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# ON QUASI-EQUATIONAL BASES FOR DIFFERENTIAL GROUPOIDS AND UNARY ALGEBRAS

A.V. KRAVCHENKO, A.M. NURAKUNOV, M.V. SCHWIDEFSKY

Abstract. As is known, there exist $2^\omega$ quasivarieties of differential groupoids and unary algebras with no independent quasi-equational basis. In the present article, we show that there exist $2^\omega$ such quasivarieties with an $\omega$-independent quasi-equational basis. We also find a recursive independent quasi-equational basis for the intersection of those quasivarieties.

**Keywords:** quasivariety, quasi-equational basis, differential groupoid, unary algebra.

## 1. Introduction

For a quasivariety $\mathbf{K}$, let $\mathrm{L_q}(\mathbf{K})$ denote the quasivariety lattice of $\mathbf{K}$. In 1977, V. A. Gorbunov [2] proved the following assertion.

**Proposition 1.** *There exists a quasivariety* $\mathbf{K}$ *of unary algebras such that*

(a) *the lattice* $\mathrm{L_q}(\mathbf{K})$ *has* $2^\omega$ *elements with no cover* (*hence, no independent quasi-equational basis*);
(b) *among them, there exist* $2^\omega$ *quasivarieties with an* $\omega$*-independent quasi-equational basis.*

Later, analogs of assertion (a) of Proposition 1 were proven for quasivarieties of monounary algebras [5], directed graphs [14], unary algebras of a particular type [8], differential groupoids and pointed Abelian groups [1], and (undirected loopless) graphs [10], as well as for antivarieties of monounary algebras [6].

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For a more detailed study of independent and $\omega$-independent quasi-equational bases, we refer to [9]. In particular, sufficient conditions are found in [9] for a quasivariety to contain $2^\omega$ subquasivarieties with an $\omega$-independent quasi-equational basis but no independent quasi-equational basis relative to a given quasivariety. However, the variety of differential groupoids and the variety of unary algebras considered in [1, 8] do not satisfy these sufficient conditions. In the present article, we show that assertion (b) of Proposition 1 holds for these varieties too.

In the proofs of Theorems 4 and 8, we use some ideas from the proof of [2, Theorem 3*]. In the proofs of Propositions 6 and 9, we use some ideas from the proof of [5, Theorem 2].

For all definitions and notation concerning algebraic structures and quasivarieties, we refer to the monograph [3].

*Quasi-identities* are universal sentences of the form

$$\forall \overline{x}\ \varphi_1(\overline{x})\ \&\ldots\&\ \varphi_k(\overline{x})\ \rightarrow\ \varphi_0(\overline{x}),$$

where $\varphi_i(\overline{x})$ is an atomic formula for each $i \leqslant k$. A class $\mathbf{K}$ is a *quasivariety* if it coincides with the class of all models of some set $\Phi$ of quasi-identities. Then the set $\Phi$ is called a *quasi-equational basis* for $\mathbf{K}$. A basis $\Phi$ is *independent* if, for every $\varphi \in \Phi$, there is a structure satisfying each quasi-identity from the set $\Phi \setminus \{\varphi\}$ and violating the quasi-identity $\varphi$. Let $\mathbb{N}$ denote the set of natural numbers. A basis $\Phi$ is *$\omega$-independent* if there is a partition $\Phi = \bigcup_{n\in\mathbb{N}} \Phi_n$ such that, for every $n \in \mathbb{N}$, there is a structure satisfying each quasi-identity from the set $\Phi \setminus \Phi_n$ and violating some quasi-identity from the set $\Phi_n$.

We denote structures by calligraphic letters. The universe of a structure is denoted by the corresponding italic letter. For classes of algebraic structures, we use boldface letters. We assume that all classes are *abstract*, i.e., closed under isomorphism.

## 2. Differential groupoids

A *differential groupoid* is an algebra endowed with one binary operation $\cdot$ that satisfies the following identities:

$$\forall x\ x\cdot x = x, \quad \forall x\,\forall y\ x\cdot(x\cdot y) = x,$$
$$\forall x\,\forall y\,\forall z\,\forall t\ (x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t).$$

Let $\mathbf{Dm}$ denote the variety of all differential groupoids. For brevity, we write $x_1x_2\ldots x_n$ for $(\ldots(x_1\cdot x_2)\cdot\ldots)\cdot x_n$ and $xy^n$ for $x\underbrace{y\ldots y}_{n}$.

We use the following representation of differential groupoids which is due to [11]. A groupoid $\mathcal{G}$ is an $\mathbf{Lz}$-$\mathbf{Lz}$-*sum* (of orbits $\mathcal{G}_i$ over a groupoid $\mathcal{I}$) *satisfying the left normal law* if there is a partition $G = \bigcup_{i\in I} G_i$ such that, for every pair $(i,j) \in I^2$, there is a mapping $h_i^j\colon G_i \to G_i$ satisfying the following conditions:

(i) for every $i \in I$, $h_i^i$ is the identity mapping;
(ii) we have $h_i^j(h_i^k(x)) = h_i^k(h_i^j(x))$ for all $i,j,k \in I$ and $x \in G_i$;
(iii) we have $a_i \cdot a_j = h_i^j(a_i)$ for all $i,j \in I$, $a_i \in G_i$ and $a_j \in G_j$.

According to [11, Theorem 2.2], a groupoid is differential if and only if it can be represented as an $\mathbf{Lz}$-$\mathbf{Lz}$-sum satisfying the left normal law. For more detailed information on differential groupoids, the reader is referred to the monograph [12].

Let $n > 0$. The structure defined in **Dm** by the generators $\{x, y\}$ and the defining relations $\{yx = y, xy^n = x\}$ is called the *cycle* of length $n$ and is denoted by $\mathcal{D}_n$. It is convenient to regard $D_n$ as $G_0 \cup G_1$, where $G_1$ is the singleton orbit $\{b\}$ and $G_0 = \{a, ab, ab^2, \ldots, ab^{n-1}\}$. By $\mathcal{D}_0$, we denote the trivial groupoid.

The following properties of the cycles can be deduced from [7, Lemma 3].

**Lemma 2.** *Let $n > 0$.*

(a) *The class $\{\mathcal{D}_m \mid m$ divides $n\}$ coincides with the class of nontrivial homomorphic images of $\mathcal{D}_n$.*

(b) *If $m \in \mathbb{N}$ and $\varphi\colon \mathcal{D}_n \to \mathcal{D}_m$ is a homomorphism then either $\varphi(\mathcal{D}_n) \in \{\mathcal{D}_0, \mathcal{D}_1\}$ or $m$ divides $n$.*

The structure of the variety lattice of differential groupoids is explicitly described in [11] (see also [12, Theorem 8.4.14]). In particular, each subvariety of **Dm** is defined by a single identity. In contrast to that description, the structure of the quasivariety lattice $\mathrm{L_q}(\mathbf{Dm})$ is much more complicated. Namely, the variety **Dm** is $Q$-universal [7], there exist $2^\omega$ classes **K** of differential groupoids such that the set of (isomorphism types of) finite sublattices of the lattice $\mathrm{L_q}(\mathbf{K})$ is not computable [13], and there exist $2^\omega$ quasivarieties of differential groupoids with no independent quasi-equational basis [1].

Let $\mathbb{P}$ denote the set of all primes and let $P_{\mathrm{fin}}(\mathbb{P})$ denote the set of all finite subsets of $\mathbb{P}$. For a nonempty set $F \in P_{\mathrm{fin}}(\mathbb{P})$, let $[F]$ denote the product of all primes belonging to $F$. For the empty set $\varnothing$, we put $[\varnothing] = 1$. Let $I \subseteq \mathbb{P}$. We denote by $\varphi_F^I$ the quasi-identity

$$\forall x\, \forall y\ xy^{[F]} = x \ \&\ yx = y \ \to\ xy^{[F\cap I]} = x.$$

Let $\Phi_I = \{\varphi_F^I \mid F \in P_{\mathrm{fin}}(\mathbb{P})\}$ and let $\mathbf{K}_I$ be the subquasivariety of **Dm** defined by the set of quasi-identities $\Phi_I$. The following assertion is proven in [1, Theorem 5].

**Proposition 3.** *For every infinite proper subset $I \subseteq \mathbb{P}$, the quasivariety $\mathbf{K}_I$ has no independent quasi-equational basis relative to* **Dm**.

We fix an infinite subset $I \subseteq \mathbb{P}$ such that the complement $\mathbb{P}\backslash I$ is infinite too. For all $F \in P_{\mathrm{fin}}(\mathbb{P})$ and $p \notin F$, we denote by $\psi_F^p$ the quasi-identity

$$\forall x\, \forall y\ xy^{[F\cup\{p\}]} = x \ \&\ yx = y \ \to\ xy^{[F]} = x.$$

We put $\Psi_p = \{\psi_F^p \mid F \in P_{\mathrm{fin}}(\mathbb{P}),\ p \notin F\}$ and $\Psi_I = \bigcup_{p\in\mathbb{P}\backslash I} \Psi_p$. We denote by $\mathbf{K}'_I$ the subquasivariety of **Dm** defined by the set of quasi-identities $\Psi_I$.

**Theorem 4.** *There exist $2^\omega$ quasivarieties of differential groupoids with an $\omega$-independent quasi-equational basis and no independent quasi-equational basis relative to* **Dm**.

*Proof.* We prove that the quasivarieties of the form $\mathbf{K}'_I$ are the required ones. We first show that $\mathbf{K}_I = \mathbf{K}'_I$.

Indeed, let $\mathcal{A} \in \mathbf{K}_I$, i.e., let $\mathcal{A} \models \Phi_I$. We consider $\psi_F^p \in \Psi_I$. Let the premise of $\psi_F^p$ hold on some elements $a, b \in A$, i.e., assume that $ba = b$ and $ab^{[F\cup\{p\}]} = a$ in $\mathcal{A}$. Since $p \notin I$, we have $(F \cup \{p\}) \cap I = F \cap I \subseteq F$. In particular, $m = \big[(F \cup \{p\}) \cap I\big]$ divides $[F]$. Since the quasi-identity $\varphi^I_{F\cup\{p\}}$ holds in $\mathcal{A}$, we obtain $ab^m = a$. Therefore, we have $ab^{[F]} = a$ and the conclusion of $\psi_F^p$ holds on $a$ and $b$. Thus, $\mathcal{A} \models \Psi_I$ and $\mathcal{A} \in \mathbf{K}'_I$.

Conversely, let $\mathcal{A} \in \mathbf{K}'_I$, i.e., let $\mathcal{A} \models \Psi_I$. We consider $\varphi_F^I \in \Phi_I$. Let the premise of $\varphi_F^I$ hold on some elements $a, b \in A$, i.e., assume that $ba = b$ and $ab^{[F]} = a$ in $\mathcal{A}$. If $F \subseteq I$ then $F \cap I = F$ and the conclusion of $\varphi_F^I$ holds on $a$ and $b$. Otherwise, we have $F \backslash I \neq \varnothing$. Let $F \backslash I = \{p_0, \dots, p_k\}$, let $G_0 = F$, and let $G_{i+1} = G_i \backslash \{p_i\}$, where $0 \leqslant i \leqslant k$. Then $G_{k+1} = F \cap I$. Since $ab^{[F]} = a$, the premise of the quasi-identity $\psi_{G_0}^{p_0}$ holds in $\mathcal{A}$. Let $0 \leqslant i \leqslant k$. Since $p_i \notin I$, the quasi-identity $\psi_{G_i}^{p_i}$ belongs to $\Psi_I$. It is clear that the conclusion of $\psi_{G_i}^{p_i}$ coincides with the first atomic formula in the premise of $\psi_{G_{i+1}}^{p_{i+1}}$. Therefore, we have $ab^{[G_{k+1}]} = a$ in $\mathcal{A}$; hence, the conclusion of $\varphi_F^I$ holds. Thus, $\mathcal{A} \models \Phi_I$ and $\mathcal{A} \in \mathbf{K}_I$.

According to Proposition 3, there exist $2^\omega$ quasivarieties of the form $\mathbf{K}'_I$ with no independent quasi-equational basis. We show that $\Psi_I = \bigcup_{p \in \mathbb{P} \backslash I} \Psi_p$ is an $\omega$-independent quasi-equational basis for $\mathbf{K}'_I$ relative to $\mathbf{Dm}$.

We fix $p \in \mathbb{P} \backslash I$ and prove that the cycle $\mathcal{D}_p$ satisfies each quasi-identity from $\Psi_I \backslash \Psi_p$. Indeed, let $q \in \mathbb{P} \setminus I$, $q \neq p$, $F \in P_{\mathrm{fin}}(\mathbb{P})$, and $q \notin F$. Let the premise of $\psi_F^q$ hold in $\mathcal{D}_p$ under some interpretation of variables. Then there is a homomorphism from $\mathcal{D}_{[F \cup \{q\}]}$ into $\mathcal{D}_p$. If the image of this homomorphism is isomorphic to either $\mathcal{D}_0$ or $\mathcal{D}_1$ then the conclusion of $\psi_F^q$ holds under the same interpretation of variables. Otherwise, according to Lemma 2 (a), this image is isomorphic to $\mathcal{D}_p$. Applying Lemma 2 (b), we find that $p$ divides $[F \cup \{q\}]$. Since $p \neq q$, we have $p \in F$; hence, $p$ divides $[F]$. Therefore, the conclusion of $\psi_F^q$ holds in $\mathcal{D}_p$ under the same interpretation of variables.

Since $\mathcal{D}_p$ does not satisfy $\psi_\varnothing^p \in \Psi_p$, the basis $\Psi_I$ is $\omega$-independent. □

Let $\mathbf{K}$ denote the intersection of all quasivarieties $\mathbf{K}_I$, where $I$ ranges over the set $P_\infty(\mathbb{P})$ of all infinite proper subsets of $\mathbb{P}$. The set of quasi-identities $\varphi_F^I$, where $I$ again ranges over $P_\infty(\mathbb{P})$ and $F$ ranges over the set of all finite subsets of $\mathbb{P}$, constitutes a quasi-equational basis for $\mathbf{K}$. The following assertion is straightforward to verify.

**Lemma 5.** *The equality*

$$\mathbf{K} = \bigcap \{\mathbf{K}'_I \mid I \subseteq \mathbb{P},\ |I| = |\mathbb{P} \backslash I| = \omega\}$$

*holds. Moreover, $\mathcal{A} \in \mathbf{K}$ if and only if no cycle $\mathcal{D}_{[F]}$ with $F \neq \varnothing$ embeds into $\mathcal{A}$.*

By Theorem 4 and Lemma 5, each of the sets $\bigcup_I \Phi_I$ and $\bigcup_I \Psi_I$ forms a quasi-equational basis for $\mathbf{K}$. However, these two bases are not independent. Nevertheless, the following assertion holds.

**Proposition 6.** *The quasivariety $\mathbf{K}$ has a recursive independent quasi-equational basis relative to $\mathbf{Dm}$.*

*Proof.* Consider the natural order $p_0 < p_1 < \ldots < p_n < \ldots$ on the set $\mathbb{P}$. We put $F_{-1} = \varnothing$ and $F_n = \{p_k \in \mathbb{P} \mid k \leqslant n\}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we denote by $\xi_n$ the following quasi-identity:

$$\forall x \, \forall y \ xy^{[F_n]} = x \ \&\ yx = y \ \to xy^{[F_{n-1}]} = x.$$

We show that $\Xi = \{\xi_n \mid n \in \mathbb{N}\}$ is an independent quasi-equational basis for $\mathbf{K}$.

By Lemma 5, for each structure $\mathcal{A} \notin \mathbf{K}$, there is a nonempty finite set $F \subseteq \mathbb{P}$ such that $\mathcal{D}_{[F]}$ embeds into $\mathcal{A}$. Consider the greatest $n \in \mathbb{N}$ such that $p_n \in F$. Then $\mathcal{D}_{[F]}$ (and, consequently, $\mathcal{A}$) does not satisfy $\xi_n$. Conversely, if $n \in \mathbb{N}$ and the premise of $\xi_n$ holds in a differential groupoid $\mathcal{A} \in \mathbf{K}$ under some interpretation of variables then there exists a homomorphism from $\mathcal{D}_{[F_n]}$ into $\mathcal{A}$. If the image of this homomorphism is isomorphic to either $\mathcal{D}_0$ or $\mathcal{D}_1$ then the conclusion of $\xi_n$ holds in $\mathcal{A}$ under the same interpretation of variables. Otherwise, by Lemma 2 (a), the cycle $\mathcal{D}_{[G]}$, where $\varnothing \neq G \subseteq F_n$, embeds into $\mathcal{A}$. In view of Lemma 5, this is impossible; hence, $\mathcal{A}$ satisfies $\xi_n$. Therefore, the set $\Xi$ is a quasi-equational basis for $\mathbf{K}$. Since the set $\{([F_n], [F_{n-1}]) \mid n \geqslant 0\}$ is computable, this basis is recursive.

We show that $\Xi$ is an independent basis for $\mathbf{K}$. In view of Lemma 5, it suffices to establish that, for every $n \in \mathbb{N}$, the cycle $\mathcal{D}_{[F_n]}$ satisfies each quasi-identity $\xi_m$ with $m \neq n$. If $m < n$ then the premise of $\xi_m$ holds on some elements $a, b \in D_{[F_n]}$ if and only if the subgroupoid of $\mathcal{D}_{[F_n]}$ generated by these elements is isomorphic to either $\mathcal{D}_0$ or $\mathcal{D}_1$. In this case, the conclusion of $\xi_m$ holds on $a$ and $b$ in $\mathcal{D}_{[F_n]}$. If $m > n$ then $F_n \subseteq F_{m-1}$. Therefore, the conclusion of $\xi_m$ holds on arbitrary elements $a, b \in D_{[F_n]}$. Thus, we have $\mathcal{D}_{[F_n]} \not\models \xi_n$ and $\mathcal{D}_{[F_n]} \models \xi_m$ if $m \neq n$, which is our desired conclusion. □

## 3. Unary algebras

Consider the signature $\sigma = \{f, g\}$ that consists of two unary function symbols. In the sequel, we write $f^2(x)$ instead of $f\bigl(f(x)\bigr)$ and $g^2(x)$ instead of $g\bigl(g(x)\bigr)$. Let $\mathbf{K}_3$ denote the variety of unary algebras of the signature $\sigma$ defined by the identities

$$\forall x\, \forall y\ f^2(x) = g^2(x) = fg(y) = gf(y).$$

Let $\mathbf{W}$ denote the proper subquasivariety of $\mathbf{K}_3$ defined by the quasi-identities

$$\begin{aligned}
\forall x\ f(x) = f^2(x) &\ \rightarrow\ f(x) = g(x),\\
\forall x\ g(x) = g^2(x) &\ \rightarrow\ f(x) = g(x),\\
\forall x\ f(x) = g(x) &\ \rightarrow\ f(x) = f^2(x),\\
\forall x\, \forall y\ f(x) = f(y) &\ \rightarrow\ g(x) = g(y),\\
\forall x\, \forall y\ g(x) = g(y) &\ \rightarrow\ f(x) = f(y).
\end{aligned}$$

Let $\mathbf{W}'$ denote the elementary subclass of $\mathbf{W}$ defined by the sentences

$$\forall x\, \forall y\ g(x) = g(y)\ \&\ x \neq y\ \rightarrow\ g(x) = g^2(x), \tag{1}$$

$$\bigl[\forall x\ g(x) = g^2(x)\bigr]\ \rightarrow\ \bigl[\forall x\, \forall y\ x = y\bigr]. \tag{2}$$

Let $n > 1$. The structure defined in $\mathbf{W}$ by the generators

$$X_n = \{0\} \cup \{a_0, \ldots, a_{n-1}\} \cup \{b_0, \ldots, b_{n-1}\}$$

and the defining relations

$$\begin{aligned}
&f(0) = g(0) = f(a_i) = g(a_i) = 0,\\
&g(b_i) = a_i \text{ for all } 0 \leqslant i \leqslant n-1,\\
&f(b_i) = a_{i+1} \text{ for all } 0 \leqslant i \leqslant n-2 \text{ and } f(b_{n-1}) = a_0,
\end{aligned} \tag{3}$$

is called the *cycle* of length $n$ and is denoted by $\mathcal{W}_n$. Let $\mathcal{W}_1$ denote the trivial algebra of the signature $\sigma$.

We denote by $\overline{x}$ the tuple of variables from $X_n$ and by $\Delta_n(\overline{x})$ the conjunction of the formulas in (3). Let $n > 0$ and let $m > 1$. We denote by $\varphi^n_m$ the following quasi-identity:

$$\forall \overline{x}\ \Delta_{nm}(\overline{x})\ \rightarrow\ g(b_0) = a_n.$$

The following properties of the cycles are similar to those in Lemma 2 and are proven in [8, Lemmas 1 and 3].

**Lemma 7.** *Let $n > 1$.*

(a) *If $m$ divides $n$ then there exists a homomorphism from $\mathcal{W}_n$ onto $\mathcal{W}_m$. The kernels of all such homomorphisms coincide and have the form*

$$\ker\varphi = \{(0,0)\} \cup \{(a_i, a_j), (b_i, b_j) \mid i \equiv j \pmod{m}\}.$$

(b) *If $\mathcal{A} \in \mathbf{W}'$ and there exists a homomorphism from $\mathcal{W}_n$ onto $\mathcal{A}$ then $\mathcal{A}$ is isomorphic to $\mathcal{W}_m$ for a suitable divisor $m$ of $n$.*

(c) *If $\mathcal{A} \in \mathbf{W}$ and there exists a homomorphism $\varphi$ from $\mathcal{W}_n$ to $\mathcal{A}$ then one of the following conditions holds:*

(c1) *the conditions of assertion* (b) *hold;*

(c2) *we have $\{(0, a_i) \mid i \leqslant n\} \subseteq \ker\varphi$ and the substructure $\varphi(\mathcal{W}_n)$ satisfies the premise of sentence* (2);

(c3) *there exist $i \leq n$ and $k > 1$ such that $k$ divides $n$, we have $(a_i, a_{i+k}) \in \ker\varphi$ but $(a_i, a_{i+j}) \notin \ker\varphi$ for every $j$ with $0 < j < k$, and $(b_i, b_{i+k}) \notin \ker\varphi$.*

*If condition* (c3) *holds then $\mathcal{A}$ satisfies the premise of sentence* (1) *but violates its conclusion, i.e., $\mathcal{W}_k$ is a substructure of $\mathcal{A}$ and $|g^{-1}(\varphi(a_i))| > 1$.*

(d) *Let $k > 1$ and let $m > 0$. The quasi-identity $\varphi^n_m$ holds in $\mathcal{W}_k$ if and only if either $k$ divides $n$ or $k$ does not divide $mn$.*

**Theorem 8.** *There exist $2^\omega$ subsuqasivarieties of $\mathbf{W}$ with an $\omega$-independent quasi-equational basis and no independent quasi-equational basis relative to $\mathbf{W}$.*

*Proof.* Consider an infinite subset $I$ of the set $\mathbb{P}$ of primes such that the set $\mathbb{P}\backslash I$ is infinite too. Let $\mathbf{W}_I$ denote the class of structures in $\mathbf{W}$ with no substructure isomorphic to $\mathcal{W}_{p^k s}$, where $p \notin I$, $k > 0$, and $s \in \mathbb{N}$. As is shown in the proof of [8, Theorem 2], the quasivariety $\mathbf{W}_I$ has no independent quasi-equational basis relative to $\mathbf{W}$. Notice that $\mathbf{W}$ has a finite quasi-equational basis relative to both the variety $\mathbf{K}_3$ and the class $\mathbf{K}(\sigma)$ of all algebras of the signature $\sigma$. In view of [3, Theorem 6.3.1], the quasivariety $\mathbf{W}_I$ has an independent quasi-equational basis relative to neither $\mathbf{K}(\sigma)$ nor $\mathbf{K}_3$.

For every $m$ with prime divisors $p \notin I$, we find the greatest such $p$ and fix the representation

$$m = p^k s, \tag{4}$$

where $k > 0$, $s \in \mathbb{N}$, and $p$ and $s$ are relatively prime. We denote by $\psi_m$ the quasi-identity $\varphi^s_{p^k}$ and by $\Sigma_p$ the set of all quasi-identities $\psi_m$ of such a form.

We prove that $\Sigma_I = \bigcup_{p\in\mathbb{P}\backslash I} \Sigma_p$ is an $\omega$-independent quasi-equational basis for $\mathbf{W}_I$ relative to $\mathbf{W}$.

First we prove that $\Sigma_I$ is a quasi-equational basis for $\mathbf{W}_I$ relative to $\mathbf{W}$. Indeed, let $m$ have prime divisors that do not belong to $I$ and let the cycle $\mathcal{W}_m$ embed into a structure $\mathcal{A} \in \mathbf{W}$. Consider representation (4) and the corresponding quasi-identity $\psi_m$. It is clear that $\mathcal{A}$ does no satisfy $\psi_m$. Assume that no cycle $\mathcal{W}_m$, where $m$ has prime divisors that do not belong to $I$, embeds into a structure $\mathcal{A} \in \mathbf{W}$. Consider an arbitrary quasi-identity of the form $\psi_m$. If the premise of this quasi-identity holds in $\mathcal{A}$ under some interpretation of variables then there exists a homomorphism $\varphi$ from $\mathcal{W}_m$ to $\mathcal{A}$ extending this interpretation. By Lemma 7 (c), three cases are possible.

*Case* 1: the structure $\mathcal{A}$ is isomorphic to the cycle $\mathcal{W}_n$, where $n$ divides $m$. Since $\mathcal{A} \in \mathcal{W}_I$, the numbers $n$ and $p$ are relatively prime. We find that $n$ divides $s$, i.e., the conclusion of $\psi_m$ holds in $\mathcal{A}$ under the same interpretation of variables.

*Case* 2: the conditions $\varphi(a_n) = 0$ and $\varphi\big(g(x)\big) = \varphi\big(g^2(x)\big)$ hold for all $x \in W_m$. We have $g(\varphi(b_0)) = \varphi(g(b_0)) = \varphi(a_0) = 0 = \varphi(a_n)$, i.e., the conclusion of $\psi_m$ holds in $\mathcal{A}$ under the same interpretation of variables.

*Case* 3: the substructure $\varphi(\mathcal{W}_m)$ of $\mathcal{A}$ is isomorphic to the cycle $\mathcal{W}_n$, where $n$ divides $m$. This case is similar to Case 1.

Second we prove that, for every $p \in \mathbb{P}\backslash I$, there exists a natural number $n$ such that the cycle $\mathcal{W}_n$ satisfies each quasi-identity from the set $\Sigma_I\backslash\Sigma_p$ but $\mathcal{W}_n \notin \mathbf{W}_I$. Let $q$ be a natural number such that $q > 1$ and each prime divisor of $q$ belongs to $I$. Put $n = pq$. Then $\mathcal{W}_n$ does not satisfy the quasi-identity $\psi_n$.

Let $\psi_m \in \Sigma_I\backslash\Sigma_p$. Assume that the premise of this quasi-identity holds in $\mathcal{W}_n$ under some interpretation of variables. Then there exists a homomorphism from $\mathcal{W}_m$ to $\mathcal{W}_n$ extending this interpretation. By Lemma 7 (c), we find that $n$ divides $m$. In particular, we conclude that $p$ divides $m$. Hence, the set of prime divisors of $m$ that do not belong to $I$ is nonempty. We choose the greatest element $p_*$ of this set. By the choice of $q$, the numbers $p_*$ and $q$ are relatively prime. Since $n$ divides $m$, we have $p \leqslant p_*$. Since $\psi_m \notin \Sigma_p$, we obtain the strict inequality $p < p_*$. Hence, the numbers $n = pq$ and $p_*$ are relatively prime. We consider the representation $m = p_*^t r$, where $p_*$ and $r$ are relatively prime. By the above, we conclude that $n$ divides $r$. Therefore, the conclusion of $\psi_m$ holds in $\mathcal{W}_n$ under the same interpretation of variables. □

Let $\mathbf{L}$ denote the intersection of all quasivarieties $\mathbf{W}_I$, where $I$ ranges over the set of all subsets $I$ of $\mathbb{P}$ with $|I| = |\mathbb{P}\backslash I| = \omega$. Its quasi-equational basis consists of the quasi-identities $\varphi^s_{p^k}$, where $m = p^k s$ ranges over the set of all natural numbers that are greater than 1. It is easy to see that, for a unary algebra $\mathcal{A}$, we have $\mathcal{A} \in \mathbf{L}$ if and only if no cycle $\mathcal{W}_m$ with $m > 1$ embeds into $\mathcal{A}$. By Theorem 8, the set of quasi-identities $\Sigma = \bigcup_{p\in\mathbb{P}} \Sigma_p$ forms an $\omega$-independent quasi-equational basis for $\mathbf{L}$. We find an independent quasi-equational basis for $\mathbf{L}$.

**Proposition 9.** *The quasivariety* $\mathbf{L}$ *has an infinite recursive independent quasi-equational basis relative to* $\mathbf{W}$.

*Proof.* Consider the natural order $p_0 < p_1 < \ldots < p_n < \ldots$ on the set $\mathbb{P}$. For every $n \in \mathbb{N}$, we put

$$n^* = p_0^{n+1} \cdot \ldots \cdot p_n^{n+1}; \qquad n^\sharp = \begin{cases} p_0^n \cdot \ldots \cdot p_{n-1}^n & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

We denote by $\gamma_n$ the quasi-identity $\varphi^{n^\sharp}_{n^*/n^\sharp}$. We put $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$. It is straightforward to verify that $\Gamma$ is a quasi-equational basis for $\mathbf{L}$ relative to $\mathbf{W}$. Since the set $\{(n^*, n^\sharp) | n \geqslant 0\}$ is computable, this basis is recursive.

We show that $\Gamma$ is an independent basis. For each $n \in \mathbb{N}$, the quasi-identity $\gamma_n$ does not hold in $\mathcal{W}_{n^*}$. We prove that $\mathcal{W}_{n^*} \models \gamma_m$ if $m \neq n$. Assume that the premise of $\gamma_m$ holds in $\mathcal{W}_{n^*}$ under some interpretation of variables. Hence, there exists a homomorphism from $\mathcal{W}_{m^*}$ to $\mathcal{W}_{n^*}$. By Lemma 7 (c), we find that $n^*$ divides $m^*$. Since $m \neq n$, we obtain $n < m$. Therefore, the conclusion of $\gamma_m$ holds in $\mathcal{W}_{n^*}$ under the same interpretation of variables. Thus, we have $\mathcal{W}_{n^*} \not\models \gamma_n$ and $\mathcal{W}_{n^*} \models \gamma_m$ if $m \neq n$, which is our desired conclusion. □

Notice that only three examples are known of quasivarieties with no $\omega$-independent quasi-equational basis; namely, two quasivarieties of unary algebras [3, Corollary 6.4.2] and the quasivariety of bipartite graphs [4, Theorem 4.5].

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Aleksandr Vladimirovich Kravchenko
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Novosibirsk State University,
ul. Pirogova, 1,
630090, Novosibirsk, Russia
Siberian Institute of Management,
ul. Nizhegorodskaya, 6,
630102, Novosibirsk, Russia
*E-mail address*: a.v.kravchenko@mail.ru

Anvar Mukhparovich Nurakunov
Institute of Mathematics NAS RK,
pr. Chui 265 a,
720071, Bishkek, Kyrgyzstan
*E-mail address*: a.nurakunov@gmail.com

Marina Vladimirovna Schwidefsky
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Novosibirsk State University,
ul. Pirogova, 1,
630090, Novosibirsk, Russia
*E-mail address*: udav17@gmail.com