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EXISTENCE OF SOLUTION FOR A NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we study the existence of nontrivial solution for the fourth-order three-point boundary value problem given as follows

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u^{'}(0) - \alpha u^{'}(\eta) = 0, \quad u(0) = u^{'''}(0) = 0, \quad u^{'}(1) - \beta u^{'}(\eta) = 0,$$

where $\eta \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. We give sufficient conditions that allow us to obtain the existence of a nontrivial solution. And by using the Leray-Schauder nonlinear alternative we prove the existence of at least one solution of the posed problem. As an application, we also given some examples to illustrate the results obtained.

Keywords: Green's function, Nontrivial solution, Leary-Schauder nonlinear alternative, Fixed point theorem, Boundary value problem.

1. Introduction

The study of fourth-order three-point boundary value problems (BVP) for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. For examples, fluid dynamics ([2]), beam theory ([9, 11]), electric circuits ([3]), ship dynamics ([5, 17, 19]), neural networks ([13]), and the mathematical model of beam's deflection ([4, 18]).

Many authors studied the existence of positive solutions for nth-order m-point boundary value problems using different methods such that fixed point theorems in cones, nonlinear alternative of Leray-Schauder, and Krasnoselskii's fixed point theorem, see ([1, 8, 10, 21]) and the references therein.

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In 2003, by using the Leray-Schauder degree theory, Yuji Liu and Weigao Ge ([12]) proved the existence of positive solutions for $(n-1,1)$ three-point boundary value problems with coefficient that changes sign given as follows

$$u^{(n)}(t)+\lambda a(t)f(u(t))=0, \quad \text{t} \in (0,1),$$
$$u(0)=\alpha u(\eta), \quad u(1)=\beta u(\eta), \quad u^{(i)}(0)=0 \;\; for \;\; i=1,2,...,n-2,$$
$$and$$
$$u^{(n-2)}(0)=\alpha u^{(n-2)}(\eta),\ u^{(n-2)}(1)=\beta u^{(n-2)}(\eta),\ u^{(i)}(0)=0\ for\ i=1,2,..,n-3,$$

where $\eta \in (0,1)$, $\alpha \geq 0$, $\beta \geq 0$, and $a:(0,1) \to \mathbb{R}$ may change sign and $\mathbb{R} = (-\infty,\infty)$, $f(0)>0$, $\lambda>0$ is a parameter.

In 2005, Paul W. Eloea and Bashir Ahmad ([7]) studied the existence of positive solutions of a nonlinear nth-order boundary value problem with nonlocal conditions as follows

$$u^{(n)}(t)+a(t)f(u(t))=0, \quad \text{t} \in (0,1),$$
$$u(0)=0, \quad u^{'}(0)=0,...,u^{(n-2)}(0)=0, \quad \alpha u(\eta)=u(1),$$

where $0<\eta<1$, $0<\alpha\eta^{n-1}<1$, f is either superlinear or sublinear, and $a:[0,1] \longrightarrow [0,\infty)$ is continuous. The methods used is the fixed point theorem in cones due to Krasnoselkiî and Guo.

Then in the year 2009, Xie, Liu and Bai ([20]) used fixed-point theory to study the existence of positive solutions for a singular nth-order three-point boundary value problem on time scales represented in the following figure

$$u^{(n)}(t)+h(t)f(u(t))=0, \quad \text{t} \in (0,1),$$
$$u(a)=\alpha u(\eta), \quad u^{'}(a)=0,...,u^{(n-2)}(a)=0, \quad u(b)=\beta u(\eta),$$

where $a<\eta<b$, $0\leq\alpha<1$, $0<\beta(\eta-a)^{n-1}<(1-\alpha)(b-a)^{n-1}+\alpha(\eta-a)^{n-1}$, $f \in C([a,b]\times[0,\infty),[0,\infty))$ and $h \in C([a,b],[0,\infty))$ may be singular at $t=a$ and $t=b$.

In 2004, Yong-Ping Sun ([14]) studied the existence of nontrivial solution for the three-point boundary value problem

$$u^{''}(t)+f(t,u(t))=0, \quad 0\leq t\leq 1,$$
$$u^{'}(0)=0, \quad u(1)=\alpha u^{'}(\eta),$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, using the Leray-Schauder nonlinear alternative. The same author in ([15]), who used the same method to study solvability of a nonlinear second-order three-point boundary value problem

$$u^{''}(t)+f(t,u(t))=0, \quad 0\leq t\leq 1,$$
$$u^{'}(0)=0, \quad u(1)=\alpha u(\eta),$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$.

As well in ([16]), who used the same method to study nontrivial symmetric solution of a nonlinear second-order three-point boundary value problem

$$u^{''}(t)+f(t,u(t))=0, \quad 0\leq t\leq 1,$$
$$u(0)=u(1)=\alpha u(\eta),$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, $f(.,x)$ is symmetric on $[0,1]$ for every $x \in \mathbb{R}$.

Motivated by the above works, we extend the results obtained for second-order boundary value problem to fourth-order boundary value problem with the change a new boundary conditions, see (2), by using a style and a different method than in the papers mentioned earlier ([12, 20]), as well we prove the existence of nontrivial solution for the fourth-order three-point boundary value problem (BVP)

$$u^{(4)}(t) + f(t,u(t)) = 0, \quad 0 < t < 1. \tag{1}$$

$$u^{'}(0) - \alpha u^{'}(\eta) = 0, \quad u(0) = u^{'''}(0) = 0, \quad u^{'}(1) - \beta u^{'}(\eta) = 0, \tag{2}$$

where $\eta \in (0,1)$, $\alpha,\beta \in \mathbb{R}$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$, and $\mathbb{R} = (-\infty,+\infty)$.

This paper is organized as follows. In section 2, we present two lemmas that will be used to prove the results. Then, in section 3, we present and prove our main results which consists of existence theorems and corollary for nontrivial solution of the BVP $(1)-(2)$, and we establish some existence criteria of at least one solution by using the Leray-Schauder nonlinear alternative. Finally, in section 4, as an application, we give some examples to illustrate the results we obtained.

2. Preliminaries

Let $E = C([0,1])$ with the norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$ for any $y \in E$. A solution $u(t)$ of the BVP $(1)-(2)$ is called nontrivial solution if $u(t) \neq 0$. To get our results, we need to provide the following lemma.

Lemma 1. *Let $y \in C([0,1])$, $\alpha \neq 0$, $\beta\eta \neq 0$, and $\zeta = (1-\alpha) + \eta(\alpha-\beta) \neq 0$, then three-point BVP*

$$u^{(4)}(t) + y(t) = 0, \quad 0 < t < 1,$$
$$u^{'}(0) - \alpha u^{'}(\eta) = 0, \quad u(0) = u^{'''}(0) = 0, \quad u^{'}(1) - \beta u^{'}(\eta) = 0,$$

has a unique solution

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)ds + \frac{t^2(1-\alpha) + 2\alpha\eta t}{4\zeta}\int_0^1 (1-s)^2 y(s)ds + \frac{t^2(\alpha-\beta) - 2\alpha t}{4\zeta}\int_0^\eta (\eta-s)^2 y(s)ds.$$

Proof. Rewriting the differential equation as $u^{(4)}(t) = -y(t)$, and integrating four times from 0 to t, we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)ds + \frac{t^3}{6}c_0 + \frac{t^2}{2}c_1 + tc_2 + c_3. \tag{3}$$

By the boundary conditions (2), we have $u(0) = u^{'''}(0) = 0$, i.e. $c_3 = c_0 = 0$,

and $u^{'}(0) - \alpha u^{'}(\eta) = 0$, implies

$$c_2 = -\frac{\alpha}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 y(s)ds + \frac{\alpha\eta}{(1-\alpha)}c_1, \tag{4}$$

also $u^{'}(1) - \beta u^{'}(\eta) = 0$, we get

$$c_1 = \frac{1}{2(1-\beta\eta)}\int_0^1 (1-s)^2 y(s)ds - \frac{\beta}{2(1-\beta\eta)}\int_0^\eta (\eta-s)^2 y(s)ds + \frac{(\beta-1)}{(1-\beta\eta)}c_2. \tag{5}$$

Compensate equation (4) in the equation (5), we obtain

$$c_1 = \frac{(1-\alpha)}{2((1-\alpha)+\eta(\alpha-\beta))}\int_0^1 (1-s)^2 y(s)ds + \frac{(\alpha-\beta)}{2((1-\alpha)+\eta(\alpha-\beta))}\int_0^\eta (\eta-s)^2 y(s)ds,$$

and

$$c_2 = \frac{\alpha\eta}{2((1-\alpha)+\eta(\alpha-\beta))}\int_0^1 (1-s)^2 y(s)ds - \frac{\alpha}{2((1-\alpha)+\eta(\alpha-\beta))}\int_0^\eta (\eta-s)^2 y(s)ds.$$

Substituting c_1 and c_2 by their values in (3), we obtain the solution in the statement of the lemma. this completes the proof. □

Define the integral operator $T : E \longrightarrow E$, by

$$Tu(t) = -\frac{1}{6}\int_0^t (t-s)^3 f(s,u(s))ds + \frac{t^2(1-\alpha)+2\alpha\eta t}{4\zeta}\int_0^1 (1-s)^2 f(s,u(s))ds+$$

$$\frac{t^2(\alpha-\beta)-2\alpha t}{4\zeta}\int_0^\eta (\eta-s)^2 f(s,u(s))ds. \tag{6}$$

By Lemma 2.1, the BVP $(1)-(2)$ has a solution if and only if the operator T has a fixed point in E. So we only need to seek a fixed point of T in E. By Ascoli-Arzela theorem, we can prove that T is a completely continuous operator. Now we cite the Leray-Schauder nonlinear alternative.

Lemma 2. ([6]). *Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. $T : \overline{\Omega} \to E$ be a completely continuous operator. Then, either*
(i) there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $T(u) = \lambda u$, or
(ii) there exists a fixed point $u^* \in \overline{\Omega}$ of T.

3. Existence of nontrivial solution

In this section, we prove the existence of a nontrivial solution for the BVP $(1)-(2)$. Suppose that $f \in C([0,1]\times\mathbb{R},\mathbb{R})$.

Theorem 1. *Suppose that $f(t,0) \neq 0$, $\zeta \neq 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1]\times\mathbb{R},$$

$$\frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2 k(s)ds + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2 k(s)ds < 1.$$

Then the BVP $(1)-(2)$ has at least one nontrivial solution $u^* \in C([0,1])$.

Proof. Let

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2 k(s)\,ds + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2 k(s)ds,$$

$$N = \frac{1}{6}\int_0^1 (1-s)^3 h(s)ds + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2 h(s)\,ds + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2 h(s)ds.$$

Then $M < 1$. Since $f(t,0) \neq 0$, there exists an interval $[a,b] \subset [0,1]$ such that $\min_{a\leq t\leq b}|f(t,0)| > 0$. And as $h(t) \geq |f(t,0)|$, a.e. $t \in [0,1]$, we know that $N > 0$.

Let $A = N(1-M)^{-1}$ and $\Omega = \{u \in E : \|u\| < A\}$. Assume that $u \in \partial\Omega$ and $\lambda > 1$ such that $Tu = \lambda u$, then

$$\begin{aligned}
\lambda A &= \lambda\|u\| = \|Tu\| = \max_{0\leq t\leq 1}|(Tu)(t)| \\
&\leq \frac{1}{6}\max_{0\leq t\leq 1}\int_0^t (t-s)^3|f(s,u(s))|ds + \max_{0\leq t\leq 1}|\frac{t^2(1-\alpha)+2\alpha\eta t}{4\zeta}|\int_0^1 (1-s)^2|f(s,u(s))|ds \\
&\quad + \max_{0\leq t\leq 1}|\frac{t^2(\alpha-\beta)-2\alpha t}{4\zeta}|\int_0^\eta (\eta-s)^2|f(s,u(s))|ds \\
&\leq \frac{1}{6}\int_0^1 (1-s)^3|f(s,u(s))|ds + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2|f(s,u(s))|ds \\
&\quad + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2|f(s,u(s))|ds \\
&\leq [\frac{1}{6}\int_0^1 (1-s)^3 k(s)|u(s)|ds + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2 k(s)|u(s)|ds \\
&\quad + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2 k(s)|u(s)|ds] + [\frac{1}{6}\int_0^1 (1-s)^3 h(s)ds \\
&\quad + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2 h(s)ds + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2 h(s)ds] = M\|u\| + N.
\end{aligned}$$

Therefore,

$$\lambda \leq M + \frac{N}{A} = M + \frac{N}{N(1-M)^{-1}} = M + (1-M) = 1.$$

This contradicts $\lambda > 1$. By Lemma 2.2, T has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t,0) \neq 0$, the BVP $(1)-(2)$ has a nontrivial solution $u^* \in E$.

This completes the proof. □

Theorem 2. *Suppose that $f(t,0) \neq 0$, $\zeta > 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \quad (t,x) \in [0,1]\times\mathbb{R}.$$

If one of the following conditions is fulfilled

(1) *There exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < [\frac{12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{2\zeta(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{(1+2q)/q})}]^p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \leq \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta + (4+\mu)(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{3+\mu})}s^\mu, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta + (4+\mu)(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{3+\mu})} s^{\mu}\} > 0.$$

(3) *There exists a constant* $\mu > -3$ *such that*

$$k(s) \leq \frac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu) + 3(4+\mu)(1+2|\alpha|(2+\eta)+|\beta|)}(1-s)^{\mu}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu) + 3(4+\mu)(1+2|\alpha|(2+\eta)+|\beta|)}(1-s)^{\mu}\} > 0.$$

(4) $k(s)$ *satisfies*

$$k(s) \leq \frac{24\zeta}{\zeta + 2(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^3)}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24\zeta}{\zeta + 2(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^3)}\} > 0.$$

(5) $f(t,x)$ *satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{24\zeta}{\zeta + 2(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^3)}.$$

Then the BVP $(1)-(2)$ *has at least one nontrivial solution* $u^* \in E$.

Proof. Let M be defined as in the proof of Theorem 1. To prove Theorem 2, we only need to prove that $M < 1$. Since $\zeta > 0$, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+|\alpha|(1+2\eta)}{4\zeta}\int_0^1 (1-s)^2 k(s)\,ds + \frac{|\beta|+3|\alpha|}{4\zeta}\int_0^{\eta} (\eta-s)^2 k(s)\,ds,$$

(1) Using the Hölder inequality, we have

$$M \leq [\int_0^1 k^p(s)ds]^{1/p}\{\frac{1}{6}[\int_0^1 (1-s)^{3q}ds]^{1/q} + \frac{1+|\alpha|(1+2\eta)}{4\zeta}[\int_0^1 (1-s)^{2q}ds]^{1/q} + \frac{|\beta|+3|\alpha|}{4\zeta}[\int_0^{\eta}(\eta-s)^{2q}ds]^{1/q}\}$$

$$\leq [\int_0^1 k^p(s)ds]^{1/p}[\frac{1}{6}(\frac{1}{1+3q})^{1/q} + \frac{1+|\alpha|(1+2\eta)}{4\zeta}(\frac{1}{1+2q})^{1/q} + \frac{|\beta|+3|\alpha|}{4\zeta}(\frac{\eta^{1+2q}}{1+2q})^{1/q}]$$

$$< \frac{12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{2\zeta(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{(1+2q)/q})}$$

$$\times \frac{2\zeta(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{(1+2q)/q})}{12\zeta(1+2q)^{1/q}(1+3q)^{1/q}} = 1.$$

(2) In this case, we have

$$M < \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta+(4+\mu)(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^{3+\mu})}[\frac{1}{6}\int_0^1(1-s)^3 s^\mu\,ds$$

$$+\frac{1+|\alpha|(1+2\eta)}{4\zeta}\int_0^1(1-s)^2 s^\mu ds+\frac{|\beta|+3|\alpha|}{4\zeta}\int_0^\eta(\eta-s)^2 s^\mu\,ds]$$

$$\leq \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta+(4+\mu)1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^{3+\mu})}[\frac{1}{(1+\mu)(2+\mu)(3+\mu)(4+\mu)}$$

$$+\frac{1+|\alpha|(1+2\eta)}{2\zeta}.\frac{1}{(1+\mu)(2+\mu)(3+\mu)}+\frac{|\beta|+3|\alpha|}{2\zeta}.\frac{\eta^{3+\mu}}{(1+\mu)(2+\mu)(3+\mu)}]$$

$$=\frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta+(4+\mu)(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^{3+\mu})}$$

$$\times\frac{2\zeta+(4+\mu)(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^{3+\mu})}{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}=1.$$

(3) In this case, we have

$$M < \frac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu)+3(4+\mu)(1+2|\alpha|(2+\eta)+|\beta|)}[\frac{1}{6}\int_0^1(1-s)^{3+\mu}\,ds$$

$$+\frac{1+|\alpha|(1+2\eta)}{4\zeta}\int_0^1(1-s)^{2+\mu}ds+\frac{|\beta|+3|\alpha|}{4\zeta}\int_0^\eta(\eta-s)^2(1-s)^\mu\,ds]$$

$$\leq \tfrac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu)+3(4+\mu)(1+2|\alpha|(2+\eta)+|\beta|)}[\tfrac{1}{6}\int_0^1(1-s)^{3+\mu}ds+\tfrac{1+|\alpha|(1+2\eta)}{4\zeta}\int_0^1(1-s)^{2+\mu}ds$$

$$+\frac{|\beta|+3|\alpha|}{4\zeta}\int_0^1(1-s)^{2+\mu}ds]$$

$$=\frac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu)+3(4+\mu)(1+2|\alpha|(2+\eta)+|\beta|)}$$

$$\times\frac{2\zeta(3+\mu)+3(4+\mu)(1+2|\alpha|(2+\eta)+|\beta|)}{12\zeta(3+\mu)(4+\mu)}=1.$$

(4) In this case, we have

$$M < \frac{24\zeta}{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}$$

$$\times[\frac{1}{6}\int_0^1(1-s)^3ds+\frac{(1+|\alpha|(1+2\eta)}{4\zeta}\int_0^1(1-s)^2ds+\frac{|\beta|+3|\alpha|}{4\zeta}\int_0^\eta(\eta-s)^2ds]$$

$$=\frac{24\zeta}{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}.\frac{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}{24\zeta}=1.$$

(5) Let $\epsilon=\frac{24\zeta}{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}-Q]$, then there exists $c>0$ such that

$$|f(t,x)|\leq[\frac{24\zeta}{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}-\epsilon]|x|,\quad (t,x)\in[0,1]\times\mathbb{R}/(-c,c).$$

Set $A=\max\{|f(t,x)|:(t,x)\in[0,1]\times[-c,c]\}$, then

$$|f(t,x)|\leq[\frac{24\zeta}{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}-\epsilon]|x|+A,\quad (t,x)\in[0,1]\times\mathbb{R}.$$

Set $k(s)=\frac{24\zeta}{\zeta+2(1+|\alpha|(1+2\eta)+(|\beta|+3|\alpha|)\eta^3)}-\epsilon$, $h(s)=A$, then (4) holds.

This completes the proof. □

Corollary 1. *Suppose $f(t,0) \neq 0$, $\zeta > 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \quad (t,x) \in [0,1] \times \mathbb{R}.$$

If one of following conditions is fulfilled

(1) *There exists a constant $p > 1$ such that*

$$\int_0^1 k^p(s)ds < [\frac{12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{2\zeta(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+6|\alpha|+|\beta|)}]^p, \quad (\frac{1}{p} + \frac{1}{q} = 1).$$

(2) *There exists a constant $\mu > -1$ such that*

$$k(s) \leq \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta + (4+\mu)(1+6|\alpha|+|\beta|)} s^\mu, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta + (4+\mu)(1+6|\alpha|+|\beta|)} s^\mu\} > 0.$$

(3) *$k(s)$ satisfies*

$$k(s) \leq \frac{24\zeta}{\zeta + 2(1+6|\alpha|+|\beta|)}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24\zeta}{\zeta + 2(1+6|\alpha|+|\beta|)}\} > 0.$$

(4) *$f(t,x)$ satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{24\zeta}{\zeta + 2(1+6|\alpha|+|\beta|)}.$$

Then the BVP $(1)-(2)$ has at least one nontrivial solution $u^* \in E$.

Proof. In this case, we have

$$\begin{aligned} M = &\frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+|\alpha|(1+2\eta)}{4\zeta}\int_0^1 (1-s)^2 k(s)\,ds \\ &+ \frac{|\beta|+3|\alpha|}{4\zeta}\int_0^\eta (\eta-s)^2 k(s)\,ds \\ \leq &\frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+3|\alpha|}{4\zeta}\int_0^1 (1-s)^2 k(s)ds + \frac{|\beta|+3|\alpha|}{4\zeta}\int_0^1 (1-s)^2 k(s)\,ds \\ = &\frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+6|\alpha|+|\beta|}{4\zeta}\int_0^1 (1-s)^2 k(s)ds. \end{aligned}$$

Proof of this corollary 1 is the same method in the proof theorem 2. This completes the proof. □

Theorem 3. *Suppose that $f(t,0) \neq 0$, $\alpha > 0$, $\beta > 0$, $\zeta < 0$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \quad (t,x) \in [0,1] \times \mathbb{R}.$$

If one of the following conditions is fulfilled

(1) *There exists a constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)\,ds < [\frac{-12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{-2\zeta(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha(1+2\eta)+(\beta+3\alpha)\eta^{(1+2q)/q})}]^p, \quad \frac{1}{p}+\frac{1}{q}=1.$$

(2) *There exists a constant* $\mu > -1$ *such that*

$$k(s) \le \frac{-2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{-2\zeta+(4+\mu)(1+\alpha(1+2\eta)+(\beta+3\alpha)\eta^{3+\mu})} s^\mu, \quad a.e. \quad s\in[0,1],$$

$$meas\{s\in[0,1]: k(s) < \frac{-2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{-2\zeta+(4+\mu)(1+\alpha(1+2\eta)+(\beta+3\alpha)\eta^{3+\mu})} s^\mu\} > 0.$$

(3) *There exists a constant* $\mu > -3$ *such that*

$$k(s) \le \frac{-12\zeta(3+\mu)(4+\mu)}{-2\zeta(3+\mu)+3(4+\mu)(1+2\alpha(2+\eta)+\beta)}(1-s)^\mu, \quad a.e. \quad s\in[0,1],$$

$$meas\{s\in[0,1]: k(s) < \frac{-12\zeta(3+\mu)(4+\mu)}{-2\zeta(3+\mu)+3(4+\mu)(1+2\alpha(2+\eta)+\beta)}(1-s)^\mu\} > 0.$$

(4) $k(s)$ *satisfies*

$$k(s) \le \frac{-24\zeta}{-\zeta+2(1+\alpha(1+2\eta)+(\beta+3\alpha)\eta^3)}, \quad a.e. \quad s\in[0,1],$$

$$meas\{s\in[0,1]: k(s) < \frac{-24\zeta}{-\zeta+2(1+\alpha(1+2\eta)+(\beta+3\alpha)\eta^3)}\} > 0.$$

(5) $f(t,x)$ *satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{-24\zeta}{-\zeta+2(1+\alpha(1+2\eta)+(\beta+3\alpha)\eta^3)}.$$

Then the BVP $(1)-(2)$ *has at least one nontrivial solution* $u^* \in E$.

Proof. Let M be given as in the proof of Theorem 1. To prove Theorem 3, we only need to prove that $M < 1$. Since $\alpha > 0$, $\beta > 0$, and $\zeta < 0$, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+\alpha(1+2\eta)}{-4\zeta}\int_0^1 (1-s)^2 k(s)ds + \frac{\beta+3\alpha}{-4\zeta}\int_0^\eta (\eta-s)^2 k(s)ds,$$

Now, the proof follows, by using the same method as the one used in the proof of Theorem 2. The proof is complete. □

Corollary 2. *Suppose* $f(t,0)\neq 0$, $\alpha > 0$, $\beta > 0$, $\zeta < 0$, *and there exist nonnegative functions* $k, h \in L^1[0,1]$ *such that*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e. \quad (t,x)\in[0,1]\times\mathbb{R}.$$

If one of following conditions is fulfilled

(1) *There exists a constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < [\frac{-12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{-2\zeta(1+2q)^{1/q}+3(1+3q)^{1/q}(1+6\alpha+\beta)}]^p, \quad (\frac{1}{p}+\frac{1}{q}=1).$$

(2) *There exists a constant* $\mu > -1$ *such that*

$$k(s) \leq \frac{-2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{-2\zeta+(4+\mu)(1+6\alpha+\beta)} s^{\mu}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{-2\zeta+(4+\mu)(1+6\alpha+\beta)} s^{\mu}\} > 0.$$

(3) $k(s)$ *satisfies*

$$k(s) \leq \frac{-24\zeta}{-\zeta+2(1+6\alpha+\beta)}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{-24\zeta}{-\zeta+2(1+6\alpha+\beta)}\} > 0.$$

(4) $f(t,x)$ *satisfies*

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{-24\zeta}{-\zeta+2(1+6\alpha+\beta)}.$$

Then the BVP $(1)-(2)$ *has at least one nontrivial solution* $u^* \in E$.

Proof. In this case, we have

$$\begin{aligned} M &= \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+\alpha(1+2\eta)}{-4\zeta}\int_0^1 (1-s)^2 k(s)ds + \frac{\beta+3\alpha}{-4\zeta}\int_0^{\eta} (\eta-s)^2 k(s)ds \\ &\leq \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+3\alpha}{-4\zeta}\int_0^1 (1-s)^2 k(s)ds + \frac{\beta+3\alpha}{-4\zeta}\int_0^1 (1-s)^2 k(s)ds \\ &= \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+6\alpha+\beta}{-4\zeta}\int_0^1 (1-s)^2 k(s)ds. \end{aligned}$$

The rest of the proof follows in the same way as in the proof of Theorem 3. This completes the proof. □

4. Examples

In order to illustrate the above results, we consider some examples.

Example 1. *Consider the three-point boundary value problem*

$$\begin{aligned} &u^{(4)} + \tfrac{t}{\sqrt[3]{5}} u \cos\sqrt{u} + e^t + 1 = 0, \quad 0 < t < 1, \\ (7) \qquad & u'(0) - 3u'(1/2) = 0, \quad u(0) = u'''(0) = 0, \quad u'(1) + 3u'(1/2) = 0. \end{aligned}$$

Set $\eta = \frac{1}{2}$, $\alpha = 3$, $\beta = -3$, *and*

$$f(t,x) = \frac{t}{\sqrt[3]{5}} x \cos\sqrt{x} + e^t + 1,$$

$$k(t) = t, \quad h(t) = e^t + 1,$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R},$$

and

$$\zeta = (1-\alpha) + \eta(\alpha-\beta) = 1 \neq 0.$$

Moreover, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)ds + \frac{1+|\alpha|(1+2\eta)}{4|\zeta|}\int_0^1 (1-s)^2 k(s)ds + \frac{|\beta|+3|\alpha|}{4|\zeta|}\int_0^\eta (\eta-s)^2 k(s)ds$$

$$M = \frac{1}{6}\int_0^1 (1-s)^3 sds + \frac{7}{4}\int_0^1 (1-s)^2 sds + 3\int_0^{1/2} (\frac{1}{2}-s)^2 sds = \frac{1}{120} + \frac{7}{48} + \frac{3}{640} \approx 0.157 < 1.$$

Hence, by Theorem 1, the BVP (7) has at least one nontrivial solution u^* in E.

Example 2. *Consider the three-point boundary value problem*

(8)
$$\begin{gathered} u^{(4)} + \tfrac{1/9(2+t)}{1+u} u \sin u^3 - t - 2 = 0, \quad 0 < t < 1, \\ u'(0) - 1/2u'(1/3) = 0, \quad u(0) = u'''(0) = 0, \quad u'(1) - u'(1/3) = 0. \end{gathered}$$

Set $\eta = 1/3$, $\alpha = 1/2$, $\beta = 1$, and

$$f(t,x) = \frac{1/9(2+t)}{1+x} x \sin x^3 - t - 2,$$

$$k(t) = \frac{1}{9}(2+t), \quad h(t) = t+2.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions, and

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta(\alpha - \beta) = \frac{1}{3} > 0.$$

Let $p = q = 2$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^1 k(s)^p ds = \int_0^1 \frac{1}{81}(2+s)^2 ds = \frac{19}{243} \approx 0.078.$$

Moreover, we have

$$[\frac{12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{2\zeta(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{(1+2q)/q})}]^p \approx 1.874.$$

Therefore,

$$\int_0^1 k(s)^p ds < [\frac{12\zeta(1+2q)^{1/q}(1+3q)^{1/q}}{2\zeta(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|(1+2\eta) + (|\beta|+3|\alpha|)\eta^{(1+2q)/q})}]^p.$$

Hence, by Theorem 2 (1), the BVP (8) has at least one nontrivial solution u^* in E.

Example 3. *Consider the three-point boundary value problem*

(9)
$$\begin{gathered} u^{(4)} + \tfrac{u}{(5+3u)\sqrt{t}} \cos e^u + t^3 - 1 = 0, \quad 0 < t < 1, \\ u'(0) - 1/4u'(1/4) = 0, \quad u(0) = u'''(0) = 0, \quad u'(1) + 2u'(1/4) = 0. \end{gathered}$$

Set $\eta = 1/4$, $\alpha = 1/4$, $\beta = -2$, and

$$f(t,x) = \frac{x}{(5+3x)\sqrt{t}} \cos e^x + t^3 - 1,$$

$$k(t) = \frac{1}{5\sqrt{t}}, \quad h(t) = t^3 + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta(\alpha - \beta) = \frac{21}{16} > 0.$$

Let $\mu = -\frac{1}{2} > -1$*, then*

$$\frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta + (4+\mu)(1+|\alpha|(1+2\eta) + (|\beta| + 3|\alpha|)\eta^{3+\mu})} \approx 2.227.$$

Therefore,

$$k(s) = \frac{1}{5\sqrt{s}} = \frac{1}{5}s^{-\frac{1}{2}} < 2.227.s^{-\frac{1}{2}},$$

$$meas\{s \in [0,1] : k(s) < \frac{2\zeta(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{2\zeta + (4+\mu)(1+|\alpha|(1+2\eta) + (|\beta| + 3|\alpha|)\eta^{3+\mu})} s^{\mu}\} > 0.$$

Hence, by Theorem 2 (2)*, the BVP* (9) *has at least one nontrivial solution* u^* *in* E*.*

Example 4. *Consider the three-point boundary value problem*

$$(10) \qquad \begin{aligned} &u^{(4)} + \tfrac{u^2}{7(3+u)(1-t)^{-2}} \cos u + e^{2t} - 2 = 0, \quad 0 < t < 1, \\ &u'(0) + 5u'(1/2) = 0, \quad u(0) = u'''(0) = 0, \quad u'(1) + 3u'(1/2) = 0. \end{aligned}$$

Set $\eta = 1/2$*,* $\alpha = -5$*,* $\beta = -3$*, and*

$$f(t,x) = \frac{x^2}{7(3+x)(1-t)^{-2}} \cos x + e^{2t} - 2,$$

$$k(t) = \frac{1}{7(1-t)^{-2}}, \quad h(t) = t^3 + 2.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta(\alpha - \beta) = 2 > 0.$$

Let $\mu = 2 > -3$*, then*

$$\frac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu) + 3(4+\mu)(1+|\alpha|(1+2\eta) + |\beta|)} = \frac{360}{271}.$$

Therefore,

$$k(s) = \frac{1}{7(1-s)^{-2}} = \frac{1}{7}(1-s)^2 < \frac{360}{271}(1-s)^2,$$

$$meas\{s \in [0,1] : k(s) < \frac{12\zeta(3+\mu)(4+\mu)}{2\zeta(3+\mu) + 3(4+\mu)(1+|\alpha|(1+2\eta) + |\beta|)}(1-s)^{\mu}\} > 0.$$

Hence, by Theorem 2 (3)*, the BVP* (10) *has at least one nontrivial solution* u^* *in* E*.*

Example 5. *Consider the three-point boundary value problem*

$$(11)\quad \begin{gathered} u^{(4)} + \tfrac{t^2 u}{3(2+u^2)} \sin\sqrt{u} + t - 1 = 0, \quad 0 < t < 1, \\ u^{'}(0) + u^{'}(1/5) = 0, \quad u(0) = u^{'''}(0) = 0, \quad u^{'}(1) + 2u^{'}(1/5) = 0. \end{gathered}$$

Set $\eta = 1/5$, $\alpha = -1$, $\beta = -2$, *and*

$$f(t,x) = \frac{t^2 x}{3(2+x^2)} \sin\sqrt{x} + t - 1,$$

$$k(t) = \frac{t^2}{3}, \quad h(t) = t + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta(\alpha - \beta) = \frac{11}{5} > 0.$$

Moreover, we have

$$\frac{24\zeta}{\zeta + 2(1 + |\alpha|(1+2\eta) + (|\beta| + 3|\alpha|)\eta^3)} = \frac{1320}{177}.$$

Therefore,

$$k(s) = \frac{s^2}{3} < \frac{1320}{177}, \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24\zeta}{\zeta + 2(1 + |\alpha|(1+2\eta) + (|\beta| + 3|\alpha|)\eta^3)} > 0.$$

Hence, by Theorem 2 (4), the BVP (11) has at least one nontrivial solution u^* *in* E.

Example 6. *Consider the three-point boundary value problem*

$$(12)\quad \begin{gathered} u^{(4)} + \tfrac{5tu}{4(3+7e^t)} - \sqrt{t} + 3 = 0, \quad 0 < t < 1, \\ u^{'}(0) + 4u^{'}(1/2) = 0, \quad u(0) = u^{'''}(0) = 0, \quad u^{'}(1) + u^{'}(1/2) = 0. \end{gathered}$$

Set $\eta = 1/2$, $\alpha = -4$, $\beta = -1$, *and*

$$f(t,x) = \frac{5tx}{4(3+7e^t)} - \sqrt{t} + 3,$$

$$k(t) = 5t, \quad h(t) = \sqrt{t} + 3.$$

It is easy to prove that $k, h \in L^1[0,1]$ *are nonnegative functions, and*

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbb{R}.$$

and

$$\zeta = (1-\alpha) + \eta(\alpha - \beta) = \frac{7}{2} > 0.$$

Moreover, we have

$$\frac{24\zeta}{\zeta + 2(1 + |\alpha|(1+2\eta) + (|\beta| + 3|\alpha|)\eta^3)} = \frac{336}{115}.$$

and

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| = \limsup_{|x|\to\infty} (\frac{5}{4(3+7e)} + \frac{4}{|x|}) = \frac{5}{4(3+7e)} \approx 0.057.$$

Therefore,

$$Q = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{24\zeta}{\zeta + 2(1+|\alpha|(1+2\eta) + (|\beta| + 3|\alpha|)\eta^3)}.$$

Hence, by Theorem 2 (5)*, the BVP* (12) *has at least one nontrivial solution* u^* *in* E.

Remark. We can give examples similar in relation to the Corollary 1, Theorem 3, and Corollary 2.

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