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ON THE VOLUME OF DOUBLE TWIST LINK CONE-MANIFOLDS

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Abstract. We consider the double twist link $J(2m+1, 2n+1)$ which is the two-bridge link corresponding to the continued fraction $(2m+1) - 1/(2n+1)$. It is known that $J(2m+1, 2n+1)$ has reducible nonabelian $SL_2(\mathbb{C})$-character variety if and only if $m = n$. In this paper we give a formula for the volume of hyperbolic cone-manifolds of $J(2m+1, 2m+1)$. We also give a formula for the A-polynomial 2-tuple corresponding to the canonical component of the character variety of $J(2m+1, 2m+1)$.

Keywords: canonical component, cone-manifold, hyperbolic volume, the A-polynomial, two-bridge link, double twist link.

1. Introduction

For a hyperbolic link $\mathcal{L}$ in S^3, let $E_{\mathcal{L}} = S^3 \setminus \mathcal{L}$ be the link exterior and let ρ_{hol} be a holonomy representation of $\pi_1(E_{\mathcal{L}})$ into $PSL_2(\mathbb{C})$. Thurston [16] showed that ρ_{hol} can be deformed into an ℓ-parameter family $\{\rho_{\alpha_1,\cdots,\alpha_\ell}\}$ of representations to give a corresponding family $\{E_{\mathcal{L}}(\alpha_1, \cdots, \alpha_\ell)\}$ of singular complete hyperbolic manifolds, where ℓ is the number of components of $\mathcal{L}$. In this paper we consider only the case where all of α_j's are equal to a single parameter α. In which case we also denote $E_{\mathcal{L}}(\alpha_1, \cdots, \alpha_\ell)$ by $E_{\mathcal{L}}(\alpha)$. These α's and $E_{\mathcal{L}}(\alpha)$'s are called the cone-angles and hyperbolic cone-manifolds of $\mathcal{L}$, respectively. We consider the complete hyperbolic structure on a link complement as the cone-manifold structure with cone-angle zero. It is known that for a two-bridge link $\mathcal{L}$ there exists an angle $\alpha_{\mathcal{L}} \in [\frac{2\pi}{3}, \pi)$ such that $E_{\mathcal{L}}(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_{\mathcal{L}})$, Euclidean for $\alpha = \alpha_{\mathcal{L}}$, and spherical for $\alpha \in (\alpha_{\mathcal{L}}, \pi)$ [5, 7, 13, 14]. A method for computing the volume of hyperbolic cone-manifolds of links was outlined in [5], and explicit volume formulas have been

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known for hyperbolic cone-manifolds of the links 5_1^2, 6_2^2, 6_3^2, 7_3^2 (see [4, 6, 11] and references therein) and of twisted Whitehead links [17].

For integers m and n, consider the double twist link $J(2m+1, 2n+1)$ which is the two-bridge link corresponding to the continued fraction $(2m+1) - 1/(2n+1)$ (see Figure 1). It was shown by Petersen and the author [12] that $J(2m+1, 2n+1)$ has reducible nonabelian $SL_2(\mathbb{C})$-character variety if and only if $m = n$. In this paper we are interested in the double twist link $\mathcal{L}_m = J(2m+1, 2m+1)$, since the canonical component of the character variety of $\mathcal{L}_m$ has a rather nice form (see Remark 1). Here a canonical component of the character variety of a hyperbolic link $\mathcal{L}$ is a component containing the character of a lift of a holonomy representation of $\pi_1(E_{\mathcal{L}})$ to $SL_2(\mathbb{C})$.

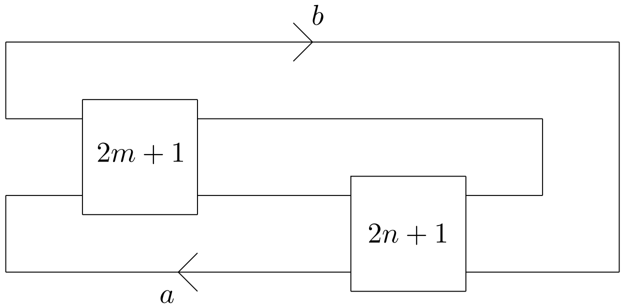

Рис. 1. The double twist link $J(2m+1, 2n+1)$. Here $2m+1$ and $2n+1$ denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left handed) twists.

Let $\{S_j(v)\}_{j\in\mathbb{Z}}$ be the sequence of Chebychev polynomials of the second kind defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers j.

Let

$$R_{\mathcal{L}_m}(s,z) = (s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) - z.$$

The volume of the hyperbolic cone-manifold of $\mathcal{L}_m$ is computed as follows.

Theorem 1. *For $\alpha \in (0, \alpha_{\mathcal{L}_m})$ we have*

$$\operatorname{Vol} E_{\mathcal{L}_m}(\alpha) = \int_\alpha^\pi \log \left| \frac{S_m(z) - e^{-i\omega} S_{m-1}(z)}{S_m(z) - e^{i\omega} S_{m-1}(z)} \right| d\omega$$

where z, with $\operatorname{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$, is a certain root of $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$.

Note that the above volume formula for the hyperbolic cone-manifold $E_{\mathcal{L}_m}(\alpha)$ depends on the choice of a root z, with $\operatorname{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$, of $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$. In numerical approximations, we choose the root z which gives the maximal volume.

It is known that the volume of the k-fold cyclic covering over a hyperbolic link $\mathcal{L}$ is k times the volume of the hyperbolic cone-manifold of $\mathcal{L}$ with cone-angle $2\pi/k$. As a direct consequence of Theorem 1, we obtain the following.

Corollary 1. *The hyperbolic volume of the k-fold cyclic covering over the two-bridge link $\mathcal{L}_m$, with $k \geq 3$, is given by the following formula*

$$k \operatorname{Vol} E_{\mathcal{L}_m}(\frac{2\pi}{k}) = k \int_{\frac{2\pi}{k}}^{\pi} \log \left| \frac{S_m(z) - e^{-i\omega} S_{m-1}(z)}{S_m(z) - e^{i\omega} S_{m-1}(z)} \right| d\omega$$

where z, with $\operatorname{Im}(S_{m-1}(z)S_m(z)) \geq 0$, is a certain root of $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$.

The A-polynomial of a knot in S^3 was introduced by Cooper, Culler, Gillet, Long and Shalen [2] in the 90's. It describes the $SL_2(\mathbb{C})$-character variety of the knot complement as viewed from the boundary torus. The A-polynomial carries a lot of information about the topology of the knot. For example, the sides of the Newton polygon of the A-polynomial of a knot in S^3 give rise to incompressible surfaces in the knot complement [2]. A generalization of the A-polynomial to links in S^3 was proposed by Zhang [18]. For an ℓ-component link in S^3, Zhang defined a polynomial ℓ-tuple link invariant called the A-polynomial ℓ-tuple. The A-polynomial 1-tuple of a knot is just its A-polynomial. The A-polynomial ℓ-tuple also caries important information about the topology of the link. For example, it can be used to construct concrete examples of hyperbolic link manifolds with non-integral traces [18].

The A-polynomial 2-tuple has been computed for a family of two bridge links called twisted Whitehead links [17]. In this paper we compute the A-polynomial 2-tuple for the canonical component of the character variety of $\mathcal{L}_m$.

Theorem 2. *Let $\{Q_j(s,w)\}_{j\in\mathbb{Z}}$ be the sequence of polynomials in two variables s, w defined by $Q_{-1} = Q_0 = 2$ and*

$$Q_j = \alpha Q_{j-1} - Q_{j-2} + \beta$$

where

$$\begin{aligned}
\alpha &= (s^8+s^4)w^4 + (-2s^8+6s^6+6s^4-2s^2)w^3 + (s^8-12s^6+34s^4-12s^2+1)w^2 \\
&\quad + (-2s^6+6s^4+6s^2-2)w + s^4 + 1, \\
\beta &= -2(s^2-1)^2\left(s^4w^4 - (s^4+s^2)w^3 - 6s^2w^2 - (s^2+1)w + 1\right).
\end{aligned}$$

Then the A-polynomial 2-tuple corresponding to the canonical component of the character variety of $\mathcal{L}_m$ is $[A(M,L), A(M,L)]$ where $A(M,L) = (L-1)Q_m(M, LM^{2m})$.

The paper is organized as follows. In Section 2 we review the definition of the A-polynomial ℓ-tuple of an ℓ-component link in S^3. In Section 3 we compute the nonabelian $SL_2(\mathbb{C})$-representations of the double twist link $J(2m+1, 2n+1)$. In Section 4 we compute the volume of hyperbolic cone-manifolds of $\mathcal{L}_m = J(2m+1, 2m+1)$ and give a proof of Theorem 1. The last section is devoted to the computation of the A-polynomial 2-tuple for the canonical component of the character variety of $\mathcal{L}_m$.

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2. The A-polynomial ℓ-tuple of a link

2.0.1. *Character varieties.* The set of characters of representations of a finitely generated group G into $SL_2(\mathbb{C})$ is known to be a algebraic set over $\mathbb{C}$ [3, 9]. It is called the character variety of G and denoted by $\chi(G)$. For example, the character variety $\chi(\mathbb{Z}^2)$ of the free abelian group on 2 generators μ, λ is isomorphic to $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (M, L) and $\tau : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ is the involution defined by $\tau(M, L) = (M^{-1}, L^{-1})$. This fact can be proved by noting that every representation $\rho : \mathbb{Z}^2 \to SL_2(\mathbb{C})$ is conjugate to an upper diagonal one, with M and L being the upper left entries of $\rho(\mu)$ and $\rho(\lambda)$ respectively.

2.0.2. *The A-polynomial.* Suppose $\mathcal{L} = K_1 \sqcup \cdots \sqcup K_\ell$ be an ℓ-component link in S^3. Let $E_\mathcal{L} = S^3 \setminus \mathcal{L}$ be the link exterior and $T_1, \dots, T_\ell$ the boundary tori of $E_\mathcal{L}$ corresponding to $K_1, \dots, K_\ell$ respectively. Each T_j is a torus whose fundamental group is free abelian of rank two. An orientation of K_j will define a unique pair of an oriented meridian μ_j and an oriented longitude λ_j such that the linking number between the longitude λ_j and the knot K_j is 0. The pair provides an identification of $\chi(\pi_1(T_j))$ and $(\mathbb{C}^*)^2_j/\tau_j$, where $(\mathbb{C}^*)^2_j$ is the set of non-zero complex pairs (M_j, L_j) and τ_j is the involution $\tau(M_j, L_j) = (M_j^{-1}, L_j^{-1})$, which actually does not depend on the orientation of K_j.

The inclusion $T_j \hookrightarrow E_\mathcal{L}$ induces the restriction map

$$\rho_j : \chi(\pi_1(E_\mathcal{L})) \longrightarrow \chi(\pi_1(T_j)) \equiv (\mathbb{C}^*)^2_j/\tau_j.$$

For each $\gamma \in \pi_1(E_\mathcal{L})$ let f_γ be the regular function on $\chi(\pi_1(E_\mathcal{L}))$ defined by

$$f_\gamma(\chi_\rho) = (\chi_\rho(\gamma))^2 - 4 = (\operatorname{tr}\rho(\gamma))^2 - 4,$$

where χ_ρ denotes the character of a representation $\rho : \pi_1(E_\mathcal{L}) \to SL_2(\mathbb{C})$. Let $\chi_j(\pi_1(E_\mathcal{L}))$ be the subvariety of $\chi(\pi_1(E_\mathcal{L}))$ defined by $f_{\mu_k} = 0,\ f_{\lambda_k} = 0$ for all $k \neq j$. Let Z_j be the image of $\chi_j(\pi_1(E_\mathcal{L}))$ under ρ_j and $\hat{Z}_j \subset (\mathbb{C}^*)^2_j$ the lift of Z_j under the projection $(\mathbb{C}^*)^2_j \to (\mathbb{C}^*)^2_j/\tau_i$. It is known that the Zariski closure of $\hat{Z}_j \subset (\mathbb{C}^*)^2_j \subset \mathbb{C}^2_j$ in $\mathbb{C}^2_j$ is an algebraic set consisting of components of dimension 0 or 1 [18]. The union of all the 1-dimension components is defined by a single polynomial $A_j \in \mathbb{Z}[M_j, L_j]$ whose coefficients are co-prime. Note that A_j is defined up to ± 1. We will call $[A_1(M_1, L_1), \cdots, A_\ell(M_\ell, L_\ell)]$ the A-polynomial ℓ-tuple of $\mathcal{L}$. For brevity, we also write $A_j(M, L)$ for $A_j(M_j, L_j)$. We refer the reader to [18] for properties of the A-polynomial ℓ-tuple.

3. Double twist links $J(2m+1, 2n+1)$

In this section we compute nonabelian $SL_2(\mathbb{C})$-representations of the double twist link $J(2m+1, 2n+1)$. They are described by the Chebyshev polynomials of the second kind, and so we first recall some properties of these polynomials.

3.1. **Chebyshev polynomials.** Recall that $\{S_j(v)\}_{j\in\mathbb{Z}}$ is the sequence of the Chebychev polynomials of the second kind defined by $S_0(v) = 1,\ S_1(v) = v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers j. The following two lemmas are elementary, see e.g. [17].

Lemma 1. *For any integer j we have*

$$S_j^2(v) + S_{j-1}^2(v) - vS_j(v)S_{j-1}(v) = 1.$$

Lemma 2. *Suppose $V \in SL_2(\mathbb{C})$ and $v = \operatorname{tr} V$. For any integer j we have*

$$V^j = S_j(v)\mathbf{1} - S_{j-1}(v)V^{-1}$$

where $\mathbf{1}$ denotes the 2×2 identity matrix.

We will need the following lemma in the last section of the paper.

Lemma 3. *For any integer j we have*

$$S_j(z)S_{j-1}(z) = (z^2-2)S_{j-1}(z)S_{j-2}(z) - S_{j-2}(z)S_{j-3}(z) + z.$$

Proof. We have $S_j(z)S_{j-1}(z) + S_{j-2}(z)S_{j-3}(z)$

$$\begin{aligned} &= (zS_{j-1}(z) - S_{j-2}(z))S_{j-1}(z) + S_{j-2}(z)(zS_{j-2}(z) - S_{j-1}(z)) \\ &= z(S^2_{j-1}(z) + S^2_{j-2}(z)) - 2S_{j-1}(z)S_{j-2}(z). \end{aligned}$$

The lemma follows, since $S^2_{j-1}(z)+S^2_{j-2}(z) = 1+zS_{j-1}(z)S_{j-2}(z)$ by Lemma 1. □

3.2. **Nonabelian reprsentations.** In this subsection we study representations of link groups into $SL_2(\mathbb{C})$. A representation is called nonabelian if its image is a nonabelian subgroup of $SL_2(\mathbb{C})$. Let $\mathcal{L} = J(2m+1, 2n+1)$ and $E_{\mathcal{L}} = S^3 \setminus \mathcal{L}$ the link exterior. By [12] (and [10] also) the link group of $\mathcal{L}$ has a two-generator presentation

$$\pi_1(E_{\mathcal{L}}) = \langle a, b \mid aw = wa \rangle,$$

where $w = (b^{-1}a)^m\big[(ba^{-1})^m ba(b^{-1}a)^m\big]^n$ and a, b are meridians depicted in Figure 1.

Suppose $\rho : \pi_1(E_{\mathcal{L}}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix} \tag{1}$$

where $(u, s_1, s_2) \in (\mathbb{C}^*)^3$ satisfies the matrix equation $\rho(aw) = \rho(wa)$. For any word v in 2 letters a and b, we write $\rho(v) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$. Then, by Riley [15], w_{12} can be written as $w_{21} = uw'_{21}$ for some $w'_{12} \in \mathbb{C}[s_1^{\pm 1}, s_2^{\pm 1}, u]$ and the matrix equation $\rho(aw) = \rho(wa)$ is equivalent to the single equation $w'_{12} = 0$. We call w'_{12} the Riley polynomial of $\mathcal{L}$.

We now compute w'_{12} explicitly. Let $x = \operatorname{tr}\rho(a) = s_1+s_1^{-1}$, $y = \operatorname{tr}\rho(b) = s_2+s_2^{-1}$ and $z = \operatorname{tr}\rho(ab^{-1}) = s_1s_2^{-1} + s_1^{-1}s_2 - u$.

Let $c = (b^{-1}a)^m$ and $d = (ba^{-1})^m ba(b^{-1}a)^m = bc^{-1}ac$. Then $w = cd^n$. Since

$$\rho(b^{-1}a) = \begin{bmatrix} s_1s_2^{-1} & s_2^{-1} \\ -s_1u & s_1^{-1}s_2 - u \end{bmatrix},$$

by Lemma 2 we have $\rho(c) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ where

$$\begin{aligned} c_{11} &= S_m(z) - (s_1^{-1}s_2 - u)S_{m-1}(z), \\ c_{12} &= s_2^{-1}S_{m-1}(z), \\ c_{21} &= -s_1uS_{m-1}(z), \\ c_{22} &= S_m(z) - s_1s_2^{-1}S_{m-1}(z). \end{aligned}$$

By a direct computation we then have $\rho(d) = \rho(bc^{-1}ac) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ where

$$\begin{aligned} d_{11} &= s_1s_2S^2_m(z) - (s_1^2 + s_2^2)S_m(z)S_{m-1}(z) + (s_1s_2 + u)S^2_{m-1}(z), \\ d_{12} &= s_2S^2_m(z) - (s_1 + s_1^{-1})S_m(z)S_{m-1}(z) + s_2^{-1}S^2_{m-1}(z), \\ d_{21} &= u\big(s_1S^2_m(z) - (s_2 + s_2^{-1})S_m(z)S_{m-1}(z) + s_1^{-1}S^2_{m-1}(z)\big), \\ d_{22} &= (s_1^{-1}s_2^{-1} + u)S^2_m(z) - (s_1^{-2} + s_2^{-2})S_m(z)S_{m-1}(z) + s_1^{-1}s_2^{-1}S^2_{m-1}(z). \end{aligned}$$

Let $t = \operatorname{tr}\rho(d)$. From the above computations we have

$$\begin{aligned} t &= (s_1s_2 + s_1^{-1}s_2^{-1} + u)\big(S_m^2(z) + S_{m-1}^2(z)\big) - (s_1^2 + s_1^{-2} + s_2^2 + s_2^{-2})S_m(z)S_{m-1}(z) \\ &= (xy - z)\big(S_m^2(z) + S_{m-1}^2(z)\big) - (x^2 + y^2 - 4)S_m(z)S_{m-1}(z). \end{aligned}$$

Since $w = cd^n$, by Lemma 2 we have

$$\rho(w) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} S_n(t) - d_{22}S_{n-1}(t) & d_{12}S_{n-1}(t) \\ d_{21}S_{n-1}(t) & S_n(t) - d_{11}S_{n-1}(t) \end{bmatrix}.$$

With $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ we obtain

$$\begin{aligned} w_{11} &= c_{11}\big(S_n(t) - d_{22}S_{n-1}(t)\big) + c_{12}d_{21}S_{n-1}(t), \\ w_{21} &= c_{21}\big(S_n(t) - d_{22}S_{n-1}(t)\big) + c_{22}d_{21}S_{n-1}(t). \end{aligned}$$

By direct computations we have $w_{21} = us_1\big(S_m(z)S_{n-1}(t) - S_{m-1}(z)S_n(t)\big)$ and

$$\begin{aligned} w_{11} = &-S_{n-1}(t)\big\{(s_1s_2^{-1} + s_1^{-1}s_2 + s_1^{-1}s_2^{-1} - z)S_m(z) - s_1^{-2}S_{m-1}(z)\big\} \\ &+ S_n(t)\big(S_m(z) + (s_1s_2^{-1} - z)S_{m-1}(z)\big). \end{aligned}$$

Hence, the Riley polynomial of $\mathcal{L} = J(2m+1, 2n+1)$ is

$$w_{21}' = S_m(z)S_{n-1}(t) - S_{m-1}(z)S_n(t).$$

It determines the nonabelian $SL_2(\mathbb{C})$-character variety of $\mathcal{L}$, which is essentially the set of all nonabelian representations $\rho : \pi_1(E_{\mathcal{L}}) \to SL_2(\mathbb{C})$ up to conjugation. Moreover, for any nonabelian representation ρ of the form (1) we have $\rho(w) = \begin{bmatrix} w_{11} & * \\ 0 & (w_{11})^{-1} \end{bmatrix}$ where

$$w_{11} = -S_{n-1}(t)\big\{(s_1^{-1}s_2 + s_1^{-1}s_2^{-1})S_m(z) - s_1^{-2}S_{m-1}(z)\big\} + S_n(t)S_m(z). \tag{2}$$

Let $\overline{w}$ is the word obtained from w by exchanging a and b, namely

$$\overline{w} = (a^{-1}b)^m\big[(ab^{-1})^m ab(a^{-1}b)^m\big]^n.$$

It is easy to see that the equation $aw = wa$ is equivalent to $\overline{w}b = b\overline{w}$. Moreover, for any nonabelian representation ρ of the form (1) we have $\rho(\overline{w}) = \begin{bmatrix} \overline{w}_{11} & 0 \\ * & (\overline{w}_{11})^{-1} \end{bmatrix}$ where

$$\overline{w}_{11} = -S_{n-1}(t)\big\{(s_1s_2^{-1} + s_1^{-1}s_2^{-1})S_m(z) - s_2^{-2}S_{m-1}(z)\big\} + S_n(t)S_m(z). \tag{3}$$

Remark 1. *The above formula for the nonabelian $SL_2(\mathbb{C})$-character variety of the double twist link $\mathcal{L} = J(2m+1, 2n+1)$ was already obtained in* [12] *by a different method. Moreover, it was also shown in* [12] *that the nonabelian character variety of $\mathcal{L}$ is reducible if and only if $m = n$. In which case, it has exactly 2 irreducible components and the canonical component is determined by the equation $t = z$.*

From now on we consider only the double twist link $\mathcal{L}_m = J(2m+1, 2m+1)$, where $m \neq -1, 0$. As mentioned above, the canonical component of the character variety of $\mathcal{L}_m$ is given by the equation $t = z$ where

$$t = (xy - z)\big(S_m^2(z) + S_{m-1}^2(z)\big) - (x^2 + y^2 - 4)S_m(z)S_{m-1}(z). \tag{4}$$

4. Volume of hyperbolic cone-manifolds of $\mathcal{L}_m$

Recall that $E_{\mathcal{L}_m}(\alpha)$ is the cone-manifold of $\mathcal{L}_m$ with cone angles $\alpha_1 = \alpha_2 = \alpha$. There exists an angle $\alpha_{\mathcal{L}_m} \in [\frac{2\pi}{3}, \pi)$ such that $E_{\mathcal{L}_m}(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_{\mathcal{L}_m})$, Euclidean for $\alpha = \alpha_{\mathcal{L}_m}$, and spherical for $\alpha \in (\alpha_{\mathcal{L}_m}, \pi)$.

For $\alpha \in (0, \alpha_{\mathcal{L}_m})$, by the Schlafli formula we have

$$\operatorname{Vol} E_{\mathcal{L}_m}(\alpha) = \int_\alpha^\pi 2\log|w_{11}|\, d\omega$$

where w_{11} is the $(1,1)$-entry of the matrix $\rho(w)$ and $\rho : \pi_1(\mathcal{L}_m) \to SL_2(\mathbb{C})$ is a representation of the form (1) such that the following 3 conditions hold:

(i) $s_1 = s_2 = s = e^{i\omega/2}$,

(ii) the character χ_ρ of ρ lies on the canonical component of the character variety of $\mathcal{L}_m$,

(iii) $|w_{11}| \geq 1$.

We refer the reader to [5, 6] and references therein for the volume formula of hyperbolic cone-manifolds of links using the Schlafli formula.

We now simplify w_{11} for representations ρ of the form (1) satisfying the conditions (i)–(iii). Consider the canonical component $t = z$ of the character variety of $\mathcal{L}_m$. With $s_1 = s_2 = s = e^{i\omega/2}$, equation (2) implies that

$$\begin{aligned} w_{11} &= -S_{m-1}(z)\big\{(1+s^{-2})S_m(z) - s^{-2}S_{m-1}(z)\big\} + S_m^2(z) \\ &= \big(S_m(z) - S_{m-1}(z)\big)\big(S_m(z) - s^{-2}S_{m-1}(z)\big). \end{aligned}$$

Moreover, the equation $t = z$ can be written as

$$(s^2 + s^{-2} + 2 - z)\big(S_m^2(z) + S_{m-1}^2(z)\big) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) = z.$$

This, together with $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$ (by Lemma 2), implies that

$$\begin{aligned} S_m(z)S_{m-1}(z) &= \frac{2z - (s^2 + s^{-2} + 2)}{(z-2)(s^2 + s^{-2} - z)}, \\ S_m^2(z) + S_{m-1}^2(z) &= \frac{z^2 - 2(s^2 + s^{-2})}{(z-2)(s^2 + s^{-2} - z)}. \end{aligned}$$

Then $\big(S_m(z) - S_{m-1}(z)\big)^2 = S_m^2(z) + S_{m-1}^2(z) - 2S_m(z)S_{m-1}(z) = \frac{z-2}{s^2+s^{-2}-z}$ and

$$\begin{aligned} &(S_m(z) - s^2 S_{m-1}(z))(S_m(z) - s^{-2}S_{m-1}(z)) \\ =\ & S_m^2(z) + S_{m-1}^2(z) - (s^2 + s^{-2})S_m(z)S_{m-1}(z) \\ =\ & \frac{s^2 + s^{-2} - z}{z-2}. \end{aligned}$$

It follows that $\big(S_m(z) - S_{m-1}(z)\big)^2\big(S_m(z) - s^2 S_{m-1}(z)\big)\big(S_m(z) - s^{-2}S_{m-1}(z)\big) = 1$ and

$$w_{11}^2 = \big(S_m(z) - S_{m-1}(z)\big)^2\big(S_m(z) - s^{-2}S_{m-1}(z)\big)^2 = \frac{S_m(z) - s^{-2}S_{m-1}(z)}{S_m(z) - s^2 S_{m-1}(z)}.$$

Note that $|S_m(z) - e^{-i\omega}S_{m-1}(z)| \geq |S_m(z) - e^{i\omega}S_{m-1}(z)|$ if and only if $\operatorname{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$. Hence, for $\alpha \in (0, \alpha_{\mathcal{L}_m})$, by the Schlafli formula we have

$$\operatorname{Vol} E_{\mathcal{L}_m}(\alpha) = \int_\alpha^\pi 2\log|w_{11}|\, d\omega = \int_\alpha^\pi \log\left|\frac{S_m(z) - s^{-2}S_{m-1}(z)}{S_m(z) - s^2 S_{m-1}(z)}\right| d\omega$$

where $s = e^{i\omega/2}$ and z, with $\mathrm{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$, satisfy

$$(s^2 + s^{-2} + 2 - z)\big(S_m^2(z) + S_{m-1}^2(z)\big) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) - z = 0.$$

This completes the proof of Theorem 1.

5. The A-polynomial 2-tuple of $\mathcal{L}_m$

The canonical longitudes corresponding to the meridians a and b of $J(2m+1, 2n+1)$ are respectively $\lambda_a = wa^{-2n}$ and $\lambda_b = \overline{w}b^{-2n}$, where $\overline{w}$ is the word obtained from w by exchanging a and b.

Consider the canonical component $t = z$ of the character variety of $\mathcal{L}_m = J(2m+1, 2m+1)$. To compute the A-polynomial 2-tuple for this component, we first consider a representation $\rho : \pi_1(\mathcal{L}_m) \to SL_2(\mathbb{C})$ of the form (1) and find a polynomial relating s_1 and w_{11} on the subvariety of $t = z$ defined by $s_2^2 = (\overline{w}_{11})^2 = 1$. Recall from Subsection 3.2 that w_{11} and $\overline{w}_{11}$ are upper left entries of $\rho(w)$ and $\rho(\overline{w})$ respectively.

With $t = z$ and $s_2 = 1$, by equations (2) and (3) we have

$$\begin{aligned} w_{11} &= -S_{m-1}(m)\big\{2s_1^{-1}S_m(z) - s_1^{-2}S_{m-1}(z)\big\} + S_m^2(z) \\ &= (S_m(z) - s_1^{-1}S_{m-1}(z))^2 \end{aligned}$$

and

$$\begin{aligned} \overline{w}_{11} &= -S_{m-1}(z)\big\{(s_1 + s_1^{-1})S_m(z) - S_{m-1}(z)\big\} + S_m^2(z). \\ &= (S_m(z) - s_1S_{m-1}(z))(S_m(z) - s_1^{-1}S_{m-1}(z)). \end{aligned}$$

Moreover, since $S_m^2(z)+S_{m-1}^2(z) = 1+zS_m(z)S_{m-1}(z)$, the equation $t = z$ becomes

$$\begin{aligned} 0 &= (2x - z)(S_m^2(z) + S_{m-1}^2(z)) - x^2S_m(z)S_{m-1}(z) - z \\ &= (2x - z)(1 + zS_m(z)S_{m-1}(z)) - x^2S_m(z)S_{m-1}(z) - z \\ &= (x - z)\left(2 + (z - x)S_m(z)S_{m-1}(z)\right). \end{aligned}$$

Suppose $z - x = 0$. Then $\overline{w}_{11} = -S_{m-1}(z)\big\{zS_m(z) - S_{m-1}(z)\big\} + S_m^2(z) = 1$ and

$$w_{11} = \big(S_m(x) - s^{-1}S_{m-1}(x)\big)^2 = s^{2m}.$$

Here we use the fact that $S_j(s_1 + s_1^{-1}) = (s_1^{j+1} - s_1^{-j-1})/(s_1 - s_1^{-1})$ for all integers j.

Suppose $2 + (z - x)S_m(z)S_{m-1}(z) = 0$. This is equivalent to

$$\big(S_m(z) - s_1S_{m-1}(z)\big)\big(S_m(z) - s_1^{-1}S_{m-1}(z)\big) = -1, \tag{5}$$

since $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$. It follows that $\overline{w}_{11} = -1$ and

$$w_{11} = \big(S_m(z) - s_1^{-1}S_{m-1}(z)\big)^2 = -\frac{S_m(z) - s_1^{-1}S_{m-1}(z)}{S_m(z) - s_1S_{m-1}(z)}.$$

Hence $S_m(z) = rS_{m-1}(z)$ where $r = \frac{s_1w_{11}+s_1^{-1}}{w_{11}+1}$. We have

$$1 = S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = S_{m-1}^2(z)(1 - zr + r^2),$$

which implies that $S_{m-1}^2(z) = (1 - zr + r^2)^{-1}$. Equation (5) then becomes

$$-1 = S_{m-1}^2(z)(r - s_1)(r - s_1^{-1}) = (r - s_1)(r - s_1^{-1})/(1 - zr + r^2).$$

By solving for z from the above equation, we obtain

$$z = 2\left(r + \frac{1}{r}\right) - (s_1 + s_1^{-1}) = 2\left(\frac{s_1 w_{11} + s_1^{-1}}{w_{11}+1} + \frac{w_{11}+1}{s_1 w_{11} + s_1^{-1}}\right) - (s_1 + s_1^{-1}).$$

Now, by plugging this expression of z into the equation $2+(z-x)S_m(z)S_{m-1}(z) = 0$ we obtain a polynomial (depending on m) relating s_1 and w_{11}. Moreover, we can find a recurrence relation between these polynomials as follows.

Let $P_m(x,z) = 2+(z-x)S_m(z)S_{m-1}(z)$. By Lemma 3 we have $S_m(z)S_{m-1}(z) = (z^2-2)S_{m-1}(z)S_{m-2}(z) - S_{m-2}(z)S_{m-3}(z) + z$. This implies that

$$\begin{aligned} P_m &= 2 + (z^2-2)(P_{m-1}-2) - (P_{m-2}-2) + z(z-x) \\ &= (z^2-2)P_{m-1} - P_{m-2} + 8 - z(z+x). \end{aligned}$$

Let $Q_m(s_1, w_{11}) = s_1^2(w_{11}+1)^2(s_1^2 w_{11}+1)^2 P_m(x,z)$. By replacing

$$z = 2\left(\frac{s_1 w_{11} + s_1^{-1}}{w_{11}+1} + \frac{w_{11}+1}{s_1 w_{11} + s_1^{-1}}\right) - (s_1 + s_1^{-1})$$

into the above recurrence relation for P_m we have

$$Q_m = \alpha Q_{m-1} - Q_{m-2} + \beta$$

where

$$\begin{aligned} \alpha &= (s_1^8 + s_1^4)w_{11}^4 + (-2s_1^8 + 6s_1^6 + 6s_1^4 - 2s_1^2)w_{11}^3 + (s_1^8 - 12s_1^6 + 34s_1^4 - 12s_1^2 + 1)w_{11}^2 \\ &\qquad + (-2s_1^6 + 6s_1^4 + 6s_1^2 - 2)w_{11} + s_1^4 + 1, \\ \beta &= -2(s_1^2-1)^2\left(s_1^4 w_{11}^4 - (s_1^4 + s_1^2)w_{11}^3 - 6s_1^2 w_{11}^2 - (s_1^2+1)w_{11} + 1\right). \end{aligned}$$

We have shown that $(\overline{w}_{11})^2 = 1$ and $(w_{11} - s_1^{2m})Q(s_1, w_{11}) = 0$ when both $t = z$ and $s_2 = 1$ occur. The same holds true when both $t = z$ and $s_2 = -1$ occur. This implies that $(w_{11} - s_1^{2m})Q(s_1, w_{11}) = 0$ when both $t = z$ and $s_2^2 = (\overline{w}_{11})^2 = 1$ occur.

Similarly, we have $(\overline{w}_{11} - s_2^{2m})Q(s_2, \overline{w}_{11}) = 0$ when both $t = z$ and $s_1^2 = (w_{11})^2 = 1$ occur. Since the canonical longitudes corresponding to the meridians a and b of $\mathcal{L}_m = J(2m+1, 2m+1)$ are respectively $\lambda_a = wa^{-2m}$ and $\lambda_b = \overline{w}b^{-2m}$, we conclude that the A-polynomial 2-tuple corresponding to the canonical component of the character variety of $\mathcal{L}_m$ is $[A(M,L), A(M,L)]$ where $A(M,L) = (L-1)Q_m(M, LM^{2m})$.

This completes the proof of Theorem 2.

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