S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 14, стр. 1299–1306 (2017) УДК 512.554

DOI 10.17377/semi.2017.14.110 MSC 18D50

SELF-DUAL BINARY QUADRATIC OPERADS

P.S. KOLESNIKOV

Abstract. We describe those binary quadratic operads generated by a two-dimensional space that are isomorphic to their Koszul dual operads.

Keywords: nonassociative algebra, operad, Koszul duality.

1. Introduction

Varieties of linear algebras defined by polylinear identities of degree 3 (e.g., associative, alternative, Novikov, Poisson, et al.) give rise to corresponding binary quadratic operads [1, 2]. The most common case in practice is related with varieties of algebras with one binary operation. The corresponding operads $\mathcal{P}$ are generated by 1- or 2-dimensional S_2-module $V(2) = \mathcal{P}(2)$. In this paper, we solve the following natural question: which binary quadratic operads $\mathcal{P}$ governing varieties of algebras with one binary operation are isomorphic to their Koszul dual operads $\mathcal{P}^!$?

For example, it is well known that the operads of associative algebras As coincides with $\mathrm{As}^!$, the same holds for the operad of Poisson algebras Pois. For the operad of (left) Novikov algebras Nov, it is known that $\mathrm{Nov}^! = \mathrm{Nov}^{op}$, the opposite operad of right Novikov algebras [3].

For $\dim V(2) = 1, 2$, the isomorphism between $\mathcal{P}$ and $\mathcal{P}^!$ is possible only if S_2-module $V(2)$ is isomorphic to its skew transpose dual $V(2)^\vee$, i.e., the decomposition of $V(2)$ into irreducible S_2-modules is $M_+ \oplus M_-$, where $M_\pm = \Bbbk u_\pm$ are 1-dimensional spaces with $(12)u_\pm = \pm u_\pm$.

We will explicitly describe *self-dual* operads $\mathcal{P}$, i.e., those isomorphic to $\mathcal{P}^!$. First, we split the operads that include self-dual ones into two disjoint classes. Operads of the first and second class are in one-to-one correspondence with the points of the Grassmannian $G(2,4) \subset \mathbb{P}^5$ and of $G(2,4) \times S(1,1)$, respectively. Here $S(1,1) \subset \mathbb{P}^3$ is the Segre variety representing $\mathbb{P}^1 \times \mathbb{P}^1$. Next, we determine equations defining

Kolesnikov, P.S., Self-dual binary quadratic operads.

Received October, 19, 2017, published December, 4, 2017.

explicitly those points in these varieties that correspond to self-dual operads. These equations turn to be linear (modulo quadratic relations defining Grassmannian and Segre variety).

In this paper, the base field $\Bbbk$ is of characteristic $\ne 2, 3$.

2. Binary quadratic operads

In this section, we mainly follow notations of [1], where all necessary definitions may be found.

Let $\mathcal{F}$ be the free (symmetric) operad generated by a $\mathbb{Z}_+$-graded space $V = \bigoplus_{n\ge 1} V(n)$, where $V(n) = 0$ for $n \ne 2$. (By the definition of an operad, $\dim \mathcal{F}(1) = 1$ since it necessarily contains the identity.) We will consider the most common case when $\dim V(2) = 2$, and the symmetric group S_2 acts on $V(2)$ by permutation of coordinates: $V(2) = \operatorname{span}(\mu, \mu' = (12)\mu)$. Then

$$\mathcal{F}(3) = \Bbbk S_3 \otimes_{\Bbbk S_2} (V(2) \otimes V(2)),$$

where $(12) \in S_2$ acts on $V(2) \otimes V(2)$ as $\mathrm{id} \otimes (12)$.

Suppose U is an S_3-invariant subspace of $\mathcal{F}(3)$, and let $I = I(U)$ be the operad ideal of $\mathcal{F}$ generated by U. The quotient operad $\mathcal{P} = \mathcal{F}/I$ is denoted by $\mathcal{P}(V, U)$.

Given an operad $\mathcal{P} = \mathcal{P}(V, U)$, its Koszul dual operad $\mathcal{P}^!$ is defined as $\mathcal{P}(V^\vee, U^\perp)$, where $U^\perp \subseteq \mathcal{F}(3)^\vee \simeq \Bbbk S_3 \otimes_{\Bbbk S_2} (V(2)^\vee \otimes V(2)^\vee)$ is the orthogonal complement of U.

Let us choose the following linear basis of $\mathcal{F}(3)$:

$$e_1 = 1 \otimes_{\Bbbk S_2} (\mu \otimes \mu), \quad e_2 = (12)e_1, \quad e_3 = 1 \otimes_{\Bbbk S_2} (\mu' \otimes \mu), \quad e_4 = (12)e_3,$$
$$e_{4+i} = (13)e_i, \quad e_{8+i} = (23)e_i, \quad i = 1, 2, 3, 4.$$

Fix a linear basis $f_1, \dots, f_{12}$ in $\mathcal{F}(3)^\vee$ constructed in the same way as $e_1, \dots, e_{12}$ with ν instead of μ. Denote by Σ the (12×12)-matrix with entries $\langle f_i, e_j\rangle$. It is easy to compute that

$$\Sigma = \operatorname{diag}(1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1).$$

The following statement is a standard exercise in the theory of characters.

Lemma 1 (c.f. [4]). *The S_3-module $\mathcal{F}(3)$ has the following decomposition into irreducible S_3-modules:*

$$\mathcal{F}(3) = 2M_+ \oplus 2M_- \oplus 4M_2,$$

where M_+ is the trivial module, M_- is the sign module, $M_2 = \pi(2, 1)$ is the 2-dimensional irreducible S_3-module determined by a partition $(2, 1)$.

Let us determine how $M_\pm$ and M_2 may be embedded into $\mathcal{F}(3)$. All coordinates below are in the basis $e_1, \dots, e_{12}$.

Lemma 2 (c.f. [4]). (1) *An S_3-submodule of $\mathcal{F}(3)$ isomorphic to $M_\pm$ is spanned by a vector*

$$u_\pm(x_1, x_2) = (x_1, \pm x_1, x_2, \pm x_2, \pm x_1, x_1, \pm x_2, x_2, \pm x_1, x_1, \pm x_2, x_2),$$

$(x_1, x_2) \in \Bbbk^2$.

(2) *An S_3-submodule of $\mathcal{F}(3)$ isomorphic to M_2 is spanned by vectors $u_2^i(\bar{x})$, $i = 1, 2$, $\bar{x} = (x_1, x_2, x_3, x_4) \in \Bbbk^4$. where*

$$u_2^1 = (x_1, -x_1, x_2, -x_2, x_3, x_3 - x_1, x_4, x_4 - x_2, x_1 - x_3, -x_3, x_2 - x_4, -x_4),$$
$$u_2^2 = (x_1 - x_3, -x_3, x_2 - x_4, -x_4, x_3 - x_1, x_3, x_4 - x_2, x_4, x_1, -x_1, x_2, -x_2).$$

Proof. Let us prove (2) for example. The only 2-dimensional irreducible S_3-module M_2 has the following structure:

$$M_2 = \Bbbk u_1 \oplus \Bbbk u_2, \quad \begin{matrix} (13) : u_1 \mapsto u_1 - u_2, & (23) : u_1 \mapsto u_2, \\ u_2 \mapsto -u_2, & u_2 \mapsto u_1. \end{matrix} \tag{1}$$

Suppose the images of u_i in $\mathcal{F}(3)$ are u_2^i, $i = 1, 2$, $u_2^1 = x_1 e_1 + \cdots + x_{12} e_{12}$, $u_2^2 = y_1 e_1 + \cdots + y_{12} e_{12}$. Then (1) implies a system of linear equations on x_i, y_i, $i = 1, \ldots, 12$, its solution provides the desired statement. □

3. Necessary and sufficient conditions of self-duality

Suppose an operad $\mathcal{P} = \mathcal{P}(V, U)$ is isomorphic to its Koszul dual. Then $\dim U = \dim U^\perp = 6$. An isomorphism of binary operads $\mathcal{P}$ and $\mathcal{P}^!$ is determined by an S_2-invariant isomorphism of linear spaces $\mathcal{P}(2) = V(2)$ and $\mathcal{P}^!(2) = V(2)^\vee$. Such an isomorphism is determined by a map

$$g(a, b) : \mu \mapsto a\nu + b\nu',$$

where $\langle \nu, \mu \rangle = 1$, $\langle \nu, \mu' \rangle = 0$, $\nu' = (12)\nu$, $a, b \in \Bbbk$. Since $g(a, b) : \mu' \mapsto (12)(a\nu + b\nu') = b\nu + a\nu'$, it is necessary to assume $a^2 - b^2 \neq 0$.

The linear map $g(a, b)$ induces a linear map $\Gamma(a, b) = \mathrm{id} \otimes_{\Bbbk S_2} (g(a, b) \otimes g(a, b)) : \mathcal{F}(3) \to \mathcal{F}(3)^\vee$. The matrix of $\Gamma(a, b)$ with respect to the bases $e_1, \ldots, e_{12}$ and $f_1, \ldots, f_{12}$ is also denoted by $\Gamma(a, b)$. It is easy to compute that

$$\Gamma(a, b) = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}, \quad A = \begin{pmatrix} a^2 & ab & ab & b^2 \\ ab & a^2 & b^2 & ab \\ ab & b^2 & a^2 & ab \\ b^2 & ab & ab & a^2 \end{pmatrix},$$

where

$$\Gamma(a, b) : \bar{e} \mapsto \Gamma(a, b)\bar{f}, \quad \bar{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_{12} \end{pmatrix}, \quad \bar{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_{12} \end{pmatrix}.$$

Obviously, $\mathcal{P} \simeq \mathcal{P}^!$ if and only if there exist $a, b \in \Bbbk$, $a^2 \neq b^2$, such that

$$\Gamma(a, b)U = U^\perp. \tag{2}$$

The linear space U may be presented by a (6×12)-matrix (also denoted by U) of coordinates with respect to the basis $e_1, \ldots, e_{12}$. Then (2) may be rewritten as

$$0 = \left\langle U\Gamma(a, b)\bar{f}, (U\bar{e})^T \right\rangle = U\Gamma(a, b)\langle \bar{f}, \bar{e}^T \rangle U^T = U\Gamma(a, b)\Sigma U^T,$$

assuming that $\operatorname{rank} U = 6$. Moreover, the matrix U should determine an S_3-invariant subspace of $\mathcal{F}(3)$. The latter condition together with Lemma 1 provides the following opportunities for the space U as for an S_3-module:

(R1) $U \simeq 2M_+ \oplus 2M_- \oplus M_2$;
(R2) $U \simeq 3M_2$;
(R3) $U \simeq 2M_+ \oplus M_2$;
(R4) $U \simeq 2M_- \oplus M_2$;
(R5) $U \simeq M_+ \oplus M_- \oplus 2M_2$.

We will consider the cases (R1)–(R5) separately.

For $u, v \in \Bbbk^{12}$, denote

$$\langle u, v \rangle_{a,b} = u\Gamma(a, b)\Sigma v^T \in \Bbbk.$$

Lemma 3. *Let $x_i, y_i \in \Bbbk$, $i = 1, \dots, 4$, $a, b \in \Bbbk$. Then*
(1) $\langle u_+(x_1, x_2), u_+(y_1, y_2)\rangle_{a,b} = 0$;
(2) $\langle u_-(x_1, x_2), u_-(y_1, y_2)\rangle_{a,b} = 0$;
(3) $\langle u_\pm(x_1, x_2), u_2^i(y_1, y_2, y_3, y_4)\rangle_{a,b} = 0$, $i = 1, 2$;
(4) $\langle u_2^i(x_1, x_2, x_3, x_4), u_\pm(y_1, y_2)\rangle_{a,b} = 0$, $i = 1, 2$;
(5) $\langle u_2^i(x_1, x_2, x_3, x_4), u_2^i(y_1, y_2, y_3, y_4)\rangle_{a,b} = 0$, $i = 1, 2$.

Proof. Straightforward computation. □

Lemma 4. *Let $x_i, y_i \in \Bbbk$, $i = 1, 2$, $a, b \in \Bbbk$, $a^2 \neq b^2$. Then the following conditions are equivalent:*

- $\langle u_+(x_1, x_2), u_-(y_1, y_2)\rangle_{a,b} = \langle u_-(y_1, y_2), u_+(x_1, x_2)\rangle_{a,b} = 0$;
- $a(x_1y_1 - x_2y_2) + b(x_2y_1 - x_1y_2) = 0$.

Proof. Straightforward computation. □

Corollary 1. *If $\mathcal{P} = \mathcal{P}(V, U)$ is isomorphic to its Koszul dual operad $\mathcal{P}^!$ then U may not have a representation type (R1) or (R2).*

Proof. If U is of type (R1) then $u_\pm(x_1, x_2) \in U$ for all $x_1, x_2 \in \Bbbk$. Lemma 4 implies $\langle U, U\rangle_{a,b} = 0$ is impossible.

For further needs, let us introduce matrices $T_i \in M_{4,12}(\Bbbk)$, $i = 1, 2$, such that $u_2^i(x_1, x_2, x_3, x_4) = (x_1\ x_2\ x_3\ x_4)T_i$. Then

$$\langle u_2^i(x_1, x_2, x_3, x_4), u_2^j(y_1, y_2, y_3, y_4)\rangle_{a,b} = (x_1\ x_2\ x_3\ x_4)A_{ij}(a, b)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},$$

where $A_{ij}(a, b) = T_i\Gamma(a, b)\Sigma T_j^T$. It is easy to check that $A_{ij}(a, b) = -A_{ji}(a, b)$.

If U is of type (R2) then there exist a matrix $\widehat{U} \in M_{3,4}(\Bbbk)$ of rank 3 such that

$$\widehat{U}A_{12}(a, b)\widehat{U}^T = 0, \tag{3}$$

this condition is necessary and sufficient for self-duality of $\mathcal{P}(V, U)$ by Lemma 3(5). Up to row transformations, $\widehat{U}$ may be chosen in one of the following forms:

$$\begin{pmatrix} t_1 & 1 & 0 & 0 \\ t_2 & 0 & 1 & 0 \\ t_3 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & t_1 & 0 & 0 \\ 0 & t_2 & 1 & 0 \\ 0 & t_3 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & t_1 & 0 \\ 0 & 1 & t_2 & 0 \\ 0 & 0 & t_3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \end{pmatrix}.$$

For either of these $\widehat{U}$, (3) turns into a system of algebraic equations on t_1, t_2, t_3 with parameters $a, b \in \Bbbk$. All four systems obtained are incompatible with the condition $a^2 - b^2 \neq 0$: we applied computer algebra system `Singular` [5] to compute a Gröbner basis of (3) together with additional relation $(a^2 - b^2)c - 1$ for a new variable c to find all these ideals to be improper. □

Proposition 1. *Suppose $\widehat{U}$ be a (2×4)-matrix of rank 2 satisfying (3) for some $a, b \in \Bbbk$ such that $a^2 \neq b^2$. Then, up to row transformations, $\widehat{U}$ is one of the following:*

(U1)

$$\widehat{U} = \begin{pmatrix} \gamma & \gamma & 0 & 1 \\ -\gamma & -\gamma & 1 & 0 \end{pmatrix}, \ \gamma \in \Bbbk, \quad \widehat{U} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

relation (3) holds for every $a, b \in \Bbbk$;

(U2)

$$\widehat{U} = \begin{pmatrix} -\gamma & \gamma+2 & 0 & 1 \\ 2-\gamma & \gamma & 1 & 0 \end{pmatrix},\ \gamma \in \Bbbk, \quad \widehat{U} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

relation (3) *holds for every* $a, b \in \Bbbk$*;*

(U3)

$$\widehat{U} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}, \tag{4}$$

where

$$x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4 = 0,$$

relation (3) *holds for* $b = 0$*;*

(U4) $\widehat{U}$ *as in* (4)*, satisfying*

$$x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_4 - x_4y_1 = 0,$$

relation (3) *holds for* $a = 0$.

Proof. Case 1. Suppose $\widehat{U}$ satisfies (3) for some $a \neq 0$, $b \neq 0$, and $\widehat{U}$ may be transformed to

$$\begin{pmatrix} t_1 & t_2 & 0 & 1 \\ t_3 & t_4 & 1 & 0 \end{pmatrix}.$$

Consider the coefficients of $\widehat{U}A_{12}(a,b)\widehat{U}^T$ as polynomials in variables $t_1, \dots, t_4$, a, b, a^{-1}, b^{-1}, c, and compute Gröbner basis of the ideal generated by these coefficients together with $aa^{-1} - 1$, $bb^{-1} - 1$, $(a^2 - b^2)c - 1$. Gröbner basis obtained consists of some polynomials in $t_1, \dots, t_4$ and in a, b, a^{-1}, b^{-1}, c only. Polynomials in $t_1, \dots, t_4$ are

$$\begin{aligned} &t_3^2 - t_4^2 - 2t_3 + 2t_4, \\ &t_2^2 - t_1^2 + 2t_1 - 2t_2, \\ &t_2t_4 + t_3t_4 - t_2 + t_3 - 2t_4, \\ &t_2t_3 + t_4^2 - t_2 - t_3, \\ &t_1 + t_4. \end{aligned}$$

Every solution of this system provides a matrix $\widehat{U}$ which satisfies (3) for all $a, b \in \Bbbk$. The first two polynomials split into linear factors:

$$(t_3 - t_4)(t_3 + t_4 - 2), \quad (t_2 - t_1)(t_2 + t_1 - 2).$$

This provides four subcases.

Case 1.1: $t_1 = t_2$, $t_3 = t_4$. Then $t_1 = t_2 = -t_3 = -t_4$ as described in (U1).

Case 1.2: $t_1 + t_2 = 2$, $t_3 + t_4 = 2$. Then $t_1 = 2 - t_2 = -t_4$, $t_3 = 2 - t_4$ as described in (U2).

Case 1.3: $t_1 = t_2$, $t_3 + t_4 = 2$. Then $t_4 = \pm 1$, $t_2 = t_1 = -t_4$, $t_3 = 2 - t_4$. If $t_4 = 1$ then we obtain $\widehat{U}$ as in Case 1.1. For $t_4 = -1$, this $\widehat{U}$ is described by Case 1.2.

Case 1.4: $t_1 + t_2 = 2$, $t_3 = t_4$. Then $t_4 = \pm 1$, $t_2 = 2 + t_4$, $t_1 = -t_4$, $t_3 = t_4$. Both these matrices are already described in Cases 1.1 and 1.2.

Case 2. Suppose $\widehat{U}$ satisfies (3) for some $a \neq 0$, $b \neq 0$, and $\widehat{U}$ may be transformed to

$$\begin{pmatrix} t_1 & 0 & t_2 & 1 \\ t_3 & 1 & t_4 & 0 \end{pmatrix}.$$

It makes sense to consider this case for $t_4 = 0$ only since for $t_4 \neq 0$ this matrix may be transformed by row transformations to a matrix from Case 1. As in Case 1, we obtain the following solution:

$$t_1 = 0, \quad t_2 = t_3 = \pm 1, \quad t_4 = 0.$$

These are the matrices described in (U1) and (U2) for $t_3 = 1$ and $t_3 = -1$, respectively.

Case 3. Suppose $\widehat{U}$ satisfies (3) for some $a \neq 0$, $b \neq 0$, and $\widehat{U}$ may be transformed to either of

$$\begin{pmatrix} 0 & t_1 & t_2 & 1 \\ 1 & t_3 & t_4 & 0 \end{pmatrix}, \quad \begin{pmatrix} t_1 & 0 & 1 & t_2 \\ t_3 & 1 & 0 & t_4 \end{pmatrix}, \quad \begin{pmatrix} 0 & t_1 & 1 & t_2 \\ 1 & t_3 & 0 & t_4 \end{pmatrix}.$$

It is enough to consider these matrices for $t_4 = 0$, they do not produce new solutions of (3).

Case 4. Suppose $\widehat{U}$ satisfies (3) for some $a \neq 0$, $b \neq 0$, and $\widehat{U}$ may be transformed to

$$\begin{pmatrix} 0 & 1 & t_1 & t_2 \\ 1 & 0 & t_3 & t_4 \end{pmatrix}.$$

This case makes sense to consider for $t_1 = t_2 = t_3 = t_4 = 0$, this does not produce solutions of (3).

Case 5. Suppose $\widehat{U}$ satisfies (3) for $ab = 0$. Then

$$\widehat{U}(A_{12}(a,b) \pm A_{12}(a,b)^T)\widehat{U}^T = 0.$$

Straightforward computation shows $A_{12} + A_{12}^T = 0$.

Case 5.1: $b = 0$, $a \neq 0$. Then $\widehat{U}(A_{12} - A_{12}^T)\widehat{U}^T = 0$ if and only if $x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4 = 0$ as in (U3).

Case 5.1: $a = 0$, $b \neq 0$. Then $\widehat{U}(A_{12} - A_{12}^T)\widehat{U}^T = 0$ if and only if $x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_4 - x_4y_1 = 0$ as in (U4). □

Every 2×4 matrix $\widehat{U}$ in Proposition 1 gives rise to a subspace of $\Bbbk^4$ generated by its rows, so this statement actually describes a subset of the Grassmannian $G = G(2,4) \subset \mathbb{P}^5$. Every point of G is defined by Plücker coordinates $(p_{ij})_{1\le i<j\le 4}$: a subspace spanned by $\widehat{U}$ of the form (4) has coordinates $p_{ij} = x_iy_j - x_jy_i$. Recall that p_{ij} satisfy Plücker identity $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.

The following statement is an easy exercise in linear algebra.

Lemma 5. *The following equations determine subspaces generated by rows of matrices in Proposition 1.*

- (U1) $p_{12} = 0$, $p_{13} = p_{14} = p_{23} = p_{24}$;
- (U2) $p_{13} = p_{24}$, $p_{14} + p_{23} + 2p_{13} = 0$, $p_{12} = 4p_{34}$, $p_{14} = p_{23} + p_{12}$;
- (U3) $p_{13} = p_{24}$;
- (U4) $p_{14} = p_{12} + p_{23}$.

3.1. **Description of self-dual operads.** Suppose $U \subset \mathcal{F}(3)$ is an S_3-submodule isomorphic to $2M_\pm \oplus 2M_2$. Then U is is spanned by

$$u_\pm(1,0),\ u_\pm(0,1),\ u_2^i(\bar{x}),\ u_2^i(\bar{y}),\ i = 1,2,$$

where $\bar{x}, \bar{y} \in \Bbbk^4$ span a 2-dimensional subspace $\widehat{U}$ of $\Bbbk^4$ which may be identified with a point $p = p(U)$ of the Grassmannian $G = G(2,4)$.

If $U \subset \mathcal{F}(3)$ is an S_3-submodule isomorphic to $M_+ \oplus M_- \oplus 2M_2$ then U is is spanned by

$$u_+(s_1, s_2),\ u_-(t_1, t_2),\ u_2^i(\bar{x}),\ u_2^i(\bar{y}),\ i = 1, 2,$$

where $\bar{x}, \bar{y} \in \Bbbk^4$ are encoded by the Plücker coordinates $(p_{ij})_{1\le i<j\le 4}$ of span $(\bar{x}, \bar{y}) \in G$ as above, and (s_1, s_2, t_1, t_2) are encoded by a point in the Segre variety $S = S(1,1) \subset \mathbb{P}^3$, $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Such a point is given by coordinates $(z_{kl})_{1\le k,l\le 2}$, $z_{kl} = s_k t_l - s_l t_k$, satisfying $z_{11}z_{22} - z_{12}z_{21} = 0$. Therefore, such subspaces U are in one-to-one correspondence with points $q(U) \in G \times S$ with coordinates $(p_{ij}, z_{kl})_{1\le i<j\le 4, k,l=1,2}$.

Let $\mathcal{P} = \mathcal{P}(V, U)$ be a binary quadratic operad generated by $V = V(2)$ which is is isomorphic to $M_+ \oplus M_-$ as an S_2-module. Assume $\mathcal{P} \simeq \mathcal{P}^!$. Then, as we have already seen, U as an S_3-module is isomorphic either to $2M_\pm \oplus 2M_2$ or to $M_+ \oplus M_- \oplus 2M_2$. In the first case, U may be identified with a point in G as above, in the second case U is encoded by a point of $G \times S$. The following statements describe these points explicitly.

Theorem 1. *Let $U \simeq 2M_\pm \oplus 2M_2$ as an S_3-module. Then $\mathcal{P} = \mathcal{P}(V, U)$ is isomorphic to $\mathcal{P}^!$ if and only if $p(U) \in Y_1 \cup Y_2 \in G = G(2, 4)$, where*

$$Y_1 = \{(p_{ij}) \in G \mid p_{13} = p_{24}\},$$
$$Y_2 = \{(p_{ij}) \in G \mid p_{14} = p_{12} + p_{23}\}.$$

Proof. Follows immediately from Lemma 3 and Proposition 1. Note that $p(U) \in Y_1$ if and only if $\mathcal{P} = \mathcal{P}^!$, and $p(U) \in Y_2$ if and only if $\mathcal{P}^! = \mathcal{P}^{op}$. □

Example. For the operad Nov governing the variety of Novikov algebras, i.e., right-symmetric left commutative algebras, the space of defining relations $U \subset \mathcal{F}(3)$ is isomorphic to $2M_- \oplus 2M_2$. The corresponding point $p(U)$ of the Grassmannian belongs to Y_2 with $p_{12} = -1$, $p_{13} = 0$, $p_{14} = 1$, $p_{23} = 2$, $p_{24} = 3$, $p_{34} = 2$.

Theorem 2. *Let $U \simeq M_+ \oplus M_- \oplus 2M_2$ as an S_3-module. Then $\mathcal{P} = \mathcal{P}(V, U)$ is isomorphic to $\mathcal{P}^!$ if and only if $q(U) \in X_1 \cup X_2 \cup X_3 \cup X_4 \subset G \times S$, where*

$$\begin{aligned}
X_1 = \{(p_{ij}, z_{kl}) \in G \times S \mid\ & p_{12} = 0, p_{13} = p_{14} = p_{23} = p_{24},\\
& z_{11}^2 + z_{22}^2 \ne z_{12}^2 + z_{21}^2\},\\
X_2 = \{(p_{ij}, z_{kl}) \in G \times S \mid\ & p_{13} - p_{24} = p_{14} + p_{23} + 2p_{13} = 0,\\
& p_{12} - 4p_{34} = p_{14} - p_{12} - p_{23} = 0,\\
& z_{11}^2 + z_{22}^2 \ne z_{12}^2 + z_{21}^2\},\\
X_3 = \{(p_{ij}, z_{kl}) \in G \times S \mid\ & p_{13} = p_{24}, z_{11} = z_{22}\},\\
X_4 = \{(p_{ij}, z_{kl}) \in G \times S \mid\ & p_{14} = p_{12} + p_{23}, z_{12} = z_{21}\}.
\end{aligned}$$

Proof. Proposition 1 together with Lemma 5 describe necessary conditions on $\widehat{U} =$ span $(\bar{x}, \bar{y}) \in G(2, 4)$. It remains to determine s_i and t_i, $i = 1, 2$, in such a way that $\langle u_+(s_1, s_2), u_-(t_1, t_2)\rangle_{a,b} = 0$ for some $a, b \in \Bbbk$, $a^2 \ne b^2$. By Lemma 4, orthogonality condition for u_+ and u_- is equivalent to $a(z_{11} - z_{22}) = b(z_{12} - z_{21})$. If $ab \ne 0$ then $a^2 \ne b^2$ implies

$$z_{11}^2 + z_{22}^2 \ne z_{12}^2 + z_{21}^2. \tag{5}$$

If $\widehat{U}$ is given by (U1) or (U2) of Proposition 1 then the only condition on s_i, t_i is given by (5). If $\widehat{U}$ is given by (U3) or (U4) of Proposition 1 then $b = 0$ or $a = 0$, respectively. In each of these cases, the second parameter may be chosen to be nonzero if and only if $z_{11} - z_{22} = 0$ or $z_{12} - z_{21} = 0$, respectively. □

Example. The operad As governing the variety of associative algebras belongs to $X_3 \cap X_4$ with $z_{11} = z_{22} = 1$, $z_{12} = z_{21} = -1$, $p_{12} = p_{13} = p_{14} = p_{24} = p_{34} = 1$, $p_{23} = 0$.

Example. The operad Pois governing the variety of Poisson algebras in terms of one operation (see [6]) belongs to $X_3 \cap X_4$ with $z_{11} = z_{22} = 1$, $z_{12} = z_{21} = -1$, $p_{12} = 0$, $p_{13} = p_{14} = p_{23} = p_{24} = p_{34} = 1$.

References

[1] V. Ginzburg, M. Kapranov, *Koszul duality for operads*, Duke Math. J., **76**:1 (1994), 203–272. MR1301191

[2] J.-L. Loday, B. Vallette, *Algebraic Operads*, Gründlehren Math. Wiss., **346**, Springer-Verlag, Heidelberg, 2012. MR2954392

[3] A. S. Dzhumadil'daev, *Codimension growth and non-Koszulity of Novikov operad*, Comm. Algebra, **39**, (2011), 2943–2952. MR2834140

[4] A.I. Mal'cev, *On algebras defined by identities* (Russian), Mat. Sbornik N. S., **26(68)** (1950), 19–33.

[5] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *Singular 3-1-3—A computer algebra system for polynomial computations*, url: http://www.singular.uni-kl.de (2011).

[6] M. Markl, E. Remm, *Algebras with one operation including Poisson and other Lie-admissible algebras*, J. Algebra, **299** (2006), 171–189. MR2225770

Pavel Sergeevich Kolesnikov
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
E-mail address: pavelsk@math.nsc.ru