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ON PARAMETERS AND DISCRETENESS OF MASKIT SUBGROUPS IN $\mathrm{PSL}(2,\mathbb{C})$

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Abstract. In 1989 B. Maskit formulated the following problem. Let G be the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ generated by the elements f and g, where f has two fixed points in $\overline{\mathbb{C}}$, and g maps one fixed point of f onto the other; when is G discrete? Partial solutions of the problem were found by B. Maskit and E. Klimenko, but complete solution is not known. In this paper, the trace parameters for such groups are considered. Properties of the parameters are used to find new necessary and sufficient discreteness conditions for the groups.

Keywords: discrete group, hyperbolic geometry.

1. Introduction

Let $\mathbb{H}^3$ be the Poincaré half-space model of the three-dimensional hyperbolic space. Identify the set $\partial\mathbb{H}^3$ with the extended complex plane $\overline{\mathbb{C}}$. The group $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$ as the group of all orientation-preserving isometries, and on $\overline{\mathbb{C}}$ as the group of all linear fractional transformations.

Let G be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. A discrete group G is a discrete set in the matrix quotient-topology. An elementary group G has a finite G-orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$. A non-elementary group G has only infinite G-orbits in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$. All elementary discrete groups were classified (see [1, §5.1]). As shown in [2], a non-elementary group G is discrete if and only if any two elements $f, g \in G$ generate a discrete group. The

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problem of classification of all discrete two-generated subgroups of $\mathrm{PSL}(2,\mathbb{C})$ is far from solution. But there are many results on this topic. In [3] all two-generated discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ were described. Discreteness criterions for groups with real parameters was found in [4, 5]. The parameters of the group generated by $f, g \in \mathrm{PSL}(2,\mathbb{C})$ are the following complex numbers: $\mathrm{tr}(fgf^{-1}g^{-1}) - 2$, $\mathrm{tr}^2(f) - 4$ and $\mathrm{tr}^2(g) - 4$.

A subgroup G of $\mathrm{PSL}(2,\mathbb{C})$ generated by two elements f and g is said to be a *Maskit group* if f has two fixed points in $\overline{\mathbb{C}}$ and g maps one fixed point of f onto the other. Discreteness conditions for these groups were firstly investigated by B. Maskit [6] in 1989. Then E. Klimenko [7] continued the research. But the discreteness criterion for groups of such kind is still unknown. The present work contains new necessary and sufficient conditions, which arise from relations between parameters of different Maskit groups.

F. Gehring and G. Martin [8] showed that Maskit groups with the first generator of order 6 and the second one of order $n \geq 3$ have the smallest distances between axes of generators over all non-elementary discrete groups with two generators of orders 6 and n. An axis of an element is the geodesic line in $\mathbb{H}^3$, which is invariant under the action of the element. H. Sato [9] established that one of other Maskit groups with the first generator of order 6 is the example of Jørgensen group. More precisely, it is a non-elementary discrete group with the pair of generators, for which the Jørgensen's inequality becomes equality. All discrete Maskit groups with the first generator of order 4 are Gehring — Martin — Tan groups and Tan groups (see [10]).

2. Definitions

Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, denote

$$\mathrm{tr}(M) = a + d \quad \text{and} \quad \|M\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

Let $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M \neq \pm\mathrm{I}$. An element M is called *elliptic* if $\mathrm{tr}^2(M) \in [0;4)$, *parabolic* if $\mathrm{tr}^2(M) = 4$, *loxodromic* if $\mathrm{tr}^2(M) \in \mathbb{C}\backslash[0;4]$. An elliptic element is called *geometrically primitive* if $\mathrm{tr}^2(M) = 4\cos^2(\pi/n)$ for some $n \in \mathbb{N}$, *geometrically non-primitive* if $\mathrm{tr}^2(M) = 4\cos^2(\pi k/n)$ for some $k, n \in \mathbb{N}$, such that $k \in (1, n/2)$ and the greatest common divisor of k and n equals 1. An loxodromic element is called *hyperbolic* if $\mathrm{tr}^2(A) \in [4, +\infty)$, *strictly loxodromic* otherwise. Endow the group $\mathrm{SL}(2,\mathbb{C})$ with the topology of the norm $\|\cdot\|$.

Recall that $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm\mathrm{I}\}$. An element of this group is called *elliptic*, *geometrically primitive* or *non-primitive*, *parabolic*, *loxodromic*, *hyperbolic* or *strictly loxodromic*, if its representative in the group $\mathrm{SL}(2,\mathbb{C})$ has this type. It is easy to prove that only geometrically primitive and geometrically non-primitive elements have finite orders in the group $\mathrm{PSL}(2,\mathbb{C})$. In what follows, we do not distinguish between a matrix $M \in \mathrm{SL}(2,\mathbb{C})$ and its equivalence class $\{\pm M\} \in \mathrm{PSL}(2,\mathbb{C})$.

A subgroup G of $\mathrm{PSL}(2,\mathbb{C})$ is said to be *discrete* if it is a discrete set in the quotient topology.

It is well known [1, §4.1] that the group of all orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$ is isomorphic to the group $\mathrm{PSL}(2,\mathbb{C})$. It will be convenient to use the Poincaré half-space model of $\mathbb{H}^3$, i.e. the set $\{(z,t) : z \in \mathbb{C},\ t > 0\}$ with the metric defined by the line element $ds^2 = (|dz|^2 + dt^2)/t^2$. In this model, the boundary of $\mathbb{H}^3$ is the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Geodesic lines are vertical rays and arcs of circles perpendicular to $\mathbb{C}$. Geodesic hyperplanes are vertical half-planes and hemispheres with center in $\mathbb{C}$. An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ PSL$(2,\mathbb{C})$ acts on $\mathbb{H}^3$ by the following rule

$$g(z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\overline{c}t^2}{|cz+d|^2 + |c|^2 t^2}, \frac{t}{|cz+d|^2 + |c|^2 t^2} \right).$$

It acts on $\overline{\mathbb{C}}$ as the linear-fractional transformation

$$g(z) = \frac{az+b}{cz+d}.$$

A parabolic element has one fixed point in $\overline{\mathbb{C}}$. A non-parabolic (non-trivial) element has two fixed points in $\overline{\mathbb{C}}$. Given a non-parabolic (non-trivial) element $g \in$ PSL$(2,\mathbb{C})$, the geodesic in $\mathbb{H}^3$ joining its fixed points in $\overline{\mathbb{C}}$ is called the *axis* of g and is denoted by ℓ_g. An elliptic element g is a rotation about ℓ_g by an angle $\varphi \in (-\pi;\pi]$. In this case, the axis ℓ_g consists of fixed points of g in $\mathbb{H}^3$. A loxodromic element g is the composition of a translation along ℓ_g by an amount $\tau \in (0,+\infty)$ and a rotation about ℓ_g by an angle $\varphi \in (-\pi;\pi]$. In this case, the axis ℓ_g is invariant under the action of g.

A subgroup G of PSL$(2,\mathbb{C})$ is called *elementary* if there exists a finite G-orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$.

3. Parameters of Maskit Groups

Let $f, g \in$ PSL$(2,\mathbb{C})$. The following values

$$\gamma(f,g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2, \quad \beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4,$$

do not depend on the choice of matrices in SL$(2,\mathbb{C})$ representing f and g. Note that $\gamma(f,g) = 0$ if and only if f and g have a common fixed point in $\overline{\mathbb{C}}$ (see [1, Theorem 4.3.5]). Denote by $\langle f, g\rangle$ the group generated by f and g. Three complex numbers $\gamma = \gamma(f,g)$, $\beta = \beta(f)$, $\beta' = \beta(g)$ are called the *parameters* of the group $\langle f, g\rangle$. The ordered triple (γ, β, β') is denoted by par$\langle f, g\rangle$. Obviously, the parameters of a group depend on the choice of generators.

As shown in [11], for any ordered triple (z_1, z_2, z_3), where $z_1 \in \mathbb{C}\setminus\{0\}$ and $z_2, z_3 \in \mathbb{C}$, there exists a subgroup $\langle f, g\rangle$ of PSL$(2,\mathbb{C})$, such that par$\langle f, g\rangle = (z_1, z_2, z_3)$. Moreover, for any other subgroup $\langle \tilde{f}, \tilde{g}\rangle$ of PSL$(2,\mathbb{C})$ with par$\langle \tilde{f}, \tilde{g}\rangle = (z_1, z_2, z_3)$, there exists an element in PSL$(2,\mathbb{C})$ which conjugates $\tilde{f}$ to f or f^{-1} and conjugates $\tilde{g}$ to g or g^{-1}.

Let us formulate the definition of Maskit group in terms of parameters.

Proposition 1. [8] *A subgroup $\langle f, g\rangle$ of* PSL$(2,\mathbb{C})$ *is Maskit group if and only if* par$\langle f, g\rangle = (\beta, \beta, \beta')$ *and $\beta \neq 0$.*

Proof. See the proof of Lemma 2.31 in [8]. □

By Proposition 1, a family of all Maskit groups are parameterized by two numbers $\beta \in \mathbb{C} \setminus \{0\}$ and $\beta' \in \mathbb{C}$. There exist relationships between different Maskit groups. They are formulated in the following two propositions.

Proposition 2. *Let* $\langle f,g\rangle$ *be a subgroup of* $\mathrm{PSL}(2,\mathbb{C})$*,* $\mathrm{par}\langle f,g\rangle = (\beta,\beta,\beta')$*,* $\beta = 4\sinh^2\left(\frac{\tau+\varphi i}{2}\right) \neq 0$ *and* $k\in\mathbb{Z}$*. Then* $\mathrm{par}\langle f, gf^k\rangle = \big(\beta,\beta,e^{\pm k\cdot(\tau+\varphi i)}\cdot(\beta'+4)-4\big)$*.*

Proof. Note that

$$\gamma(f,gf^k) = \mathrm{tr}(fgf^kf^{-1}f^{-k}g^{-1}) - 2 = \mathrm{tr}(fgf^{-1}g^{-1}) - 2 = \gamma(f,g) = \beta.$$

Let us calculate the value $\beta(gf^k)$. Up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$, we may assume that f has fixed points at $0,\infty\in\overline{\mathbb{C}}$ and that $g(0)=\infty$. Then

$$f = \begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix} \quad\text{and}\quad g = \begin{pmatrix} a & b\\ -b^{-1} & 0\end{pmatrix},$$

where $\varepsilon\in\{e^{(\tau+\varphi i)/2}, -e^{(\tau+\varphi i)/2}, e^{-(\tau+\varphi i)/2}, -e^{-(\tau+\varphi i)/2}\}$ and $a\in\mathbb{C}$, $b\in\mathbb{C}\setminus\{0\}$. Hence,

$$gf^k = \begin{pmatrix} a & b\\ -b^{-1} & 0\end{pmatrix}\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}^k = \begin{pmatrix} a & b\\ -b^{-1} & 0\end{pmatrix}\begin{pmatrix}\varepsilon^k & 0\\ 0 & \varepsilon^{-k}\end{pmatrix} = \begin{pmatrix} a\varepsilon^k & b\varepsilon^{-k}\\ -b^{-1}\varepsilon^k & 0\end{pmatrix}$$

and

$$\beta(gf^k) = \mathrm{tr}^2(gf^k) - 4 = (a\varepsilon^k)^2 - 4 = \varepsilon^{2k}a^2 - 4 = e^{\pm k\cdot(\tau+\varphi i)}\cdot(\beta'+4) - 4.$$

Thus $\mathrm{par}\langle f,gf^k\rangle = \big(\beta,\beta,e^{\pm k\cdot(\tau+\varphi i)}\cdot(\beta'+4)-4\big)$. □

Proposition 3. *Let* $\langle f,g\rangle$ *be a subgroup of* $\mathrm{PSL}(2,\mathbb{C})$*,* $\mathrm{par}\langle f,g\rangle = (\beta_1,\beta_1,\beta')$*,* $\beta_1 = 4\sinh^2\left(\frac{\tau+\varphi i}{2}\right)\neq 0$*, and* $\beta_k = 4\sinh^2\left(\frac{k\cdot(\tau+\varphi i)}{2}\right)$*. Then for any* $k\in\mathbb{Z}$ *such that* k *is not divisible by the order of* f*, it follows that* $\mathrm{par}\langle f^k, g\rangle = (\beta_k,\beta_k,\beta')$*.*

Proof. Since f^k and f have the same fixed points in $\overline{\mathbb{C}}$ and $\gamma(f,g)=\beta(f)$, g maps one fixed point of f^k onto the other. By Proposition 1,

$$\gamma(f^k,g) = \beta(f^k).$$

Up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$, we may assume that f has fixed points at $0,\infty\in\overline{\mathbb{C}}$. Therefore $f = \begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}$, where $\varepsilon\in\{e^{(\tau+\varphi i)/2}, -e^{(\tau+\varphi i)/2}, e^{-(\tau+\varphi i)/2}, -e^{-(\tau+\varphi i)/2}\}$, and $f^k = \begin{pmatrix}\varepsilon^k & 0\\ 0 & \varepsilon^{-k}\end{pmatrix}$. Hence,

$$\beta(f^k) = \mathrm{tr}^2(f^k) - 4 = (\varepsilon^k+\varepsilon^{-k})^2 - 4 = (\varepsilon^k-\varepsilon^{-k})^2 = 4\sinh^2\left(\frac{k\cdot(\tau+\varphi i)}{2}\right) = \beta_k.$$

Thus $\mathrm{par}\langle f^k,g\rangle = (\beta_k,\beta_k,\beta')$. □

4. Distance and Angle Between Axes of Generators in Maskit Groups

Let ℓ_1, ℓ_2 be geodesics in $\mathbb{H}^3$ with disjoint pairs of ends in $\overline{\mathbb{C}}$ and ℓ be its common perpendicular. The distance between ℓ_1 and ℓ_2 is defined as in any metric space. The angle between ℓ_1 and ℓ_2 is the measure of the dihedral angle between the hyperplane containing ℓ, ℓ_1 and the hyperplane containing ℓ, ℓ_2.

By definition, the first generator of Maskit group is non-parabolic and generators do not share a fixed point in $\overline{\mathbb{C}}$. The following proposition is devoted to the case when the second generator is also non-parabolic. In this case, there exist formulas for the distance and the angle between axes of generators through the translation length and the rotation angle of the second generator.

Proposition 4. *Let $\langle f, g\rangle$ be a Maskit group, $\delta(f,g)$ and $\theta(f,g)$ be the distance and the angle between axes ℓ_f and ℓ_g. Let g be a non-parabolic element with the translation length τ and the rotation angle φ. Then following formulas hold*

$$\cosh\delta(f,g) = \frac{\cosh(\tau/2)}{\sqrt{\sinh^2(\tau/2)+\sin^2(\varphi/2)}}, \quad \cos\theta(f,g) = \frac{\sinh(\tau/2)}{\sqrt{\sinh^2(\tau/2)+\sin^2(\varphi/2)}}.$$

Proof. The triple of parameters for the group $\langle f, g\rangle$ has the form (β,β,β'), where $\beta \in \mathbb{C}\setminus\{0\}$ and $\beta' = 4\sinh^2\left(\frac{\tau+\varphi i}{2}\right) \neq 0$. It follows from Lemma 4.4 [8] that

$$\cosh\delta(f,g) = \frac{1}{\sqrt{2}}\cdot\sqrt{1+\left|\frac{4+\beta'}{\beta'}\right|+\left|\frac{4}{\beta'}\right|}, \quad \cos\theta(f,g) = \frac{1}{\sqrt{2}}\cdot\sqrt{1+\left|\frac{4+\beta'}{\beta'}\right|-\left|\frac{4}{\beta'}\right|}.$$

By inserting value of β' in radicand expressions and using the hyperbolic identity, we have

$$1+\left|\frac{4+\beta'}{\beta'}\right|\pm\left|\frac{4}{\beta'}\right| = 1+\frac{\left|4+4\sinh^2\left(\frac{\tau+\varphi i}{2}\right)\right|\pm 4}{\left|4\sinh^2\left(\frac{\tau+\varphi i}{2}\right)\right|} = 1+\frac{\left|\cosh^2\left(\frac{\tau+\varphi i}{2}\right)\right|\pm 1}{\left|\sinh^2\left(\frac{\tau+\varphi i}{2}\right)\right|}.$$

Note that

$$\left|\cosh^2\left(\frac{\tau+\varphi i}{2}\right)\right| = \cosh^2(\tau/2)-\sin^2(\varphi/2)$$

and

$$\left|\sinh^2\left(\frac{\tau+\varphi i}{2}\right)\right| = \sinh^2(\tau/2)+\sin^2(\varphi/2),$$

whence

$$1+\frac{\left|\cosh^2\left(\frac{\tau+\varphi i}{2}\right)\right|\pm 1}{\left|\sinh^2\left(\frac{\tau+\varphi i}{2}\right)\right|} = \frac{\cosh^2(\tau/2)+\sinh^2(\tau/2)\pm 1}{\sinh^2(\tau/2)+\sin^2(\varphi/2)}.$$

Thus,

$$\cosh\delta(f,g) = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\cosh^2(\tau/2)+\sinh^2(\tau/2)+1}{\sinh^2(\tau/2)+\sin^2(\varphi/2)}} = \frac{\cosh(\tau/2)}{\sqrt{\sinh^2(\tau/2)+\sin^2(\varphi/2)}}$$

and

$$\cos\theta(f,g) = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\cosh^2(\tau/2)+\sinh^2(\tau/2)-1}{\sinh^2(\tau/2)+\sin^2(\varphi/2)}} = \frac{\sinh(\tau/2)}{\sqrt{\sinh^2(\tau/2)+\sin^2(\varphi/2)}}.$$

Proposition is proved. □

Corollary 1. *Let $\langle f, g\rangle$ be a Maskit group, $\delta(f,g)$ and $\theta(f,g)$ be the distance and the angle between axes ℓ_f and ℓ_g. Let g be an elliptic element with the rotation angle φ. Then following formulas hold*

$$\cosh\delta(f,g) = \frac{1}{\sin(\varphi/2)} \quad \text{and} \quad \theta(f,g) = \pi/2.$$

Corollary 2. *Let $\langle f, g\rangle$ be a Maskit group, $\delta(f,g)$ and $\theta(f,g)$ be the distance and the angle between axes ℓ_f and ℓ_g. Let g be an hyperbolic element with translation length τ. Then following formulas hold*

$$\cosh\delta(f,g) = \coth(\tau/2) \quad \text{and} \quad \theta(f,g) = 0.$$

5. Discreteness of Maskit Groups

Lemma 1. *Let* (z_1, z_1, z), $z_1 = 4\sinh^2\left(\frac{\tau+\varphi i}{2}\right) \neq 0$, $z \in \mathbb{C}$ *and* $k \in \mathbb{Z}$. *If* (z_1, z_1, z) *is an ordered triple of parameters of a discrete (non-discrete) group, then*

$$\left(z_1, z_1, e^{\pm k\cdot(\tau+\varphi i)} \cdot (z+4) - 4\right)$$

is an ordered triple of parameters of a discrete (respectively, non-discrete) group for any k.

Proof. Suppose that $\langle f, g\rangle$ is a discrete (non-discrete) subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{par}\langle f, g\rangle = (z_1, z_1, z)$. Then $\langle f, gf^k\rangle$ is a discrete (non-discrete) group and, by Proposition 2, $\mathrm{par}\langle f, gf^k\rangle = \left(z_1, z_1, e^{\pm k\cdot(\tau+\varphi i)} \cdot (z+4) - 4\right)$. $\square$

Lemma 2. *Let* (z_1, z_1, z), $z_1 = 4\sinh^2\left(\frac{\tau+i\varphi}{2}\right) \neq 0$, $z \in \mathbb{C}$ *and* $k \in \mathbb{Z}$ *such that* $z_k = 4\sinh^2\left(\frac{k\cdot(\tau+\varphi i)}{2}\right) \neq 0$. *The following properties hold.*

1. *If* (z_1, z_1, z) *is an ordered triple of parameters of a discrete group, then* (z_k, z_k, z) *is an ordered triple of parameters of a discrete group for any* k.
2. *If there exists* k *such that* (z_k, z_k, z) *is an ordered triple of parameters of a non-discrete group, then* (z_1, z_1, z) *is an ordered triple of parameters of a non-discrete group.*

Proof. Suppose that $\langle f, g\rangle$ is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{par}\langle f, g\rangle = (z_1, z_1, z)$. Since $\beta(f^k) = z_k \neq 0$, an element f^k is not trivial and the exponent k is not divisible by the order of f. By Proposition 3, $\mathrm{par}\langle f^k, g\rangle = (z_k, z_k, z)$. To conclude the proof, it remains to note that any subgroup of a discrete group is discrete. $\square$

Proposition 5. [6] *Let* $\langle f, g\rangle$ *be a discrete Maskit group. Then either* f *is an elliptic element of order 2, 3, 4, or 6, or* g *is an elliptic element of order 2.*

Corollary 3. *Let* $\langle f, g\rangle$ *be a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (\beta, \beta, \beta')$, $\beta \neq 0$. *Then either* $\beta \in \{-4, -3, -2, -1\}$, *or* $\beta' = -4$.

As shown in [6], if $\langle f, g\rangle$ is a Maskit group and g is an elliptic element of order 2, then $\langle f, g\rangle$ is an elementary group. The group $\langle f, g\rangle$ is discrete if and only if f is either a loxodromic element, or an elliptic element of finite order.

Theorem 1. [6, 7] *Let* $\langle f, g\rangle$ *be a Maskit group and* f *be an elliptic element of order* 2, 3, 4, *or* 6.

1. *If* g *is a geometrically primitive elliptic, parabolic, or hyperbolic element, then* $\langle f, g\rangle$ *is a discrete group.*
2. *If* g *is a geometrically non-primitive elliptic element, then* $\langle f, g\rangle$ *is a non-discrete group.*

Corollary 4. *Let* $\langle f, g\rangle$ *be a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (\beta, \beta, \beta')$, $\beta \in \{-4, -3, -2, -1\}$, $\beta' \in [-4; +\infty)$. *The group* $\langle f, g\rangle$ *is discrete if and only if* $\beta' \in \mathfrak{D} \cup [0, +\infty)$, *where*

$$\mathfrak{D} = \left\{ -4\sin^2(\pi/m) \;\middle|\; m \in \mathbb{Z},\ m \geq 2\right\}. \tag{$*$}$$

Necessary and discreteness conditions for Maskit groups $\langle f, g\rangle$ with elliptic generator f of order 2, 3, 4, or 6, and, in particular, with strictly loxodromic generator g are formulated below.

Theorem 2. *Let $\langle f,g\rangle$ be a subgroup of* PSL$(2,\mathbb{C})$*,* par$\langle f,g\rangle = (-4,-4,\beta')$ *and the set $\mathfrak{D}$ is defined by $(*)$. The following properties hold.*

(1) *Let $\beta' \in \mathbb{R}$. The group $\langle f,g\rangle$ is discrete if and only if $\beta' = \pm(r+4)-4$, where $r \in \mathfrak{D} \cup [0,+\infty)$.*
(2) *Let $\beta' = e^{\pi ki/6}\cdot(r+4)-4$ for some numbers $k \in \{2,3,4,8,9,10\}$ and $r \in \mathfrak{D}$. Then the group $\langle f,g\rangle$ is discrete.*
(3) *Let the inequality $0 < |\beta'+4| < 1$ holds. Then the group $\langle f,g\rangle$ is not discrete.*
(4) *Let $\beta' \in \mathbb{C}\setminus\mathbb{R}$ and the inequality $|\beta'+4| \geq 4$ holds. Then the group $\langle f,g\rangle$ is discrete.*
(5) *Let $\beta' = m+n\omega$ for some numbers $m,n \in \mathbb{Z}$, $\omega = \sqrt{p}i$ or $\omega = \frac{1+\sqrt{4p-1}i}{2}$, $p \in \mathbb{N}$. Then the group $\langle f,g\rangle$ is discrete.*

Theorem 3. *Let $\langle f,g\rangle$ be a subgroup of* PSL$(2,\mathbb{C})$*,* par$\langle f,g\rangle = (-3,-3,\beta')$ *and the set $\mathfrak{D}$ is defined by $(*)$. The following properties hold.*

(1) *Let $\beta' = e^{\pi ki/3}\cdot(r+4)-4$ for some numbers $k \in \{0,2,4\}$ and $r \in [-4,+\infty)$. The group $\langle f,g\rangle$ is discrete if and only if $r \in \mathfrak{D}\cup[0,+\infty)$.*
(2) *Let $\beta' = e^{\pi ki/3}\cdot(r+4)-4$ for some numbers $k \in \{1,3,5\}$ and $r \in \mathfrak{D}\cup[0,+\infty)$. Then the group $\langle f,g\rangle$ is discrete.*
(3) *Let the inequality $0 < |\beta'+4-e^{\pi ki/3}| < 1$ holds for some number $k \in \{0,2,4\}$. Then the group $\langle f,g\rangle$ is not discrete.*

Theorem 4. *Let $\langle f,g\rangle$ be a subgroup of* PSL$(2,\mathbb{C})$*,* par$\langle f,g\rangle = (-2,-2,\beta')$ *and the set $\mathfrak{D}$ is defined by $(*)$. The following properties hold.*

(1) *Let $\beta' = e^{\pi ki/2}\cdot(r+4)-4$ for some numbers $k \in \{0,1,2,3\}$ and $r \in [-4,+\infty)$. The group $\langle f,g\rangle$ is discrete if and only if $r \in \mathfrak{D}\cup[0,+\infty)$.*
(2) *Let one of the following inequalities*

$$0 < |\beta'+4-e^{\pi ki/2}| < 1, \quad 0 < |\beta'+4-2e^{\pi ki/2}| < \frac{\sqrt{5}-1}{2},$$
$$2 < |\beta'+4| + |\beta'+4-2e^{\pi ki/2}| < \sqrt{3}+1$$

holds for some number $k \in \{0,1,2,3\}$. Then the group $\langle f,g\rangle$ is not discrete.

Theorem 5. *Let $\langle f,g\rangle$ be a subgroup of* PSL$(2,\mathbb{C})$*,* par$\langle f,g\rangle = (-1,-1,\beta')$ *and the set $\mathfrak{D}$ is defined by $(*)$. The following properties hold.*

(1) *Let $\beta' = e^{\pi ki/3}\cdot(r+4)-4$ for some numbers $k \in \{0,1,2,3,4,5\}$ and $r \in [-4,+\infty)$. The group $\langle f,g\rangle$ is discrete if and only if $r \in \mathfrak{D}\cup[0,+\infty)$.*
(2) *Let one of the following inequalities*

$$0 < |\beta'+4-e^{\pi ki/3}| < 1, \quad 0 < |\beta'+4-2e^{\pi ki/3}| \leq \frac{1}{2}$$

holds for some number $k \in \{0,1,2,3,4,5\}$. Then the group $\langle f,g\rangle$ is not discrete.

6. Proofs of Theorems 2, 3, 4, 5

Proof of Theorem 2. Suppose that $\langle f,g\rangle$ is a subgroup of PSL$(2,\mathbb{C})$ and par$\langle f,g\rangle = (-4,-4,\beta')$.

Items (1) and (2) follow from Corollary 4, Lemma 1 and item 1 of Theorem 4, item 1 of Theorem 5, Lemma 2 respectively. Item (5) is a consequence of Theorem 7.21 and Lemma 7.23 [8].

Item (3) is proved by reductio ad absurdum. Let $\langle f, g\rangle$ be a discrete group and $0 < |\beta' + 4| < 1$. By Lemma 4.3 [12], there exists an elliptic element $\tilde{f} \in \mathrm{PSL}(2, \mathbb{C})$ of order 2 such that the group $\langle \tilde{f}, g\rangle$ is discrete with

$$\gamma(\tilde{f}, g) = \beta(g) - \gamma(f, g) = \beta' + 4$$

and

$$\beta(\tilde{f}g) = \gamma(\tilde{f}, g) - \beta(g) - 4 = \beta' + 4 - \beta' - 4 = 0.$$

Note that $\gamma(\tilde{f}g, g) = \gamma(\tilde{f}, g)$ and that the element $\tilde{f}g$ is parabolic or trivial. Let us construct explicitly this element to show that it is parabolic. As in the proof of Proposition 2, we may assume that $f = \{\pm A\}$ and $g = \{\pm B\}$, where $f = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $g = \begin{pmatrix} a & b \\ -b^{-1} & 0 \end{pmatrix}$, $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \{0\}$. Since $\beta' \neq -4$, $a \neq 0$. Denote

$$C = AB - BA = \begin{pmatrix} 0 & 2bi \\ 2b^{-1}i & 0 \end{pmatrix}, \quad h = \left\{\pm \frac{1}{\det C} \cdot C\right\} = \begin{pmatrix} 0 & bi \\ b^{-1}i & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C}).$$

The element h has many remarkable properties and is called Jørgensen's Lie product of f and g (see [13]). Following the proof of Lemma 4.3 [12], we have $\tilde{f} = fh$. By direct calculations

$$\tilde{f}g = fhg = \begin{pmatrix} 1 & 0 \\ ab^{-1} & 1 \end{pmatrix}.$$

Therefore, $\tilde{f}g$ is a parabolic element and $0 < |\gamma(\tilde{f}g, g)| < 1$. This contradicts Shimizu — Leutbecher's Lemma [14, §II.C]. Thus $\langle f, g\rangle$ is a non-discrete group.

Item (5). Let us show that if $|\beta' + 4| \geq 4$, then Corollary 3 [15] holds for the group $\langle f, g\rangle$. Obviously,

$$|\gamma| - |\gamma - \beta'| = 4 - |\beta' + 4| \leq 0, \tag{1}$$

$$\frac{|\beta' + 4| + |\beta'|}{4} \cdot |\gamma| - |\gamma - \beta'| = \frac{|\beta' + 4| + |\beta'|}{4} \cdot 4 - |\beta' + 4| = |\beta'|. \tag{2}$$

Since $\beta' \notin \mathbb{R}$, for the triangle with vertices at points -4, 0, β' of the complex plane, we have

$$|\beta'| < 4 + |\beta' + 4| \quad \text{and} \quad |\beta'| + |\beta' + 4| > 4.$$

Hence,

$$|\beta'|^2 - 8|\beta'| + 16 - |\beta' + 4|^2 = \big(|\beta'| - 4 - |\beta' + 4|\big)\big(|\beta'| - 4 + |\beta' + 4|\big) < 0,$$

and

$$4|\beta'+4| - |\beta'+4|^2 + 4|\beta'| - |\beta'+4||\beta'| + 16 - 4|\beta'+4| < 12|\beta'| - |\beta'+4||\beta'| - |\beta'|^2.$$

Using this inequality, we get

$$|\gamma| - |\gamma - \beta'| = 4 - |\beta' + 4| < \frac{|\beta'| \cdot \big(12 - |\beta' + 4| - |\beta'|\big)}{|\beta' + 4| + |\beta'| + 4}. \tag{3}$$

Inequalities (1), (2), (3) allow us to apply the second condition of Corollary 3 [15]. Thus $\langle f, g\rangle$ is a discrete group. This completes the proof of Theorem 2. □

Proof of Theorem 3. Let $\langle f, g\rangle$ be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (-3, -3, \beta')$.

Items (1) and (2) are consequences of Corollary 4, Lemma 1 and item 1 of Theorem 5, Lemma 2 respectively. To prove item (3), we need the following result.

Proposition 6. *If the inequality* $0 < |\beta' + 3| < 1$ *holds, then the group* $\langle f, g\rangle$ *is not discrete.*

Proof. We use the idea from Section 5 of [12]. Assume the converse. Let $\langle f, g\rangle$ be a discrete group and $0 < |\beta' + 3| < 1$. By Lemma 4.3 [12], there exists an elliptic element $\tilde{f} \in \mathrm{PSL}(2, \mathbb{C})$ of order 2 such that the group $\langle \tilde{f}, g\rangle$ is discrete with

$$\gamma(\tilde{f}, g) = \beta(g) - \gamma(f, g) = \beta' + 3$$

and

$$\beta(\tilde{f}g) = \gamma(\tilde{f}, g) - \beta(g) - 4 = \beta' + 3 - \beta' - 4 = -1.$$

Let us consider the discrete group $\langle \tilde{f}g, g\rangle$. Note that $\tilde{f}g$ is an elliptic element of order 6 and $0 < |\gamma(\tilde{f}g, g)| < 1$. This contradicts Theorem 3.4 [8]. □

The latter proposition and Lemma 1 yields Item (3). Theorem 3 is proved. □

Proof of Theorem 4. Suppose that $\langle f, g\rangle$ is a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ and $\mathrm{par}\langle f, g\rangle = (-2, -2, \beta')$.

Item (1) follows from Corollary 4 and Lemma 1. Item (2) is established by using the following proposition and Lemma 1.

Proposition 7. *If one of the following inequalities*

$$(4)\quad 0 < |\beta' + 3| < 1, \quad 0 < |\beta' + 2| < \frac{\sqrt{5} - 1}{2}, \quad 2 < |\beta' + 4| + |\beta' + 2| < \sqrt{3} + 1$$

holds, then the group $\langle f, g\rangle$ *is not discrete.*

Proof. As above, we use the idea from Section 5 of [12]. Assume the converse. Let $\langle f, g\rangle$ be a discrete group and one of inequalities (4) holds. By Lemma 4.3 [12], there exists an elliptic element $\tilde{f} \in \mathrm{PSL}(2, \mathbb{C})$ of order 2 such that the group $\langle \tilde{f}, g\rangle$ is discrete with

$$\gamma(\tilde{f}, g) = \beta' + 2 \quad \text{and} \quad \beta(\tilde{f}g) = -2.$$

Therefore, the discrete group $\langle \tilde{f}g, g\rangle$ with the elliptic generator $\tilde{f}g$ of order 4 is such that

$$\text{either}\quad 0 < |\gamma(\tilde{f}g, g) + 1| < 1, \quad \text{or}\quad 0 < |\gamma(\tilde{f}g, g)| < \frac{\sqrt{5} - 1}{2}$$
$$\text{or}\quad 2 < |\gamma(\tilde{f}g, g) + 2| + |\gamma(\tilde{f}g, g)| < \sqrt{3} + 1.$$

This contradicts Lemma 4.23 and results from item 5.9 of [8]. □

Theorem 4 is proved. □

Proof of Theorem 5. Let $\langle f, g\rangle$ be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$, $\mathrm{par}\langle f, g\rangle = (-1, -1, \beta')$.

Item (1) is a consequence of Corollary 4 and Lemma 1. To prove item (2), we need the following observation and Lemma 1. If either $0 < |\beta' + 3| < 1$ or $0 < |\beta' + 2| \leq 1/2$, then the group $\langle f, g\rangle$ is not discrete. Indeed, using Proposition 6 and Lemma 2, we obtain the first estimate on the parameter β'. Using Theorem 1 [16], we obtain the second estimate on the parameter β'. This concludes the proof of Theorem 5. □

References

[1] Beardon A. F., *The Geometry of Discrete Groups*, Springer-Verlag, New York, Heidelberg, and Berlin, 1983. MR0698777

[2] Jørgensen T., *A note on subgroups of SL(2,C)*, Quart. J. Math. Oxford Ser. (2), **28**:110 (1977), 209–211. MR0444839

[3] Gilman G., *Two-generator discrete subgroups of PSL(2,C)*, Memoirs of the AMS, **117**:561 (1995). MR1290281

[4] Gehring F. W., Gilman J. P., and Martin G. J., *Kleinian groups with real parameters*, Commun. Contemp. Math., **3**:2 (2001), 163–186. MR1831927

[5] Klimenko E., Kopteva N., *All discrete RP-groups whose generator have real trace*, Int. J. Algebra Comput., **15**:3 (2005), 577–618. MR2151429

[6] Maskit B., *Some special 2-generator Kleinian groups*, Proc. Am. Math. Soc., **106** (1989), 175–186.

[7] Klimenko, E. Ya., *A class of the 2-generator subgroups of PSL(2,C)*, translation in Siberian Math. J., **30**: 5 (1990), 723-–725.

[8] Gehring F. W., Martin G. J., *Commutators, collars and the geometry of Mobius groups*, J. Anal. Math., **63**:1 (1994), 175–219. MR1269219

[9] Sato H., *One-parameter families of extreme discrete groups for Jørgensen inequality*, Contemporary Math. (The First Alfors-Bers Colloquium), **256** (2000), 271–287. MR1759686

[10] Vesnin A. Yu., Masley A. V., *On Jørgensen numbers and their analogs for groups of figure-eight orbifolds*, Siberian Mathematical Journal, **55**:5 (2014), 807–816. MR3289108

[11] Gehring F. W., Martin G. J., *Stability and extremality in Jørgensen's inequality*, Complex Variables, Theory Appl., **12**:1–4 (1989), 277–282. MR1040927

[12] Gehring F. W., Martin G. J., *On the Margulis constant for Kleinian groups, I*, Ann. Acad. Sci. Fenn., Math., **21**:2 (1996), 439–462. MR1404096

[13] Jørgensen T., *A compact 3-manifolds of constant negative curvature fibering over the circle*, Ann. Math., **106** (1977), 61–72. MR0450546

[14] Maskit B., *Kleinian groups* Springer-Verlag, New York, Heidelberg, and Berlin, 1987.

[15] Maslei A. V., *Sufficient discreteness conditions for subgroups of PSL(2,C) generated by an involution and a nonparabolic element*, Math. Notes, **95**:1–2 (2014), 282—285. MR3267216

[16] Tan D., *On two-generator discrete groups of Mobius transformations*, Proc. Am. Math. Soc., **106** (1989), 763–770. MR0969527

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