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ON THE EMBEDDING OF THE FREE LATTICE OF RANK 3 IN THE LATTICE FREELY GENERATED BY THREE COMPLETELY RIGHT MODULAR ELEMENTS

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Abstract. We consider the lattice freely generated by three elements, which are right modular and dually right modular simultaneously. We prove that this lattice contains the free lattice of rank 3 as a sublattice.

Keywords: right modular element, dually right modular element, completely right modular element, free lattice.

1. Introduction

Freely generated lattices, whose generators subject to some conditions, are classical objects of the lattice theory (e.g., [1]). The most popular conditions are properties close to modularity. We follow the modern terminology from [2] to define these properties.

Definition 1. *An element a of a lattice L is called* left modular *if*

$$\forall x, y \in L\colon x < y \to x \vee (a \wedge y) = (x \vee a) \wedge y.$$

It is easy to verify the left modularity is a self-dual property.

Definition 2. *An element a of a lattice L is called* right modular *if*

$$\forall x, y \in L\colon x < a \to x \vee (y \wedge a) = (x \vee y) \wedge a.$$

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Definition 3. *An element of a lattice is called* dually right modular *if it is right modular in the dual lattice.*

For short, we will use the following definition.

Definition 4. *An element of a lattice is called* completely right modular *if it is both right modular and dually right modular*

The lattice freely generated by elements a, b, and c satisfying the condition $b \vee (a \wedge c) = c \vee (a \wedge b)$ and that a is a left modular element of this lattice is finite and has 17 elements ([3], Lemma 2.3). In [4] it was proved that the lattice freely generated by three left modular elements (without any defining relations) is isomorphic to the free modular lattice of rank 3. All combinations of the above-defined properties of generators of a 3-generated lattice which ensure the modularity of this lattice were found in [5]. It means that these conditions imply the finiteness the lattice. Note that each such combination contains at least one left modular generator. However, it is not known whether the lattice without left modular generators is finite.

2. Main result

Theorem 1. *The lattice freely generated by three completely right modular elements is infinite and contains the free lattice of rank 3 as a sublattice.*

The free lattice of rank 3 is well known to be infinite. We denote it as F_3. Moreover, it contains the free lattice generated by a countable set as a sublattice. Figure 1 shows a sketch of F_3 generated by three elements x, y, and z. Generators, their pairwise meets and joins, maximal and minimal elements are shown only (a more detailed sketch of the free lattice of rank 3 can be found in [1], p. 497).

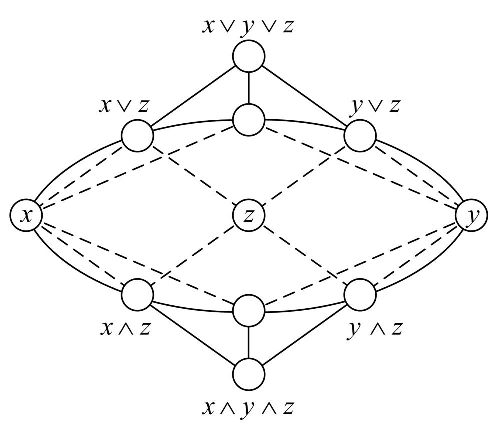

Fig. 1

In the same time, the free modular lattice of rank 3 M is finite. It is shown on Figure 2.

To prove the Theorem 1 we construct an auxiliary lattice L generated by three completely right modular elements and containing F_3 as a sublattice. To do this we replace in the lattice M the diamond $\{s, a_1, b_1, c_1, t\}$ with the free lattice F_3. The elements a_1, b_1, and c_1 coincide with generators of F_3. The elements s and t coincide with the least and the greatest elements of F_3, respectively (see Figure 3).

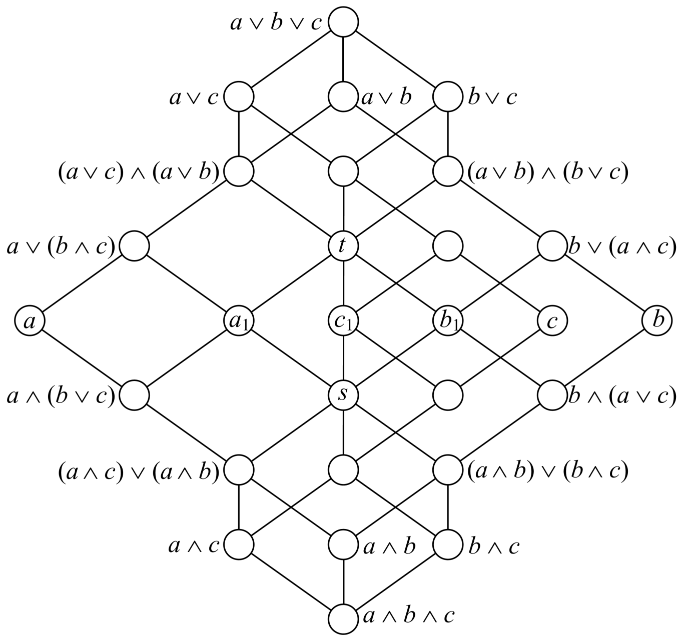

Fig. 2

The order relation both in F_3 and in M does not change. The order relation between an element from M and an element from F_3 is defined by the diagram represented on Figure 3. This poset is easy seen to be a lattice.

For example, if $x \in \{a \vee (b \wedge c), a, a \wedge (b \vee c)\}$ and $y \in F_3$ i.e. $s \leq y \leq t$, then for $y \geq a_1$ we have $(a \vee (b \wedge c)) \wedge y = a_1$, $a \wedge y = (a \wedge (b \vee c)) \wedge y = a \wedge (b \vee c)$ and for $y \ngeq a_1$ we have $(a \vee (b \wedge c)) \wedge y = a_1 \wedge y, a \wedge y = (a \wedge (b \vee c)) \wedge y = (a \wedge c) \vee (a \wedge b)$. Besides, $(a \vee (b \wedge c)) \wedge (b \vee (a \wedge c)) = a_1 \wedge b_1$. For the other elements and operation $\vee$ corresponding equalities can be discharged from considerations of a-b-c symmetry and self-duality of M and F_3.

Lemma 1. *The elements a, b, and c are completely right modular elements of the lattice L.*

Proof. By the symmetry of the elements a, b, and c in the lattice L and by the duality of the lattice L, it is sufficient to check that the element a is right modular.

By the definition, it is sufficient to verify the implication

$$\forall x, y \in L\colon x < a \to x \vee (y \wedge a) = (x \vee y) \wedge a.$$

The equality holds if x and y are comparable. If $x < s$, then the equality holds for every y because the lattice M is modular.

If $x = a \wedge (b \vee c)$, then the verification is required only for $y \in [b \wedge c; b \vee c]$. In this case $y \wedge a \leq a \wedge (b \vee c) = x$ and $x \vee (y \wedge a) = x$. And also $x \vee y \in [a_1; b \vee c]$, therefore $(x \vee y) \wedge a = a \wedge (b \vee c) = x$. □

Denote by F the lattice freely generated by three completely right modular elements x, y, and z. There is a homomorphism φ from F into L such that $\varphi(x) = a$, $\varphi(y) = b$, and $\varphi(z) = c$. Let x_1, y_1, and z_1 be arbitrary elements

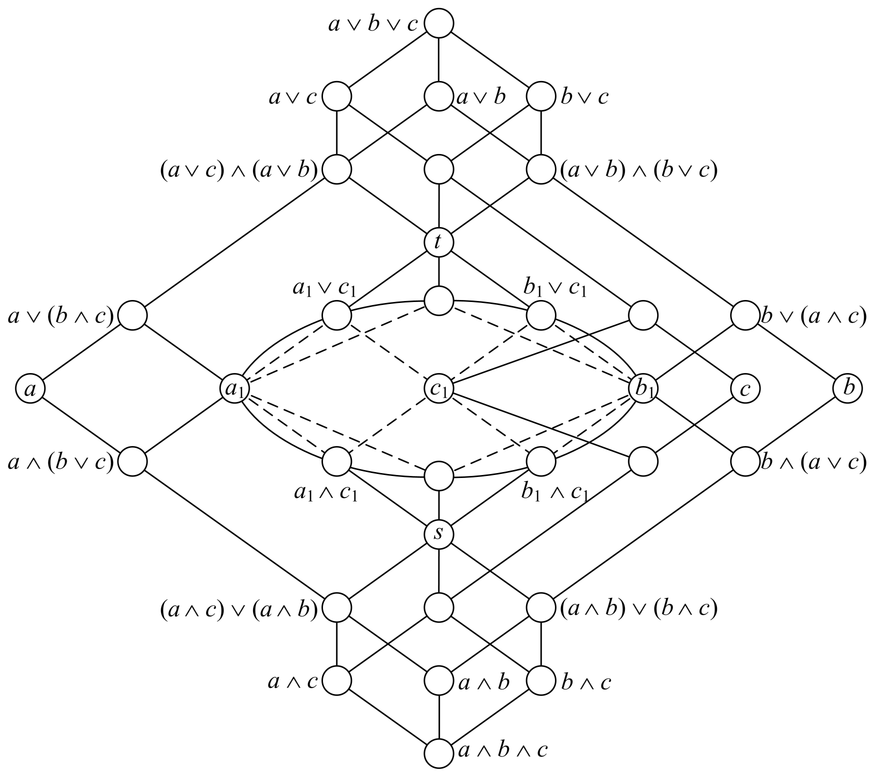

Fig. 3

from $\varphi^{-1}(a_1)$, $\varphi^{-1}(b_1)$, and $\varphi^{-1}(c_1)$ respectively. Consider the sublattice G of the lattice F generated by x_1, y_1, and z_1.

On the one hand, the restriction of φ to G is a homomorphism from G into F_3. On the other hand, there is a homomorphism ψ from F_3 into G such that $\psi(a_1) = x_1$, $\psi(b_1) = y_1$, and $\psi(c_1) = z_1$ because G is generated by three elements.

It is clear that $\varphi\psi(a_1) = a_1$, $\varphi\psi(b_1) = b_1$, and $\varphi\psi(c_1) = c_1$, therefore $\varphi\psi$ is the identity automorphism the lattice F_3. Then it is easy to see that the homomorphisms φ and ψ are isomorphisms between F_3 and G. Therefore, the lattice F contains the free lattice of rank 3 as a sublattice. In particular, F is infinite.

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