$$
$$= \lceil \log_{u(B)}(d(F)+1) \rceil - c(B).$$

These complete the proof of Lemma 4. □

Lemma 4 completes the proof of Theorem 1.

Thus, asymptotics for non-monotone complexity growth of a multi-valued logic function system is established for arbitrary basis of the form $(*)$.

Let us emphasize that the constant $c(B)$ from the lower bound from Theorem 1 is not absolute. The constant $c(B)$ depends on basis B. Indeed, for any k, $k \ge 2$ the value
$$\lceil \log_{u(B)}(d(F)+1) \rceil - I_B(F)$$
is not bounded. We can consider function f such that $d(f) \ge k^{N+1}$ and let $F = \{f\}$. If $f \in B$, then $\lceil \log_{u(B)}(d(F)+1) \rceil - I_B(F) \ge \lceil \log_k(d(F)+1) \rceil - 1 > N$.

Now let us generalize the problem. Let us consider a basis with arbitrary positive weights for the non-monotone functions in it.

Theorem 2. *Consider the basis B of the form $(*)$. Let the weights of the functions $\omega_1, \ldots, \omega_p$ be equal to $\rho_1, \ldots, \rho_p$ respectively. Let*
$$\frac{\rho_1}{\log_2 u(\omega_1)} = \min\left\{\frac{\rho_1}{\log_2 u(\omega_1)}, \ldots, \frac{\rho_p}{\log_2 u(\omega_p)}\right\}.$$

Then there exists a constant $c(B)$ such that for any k-valued logic function system F the inequalities

$$\rho_1 \left\lceil \log_{u(\omega_1)}(d(F)+1) \right\rceil - c(B) \le I_B(F) \le \rho_1 \left\lceil \log_{u(\omega_1)}(d(F)+1) \right\rceil .$$

are valid.

The upper bound of Theorem 2 follows from the upper bound of Theorem 1. Indeed, consider basis $B' = M \cup \{\omega_1\}$ such that the weight of the function ω_1 equals ρ_1, and the weight of all monotone functions are zero. Now by inequality $I_B(F) \le I_{B'}(F)$ we get the upper bound.

The proof of the lower bound from the Theorem 2 is similar to the proof of the lower bound from Theorem 1. Lemma 3 should be slightly changed. Induction should be done over the number of non-monotone elements in the circuit. Besides, [19] contains the proof of the lower bound for the non-monotone basis with arbitrary positive weights of non-monotone elements for Boolean function system realization.

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