

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 14, стр. 135–150 (2017)

УДК 510.67

DOI 10.17377/semi.2017.14.014

MSC 03C30, 03C15, 03C50, 54A05

COMBINATIONS RELATED TO CLASSES OF FINITE AND
COUNTABLY CATEGORICAL STRUCTURES AND THEIR
THEORIES

S.V. SUDOPLATOV

ABSTRACT. We consider and characterize classes of finite and countably categorical structures and their theories preserved under E -operators and P -operators. We describe e -spectra and families of finite cardinalities for structures belonging to closures with respect to E -operators and P -operators.

Keywords: finite structure, countably categorical structure, elementary theory, E -operator, P -operator, e -spectrum.

We continue to study structural properties of E -combinations and P -combinations of structures and their theories [14, 15, 16, 17, 18] applying the general context to the classes of finitely categorical and ω -categorical theories.

Approximations of structures by finite ones as well as correspondent approximations of theories were studied in a series of papers, e.g. [3, 19, 20]. We consider these approximations in the context of structural combinations.

We consider and describe e -spectra and families of finite cardinalities for structures belonging to closures with respect to E -operators and P -operators.

The paper is organized as follows. In Section 1 we recall necessary preliminary notions and results. In Section 2 we study closed classes of theories with finite models and of ω -categorical theories. Properties related to approximations of theories with finite and infinite models are described in Section 3. In Section 4 we describe e -spectra for theories with finite models and of ω -categorical theories. Relations

SUDOPLATOV, S.V., COMBINATIONS RELATED TO CLASSES OF FINITE AND COUNTABLY CATEGORICAL STRUCTURES AND THEIR THEORIES.

© 2017 SUDOPLATOV S.V.

The research is partially supported by the Grants Council (under RF President) for State Aid of Leading Scientific Schools (grant NSh-6848.2016.1) and by Committee of Science in Education and Science Ministry of the Republic of Kazakhstan (Grant No. 0830/GF4).

Received January, 31, 2017, published February, 22, 2017.

of (almost) language uniform theories and theories with finite models studies in Section 5. In Section 6 we describe families of finite cardinalities for models of theories in E -closures and P -closures.

1. PRELIMINARIES

Throughout the paper we use the following terminology in [14, 15] as well as in [9, 10].

Let $P = (P_i)_{i \in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i \in I}$ be a family of structures such that P_i is the universe of \mathcal{A}_i , $i \in I$, and the symbols P_i are disjoint with languages for the structures \mathcal{A}_j , $j \in I$. The structure $\mathcal{A}_P \equiv \bigcup_{i \in I} \mathcal{A}_i$ expanded by the predicates P_i is the P -union of the structures \mathcal{A}_i , and the operator mapping $(\mathcal{A}_i)_{i \in I}$ to \mathcal{A}_P is the P -operator. The structure \mathcal{A}_P is called the P -combination of the structures \mathcal{A}_i and denoted by $\text{Comb}_P(\mathcal{A}_i)_{i \in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright \mathcal{A}_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures \mathcal{A}' , which are elementary equivalent to $\text{Comb}_P(\mathcal{A}_i)_{i \in I}$, will be also considered as P -combinations.

Clearly, all structures $\mathcal{A}' \equiv \text{Comb}_P(\mathcal{A}_i)_{i \in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \not\equiv \text{Comb}_P(\mathcal{A}'_i)_{i \in I}$, we write $\mathcal{A}' = \text{Comb}_P(\mathcal{A}'_i)_{i \in I \cup \{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap_{i \in I} \overline{P}_i$, maybe applying Morleyzation. Moreover, we write

$$\text{Comb}_P(\mathcal{A}_i)_{i \in I \cup \{\infty\}}$$

for $\text{Comb}_P(\mathcal{A}_i)_{i \in I}$ with the empty structure \mathcal{A}_∞ .

Note that if all predicates P_i are disjoint, a structure \mathcal{A}_P is a P -combination and a disjoint union of structures \mathcal{A}_i . In this case the P -combination \mathcal{A}_P is called *disjoint*. Clearly, for any disjoint P -combination \mathcal{A}_P , $\text{Th}(\mathcal{A}_P) = \text{Th}(\mathcal{A}'_P)$, where \mathcal{A}'_P is obtained from \mathcal{A}_P replacing \mathcal{A}_i by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P -operator works for the theories $T_i = \text{Th}(\mathcal{A}_i)$ producing the theory $T_P = \text{Th}(\mathcal{A}_P)$, being P -combination of T_i , which is denoted by $\text{Comb}_P(T_i)_{i \in I}$. In general, for non-disjoint case, the theory T_P will be also called a P -combination of the theories T_i , but in such a case we will keep in mind that this P -combination is constructed with respect (and depending) to the structure \mathcal{A}_P , or, equivalently, with respect to any/some $\mathcal{A}' \equiv \mathcal{A}_P$.

For an equivalence relation E replacing disjoint predicates P_i by E -classes we get the structure \mathcal{A}_E being the E -union of the structures \mathcal{A}_i . In this case the operator mapping $(\mathcal{A}_i)_{i \in I}$ to \mathcal{A}_E is the E -operator. The structure \mathcal{A}_E is also called the E -combination of the structures \mathcal{A}_i and denoted by $\text{Comb}_E(\mathcal{A}_i)_{i \in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright \mathcal{A}_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures \mathcal{A}' , which are elementary equivalent to \mathcal{A}_E , are denoted by $\text{Comb}_E(\mathcal{A}'_j)_{j \in J}$, where \mathcal{A}'_j are restrictions of \mathcal{A}' to its E -classes. The E -operator works for the theories $T_i = \text{Th}(\mathcal{A}_i)$ producing the theory $T_E = \text{Th}(\mathcal{A}_E)$, being E -combination of T_i , which is denoted by $\text{Comb}_E(T_i)_{i \in I}$ or by $\text{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into \mathcal{A}_P and can not be represented as a disjoint P -combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E -combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E -combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of \mathcal{A}' to be the E -representability.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E -representable, we have the E' -representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T , where T is a theory of a restriction \mathcal{B} of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E -class and \mathcal{B} is not elementary equivalent to the structures \mathcal{A}_i . The resulting structure $\mathcal{A}_{E'}$ (with the E' -representability) is a e -completion, or a e -saturation, of \mathcal{A}_E . The structure $\mathcal{A}_{E'}$ itself is called e -complete, or e -saturated, or e -universal, or e -largest.

For a structure \mathcal{A}_E the number of *new* structures with respect to the structures \mathcal{A}_i , i. e., of the structures \mathcal{B} which are pairwise elementary non-equivalent and elementary non-equivalent to the structures \mathcal{A}_i , is called the e -spectrum of \mathcal{A}_E and denoted by $e\text{-Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the e -spectrum of the theory $\text{Th}(\mathcal{A}_E)$ and denoted by $e\text{-Sp}(\text{Th}(\mathcal{A}_E))$.

If \mathcal{A}_E does not have E -classes \mathcal{A}_i , which can be removed, with all E -classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\text{Th}(\mathcal{A}_E)$, then \mathcal{A}_E is called e -prime, or e -minimal.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\text{TH}(\mathcal{A}')$ the set of all theories $\text{Th}(\mathcal{A}_i)$ of E -classes \mathcal{A}_i in \mathcal{A}' .

By the definition, an e -minimal structure \mathcal{A}' consists of E -classes with a minimal set $\text{TH}(\mathcal{A}')$. If $\text{TH}(\mathcal{A}')$ is the least for models of $\text{Th}(\mathcal{A}')$ then \mathcal{A}' is called e -least.

Definition [15]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\text{Cl}_E(\mathcal{T})$ the set of all theories $\text{Th}(\mathcal{A})$, where \mathcal{A} is a structure of some E -class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \text{Comb}_E(\mathcal{A}_i)_{i \in I}$, $\text{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \text{Cl}_E(\mathcal{T})$ then \mathcal{T} is said to be E -closed.

The operator Cl_E of E -closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\text{Cl}_E(\mathcal{T})$ is the union of all $\text{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by \mathcal{T}_φ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 1.1 [15]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \text{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for \mathcal{T} with respect to E -closure Cl_E) if and only if for any formula $\varphi \in T$ the set \mathcal{T}_φ is infinite.*

Theorem 1.2 [15]. *If \mathcal{T}'_0 is a generating set for a E -closed set \mathcal{T}_0 then the following conditions are equivalent:*

- (1) \mathcal{T}'_0 is the least generating set for \mathcal{T}_0 ;
- (2) \mathcal{T}'_0 is a minimal generating set for \mathcal{T}_0 ;
- (3) any theory in \mathcal{T}'_0 is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;
- (4) any theory in \mathcal{T}'_0 is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.

Definition [15]. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\text{Cl}_P(\mathcal{T})$ the set of all theories $\text{Th}(\mathcal{A})$ such that $\text{Th}(\mathcal{A}) \in \mathcal{T}$ or \mathcal{A} is a structure of type $p_\infty(x)$ in $\mathcal{A}' \equiv \mathcal{A}_P$, where $\mathcal{A}_P = \text{Comb}_P(\mathcal{A}_i)_{i \in I}$ and $\text{Th}(\mathcal{A}_i) \in \mathcal{T}$ are pairwise distinct. As above, if $\mathcal{T} = \text{Cl}_P(\mathcal{T})$ then \mathcal{T} is said to be P -closed.

Using above only disjoint P -combinations \mathcal{A}_P we get the closure $\text{Cl}_P^d(\mathcal{T})$ being a subset of $\text{Cl}_P(\mathcal{T})$.

The closure operator $\text{Cl}_P^{d,r}$ is obtained from Cl_P^d permitting repetitions of structures for predicates P_i .

Replacing E -classes by unary predicates P_i (not necessary disjoint) being universes for structures \mathcal{A}_i and restricting models of $\text{Th}(\mathcal{A}_P)$ to the set of realizations of $p_\infty(x)$ we get the e -spectrum $e\text{-Sp}(\text{Th}(\mathcal{A}_P))$, i. e., the number of pairwise elementary non-equivalent restrictions of $\mathcal{M} \models \text{Th}(\mathcal{A}_P)$ to $p_\infty(x)$ such that these restrictions are not elementary equivalent to the structures \mathcal{A}_i .

Definition [9, 10]. A n -dimensional cube, or a n -cube (where $n \in \omega$) is a graph isomorphic to the graph \mathcal{Q}_n with the universe $\{0, 1\}^n$ and such that any two vertices $(\delta_1, \dots, \delta_n)$ and $(\delta'_1, \dots, \delta'_n)$ are adjacent if and only if these vertices differ exactly in one coordinate. The described graph \mathcal{Q}_n is called the *canonical representative* for the class of n -cubes.

Let λ be an infinite cardinal. A λ -dimensional cube, or a λ -cube, is a graph isomorphic to a graph $\Gamma = \langle X; R \rangle$ satisfying the following conditions:

(1) the universe $X \subseteq \{0, 1\}^\lambda$ is generated from an arbitrary function $f \in X$ by the operator $\langle f \rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f(i_1), \dots, f(i_m))$ by tuples $(1 - f(i_1), \dots, 1 - f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate (the i -th coordinate of function $g \in \{0, 1\}^\lambda$ is the value $g(i)$ correspondent to the argument $i < \lambda$).

The described graph $\mathcal{Q} = \mathcal{Q}_f$ with the universe $\langle f \rangle$ is a *canonical representative* for the class of λ -cubes.

Note that the canonical representative of the class of n -cubes (as well as the canonical representatives of the class of λ -cubes) are generated by any its function: $\{0, 1\}^n = \langle f \rangle$, where $f \in \{0, 1\}^n$. Therefore the universes of canonical representatives \mathcal{Q}_f of n -cubes like λ -cubes, is denoted by $\langle f \rangle$.

2. CLOSED CLASSES OF FINITELY CATEGORICAL AND ω -CATEGORICAL THEORIES

Remind that a countable complete theory T is ω -categorical if T has exactly one countable model up to isomorphisms, i.e. $I(T, \omega) = 1$. A countable theory T is n -categorical, for natural $n \geq 1$, if T has exactly one n -element model up to isomorphisms, i.e. $I(T, n) = 1$. A countable theory T is *finitely categorical* if T is n -categorical for some $n \in \omega \setminus \{0\}$.

The classes of all finitely and ω -categorical theories will be denoted by $\overline{\mathcal{T}}_{\text{fin}}$ and $\overline{\mathcal{T}}_{\omega, 1}$, respectively.

Let \mathcal{T} be a set (class) of theories in $\overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega, 1}$, T be a theory in \mathcal{T} . By Ryll-Nardzewski theorem, $S^n(T)$ is finite for any n . Then, for any n , classes $[\varphi(\bar{x})] = \{\varphi'(\bar{x}) \mid \varphi'(\bar{x}) \equiv \varphi(\bar{x})\}$ of T -formulas with n free variables and $[\varphi(\bar{x})] \leq [\psi(\bar{x})] \Leftrightarrow \varphi(\bar{x}) \vdash \psi(\bar{x})$ form a finite Boolean algebra $\mathcal{B}_n(T)$ with 2^{m_n} elements, where m_n is the number of n -types of T .

The algebra $\mathcal{B}_n(T)$ can be interpreted as a m_n -cube $\mathcal{C}_{m_n}(T)$, whose vertices form the universe $B_n(T)$ of $\mathcal{B}_n(T)$, edges $[a, b]$ link vertices a and b such that $a \leq b$ or $b \leq a$, and each vertex a is marked by some $u_a \equiv [\varphi(\bar{x})]$, where $a \leq b \Leftrightarrow u_a \leq u_b$. The label 0 is used for the vertex corresponding to $[\neg \bar{x} \approx \bar{x}]$ and 1 — for the vertex corresponding to $[\bar{x} \approx \bar{x}]$.

Obviously, the sets $[\varphi(\bar{x})]$ and the relation \leq depend on the theory T but we omit T if the theory is fixed or it is clear by the context.

Clearly, algebras $\mathcal{B}_n(T_1)$ and $\mathcal{B}_n(T_2)$, for $T_1, T_2 \in \mathcal{T}$, may be not coordinated: it is possible $\lceil \varphi(\bar{x}) \rceil < \lceil \psi(\bar{x}) \rceil$ for T_1 whereas $\lceil \psi(\bar{x}) \rceil < \lceil \varphi(\bar{x}) \rceil$ for T_2 . If $\lceil \varphi(\bar{x}) \rceil < \lceil \psi(\bar{x}) \rceil$ for T_1 and $\lceil \varphi(\bar{x}) \rceil < \lceil \psi(\bar{x}) \rceil$ for T_2 , we say that T_2 *witnesses* that $\lceil \varphi(\bar{x}) \rceil < \lceil \psi(\bar{x}) \rceil$ for T_1 (and vice versa).

At the same time, if a countable theory T_0 does not belong to $\overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$ then for some $n \geq 1$, $\mathcal{B}_n(T_0)$ is infinite and therefore there is a formula $\varphi(\bar{x})$, for instance $(\bar{x} \approx \bar{x})$, such that for the label $u = \lceil \varphi(\bar{x}) \rceil$ there is an infinite decreasing chain $(u_k)_{k \in \omega}$ of labels: $u_{k+1} < u_k < u$, witnessed by some formulas $\varphi_k(\bar{x})$. In such a case, if $T_0 \in \text{Cl}_E(\mathcal{T})$, then by Proposition 1.1 for any finite sequence (u_1, \dots, u_0, u) there are infinitely many theories in \mathcal{T} witnessing that $u_1 < \dots < u_0 < u$. In particular, cardinalities m_n for Boolean algebras $\mathcal{B}_n(T)$ and for cubes $\mathcal{C}_{m_n}(T)$ are unbounded for \mathcal{T} : distances $\rho_{n,T}(0, u)$ are unbounded for the cubes $\mathcal{C}_{m_n}(T)$, i.e., $\sup\{\rho_{n,T}(0, u) \mid T \in \mathcal{T}\} = \infty$. It is equivalent to take $(\bar{x} \approx \bar{x})$ for $\varphi(\bar{x})$ and to get $\sup\{\rho_{n,T}(0, 1) \mid T \in \mathcal{T}\} = \infty$.

Thus we get the following

Theorem 2.1. *Let \mathcal{T} be a class of theories in $\overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$. The following conditions are equivalent:*

- (1) $\text{Cl}_E(\mathcal{T}) \not\subseteq \overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$;
- (2) for some natural $n \geq 1$, Boolean algebras $\mathcal{B}_n(T)$, $T \in \mathcal{T}$, have unbounded cardinalities and, moreover, there is an infinite decreasing chain $(u_k)_{k \in \omega}$ of labels for some formulas $\varphi_k(\bar{x})$ such that any finite sequence (u_1, \dots, u_0) with $u_1 < \dots < u_0$ is witnessed by infinitely many theories in \mathcal{T} ;
- (3) the same as in (2) with $u_0 = 1$.

Corollary 2.2. *A class $\mathcal{T} \subseteq \overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$ does not generate, using the E -operator, theories, which are neither finitely categorical and ω -categorical, if and only if for the Boolean algebras $\mathcal{B}_n(T)$, $T \in \mathcal{T}$, there are no infinite decreasing chains $(u_k)_{k \in \omega}$ of labels for some formulas $\varphi_k(\bar{x})$ such that any finite sequence (u_1, \dots, u_0) with $u_1 < \dots < u_0$ is witnessed by infinitely many theories in \mathcal{T} .*

Remark 2.3. Corollary 2.2 together with Proposition 1.1 allow to determine E -closed classes of finitely categorical and ω -categorical theories. Here, since finite sets of theories are E -closed, it suffices to consider infinite sets.

Considering a set \mathcal{T} of theories with disjoint languages, for the E -closeness it suffices to add theories of the empty language describing cardinalities, in $\omega + 1$, of universes if these cardinalities meet infinitely many times in \mathcal{T} .

In such a case we obtain relative closures [17] and have the following assertions.

Proposition 2.4. *A class \mathcal{T} of theories of pairwise disjoint languages is E -closed if and only if the following conditions hold:*

- (i) for any $n \in \omega \setminus \{0\}$ whenever \mathcal{T} contains infinitely many theories with n -element models then \mathcal{T} contains the theory T_n^0 of the empty language and with n -element models;
- (ii) if \mathcal{T} contains theories with unbounded finite cardinalities of models, or infinitely many theories with infinite models, then \mathcal{T} contains the theory T_∞^0 of the empty language and with infinite models.

Corollary 2.5. *A class $\mathcal{T} \subseteq \overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$ of theories of pairwise disjoint languages is E -closed if and only if the conditions (i) and (ii) hold.*

Corollary 2.6. *A class $\mathcal{T} \subset \overline{\mathcal{T}}_{\text{fin}}$ of theories of pairwise disjoint languages is E -closed if and only if the condition (i) holds and there is $N \in \omega$ such \mathcal{T} does not have n -categorical theories for $n > N$.*

Corollary 2.7. *A class $\mathcal{T} \subset \overline{\mathcal{T}}_{\omega,1}$ of theories of pairwise disjoint languages is E -closed if and only if the condition (ii) holds.*

Corollary 2.8. *For any class $\mathcal{T} \subset \overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$ of theories of pairwise disjoint languages, $\text{Cl}_E(\mathcal{T})$ is contained in the class $\subset \overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$, moreover,*

$$\text{Cl}_E(\mathcal{T}) \subseteq \mathcal{T} \cup \{T_\lambda^0 \mid \lambda \in (\omega \setminus \{0\}) \cup \{\infty\}\}.$$

Remark 2.9. Using relative closures [17] the assertions 2.4–2.8 also hold if languages are disjoint modulo a common sublanguage Σ_0 such that all restrictions of n -categorical theories in $\mathcal{T} \cap \overline{\mathcal{T}}_{\text{fin}}$ to Σ_0 have isomorphic (finite) models \mathcal{M}_n and all restrictions of theories in $\mathcal{T} \cap \overline{\mathcal{T}}_{\omega,1}$ have isomorphic countable models \mathcal{M}_ω . In such a case, the theories T_n^0 should be replaced by $\text{Th}(\mathcal{M}_n)$ and T_∞^0 – by $\text{Th}(\mathcal{M}_\omega)$.

It is also permitted to have finitely many possibilities for each \mathcal{M}_n and for \mathcal{M}_ω .

The following example shows that (even with pairwise disjoint languages) ω -categorical theories T with unbounded $\rho_{n,T}(0,1)$ do not force theories outside the class of ω -categorical theories.

Example 2.10. Let T_n be a theory of infinitely many disjoint n -cubes with a graph relation $R_n^{(2)}$, $R_m \neq R_n$ for $m \neq n$. For the set $\mathcal{T} = \{T_n \mid n \in \omega\}$ we have $\text{Cl}_E(\mathcal{T}) = \mathcal{T} \cup \{T_\infty^0\}$. All theories in $\text{Cl}_E(\mathcal{T})$ are ω -categorical whereas $\rho_{2,T_n}(0,1) = n+2$ that witnessed by formulas describing distances $d(x,y) \in \omega \cup \{\infty\}$ between elements.

Similarly, taking for each $n \in \omega$ exactly one n -cube with a graph relation $R_n^{(2)}$, we get a set \mathcal{T} of theories such that $\text{Cl}_E(\mathcal{T}) \subset \overline{\mathcal{T}}_{\text{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$.

Remark 2.11. Assertions 2.1 – 2.5 and 2.7 – 2.9 hold for the operators Cl_P^d and $\text{Cl}_P^{d,r}$ replacing E -closures by P -closures. As non-isolated types always produce infinite structures, Corollary 2.6 holds only for Cl_P^d with finite sets \mathcal{T} of theories.

3. ON APPROXIMATIONS OF THEORIES WITH (IN)FINITE MODELS

Definition [3]. An infinite structure \mathcal{M} is *pseudofinite* if every sentence true in \mathcal{M} has a finite model.

Definition (cf. [11]). A consistent formula φ *forces* the infinity if φ does not have finite models.

By the definition, an infinite structure \mathcal{M} is pseudofinite if and only if \mathcal{M} does not satisfy formulas forcing the infinity.

We denote the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\text{fin}}$ by $\overline{\mathcal{T}}_{\text{inf}}$.

Proposition 3.1. *A theory $T \in \overline{\mathcal{T}}_{\text{inf}}$ belongs to some E -closure of theories in $\overline{\mathcal{T}}_{\text{fin}}$ if and only if T does not have formulas forcing the infinity.*

Proof. If a formula φ forces the infinity then $\mathcal{T}_\varphi \subset \overline{\mathcal{T}}_{\text{inf}}$ for any $\mathcal{T} \subseteq \overline{\mathcal{T}}$. Thus, having such a formula $\varphi \in T$, T can not be approximated by theories in $\overline{\mathcal{T}}_{\text{fin}}$ and so T does not belong to E -closures of families $\mathcal{T} \subseteq \overline{\mathcal{T}}_{\text{fin}}$.

Conversely, if any formula $\varphi \in T$ does not force the infinity then, since $T \notin \overline{\mathcal{T}}_{\text{fin}}$, $(\overline{\mathcal{T}}_{\text{fin}})_\varphi$ is infinite using unbounded finite cardinalities and we can choose infinitely

many theories in $(\overline{\mathcal{T}}_{\text{fin}})_{\varphi}$, for each $\varphi \in T$, forming a set $\mathcal{T}_0 \subset \overline{\mathcal{T}}_{\text{fin}}$ such that $T \in \text{Cl}_E(\mathcal{T}_0)$. \square

Note that, in view of Proposition 1.1, Proposition 3.1 is a reformulation of Lemma 1 in [19].

Corollary 3.2. *If a theory $T \in \overline{\mathcal{T}}_{\text{inf}}$ belongs to some E -closure of theories in $\overline{\mathcal{T}}_{\text{fin}}$ then T is not finitely axiomatizable.*

Proof. If T is finitely axiomatizable by some formula φ then $|\mathcal{T}_{\varphi}| \leq 1$ for any $\mathcal{T} \subseteq \overline{\mathcal{T}}$ and φ forces the infinity. Thus, in view of Proposition 3.1, T can not be approximated by theories in $\overline{\mathcal{T}}_{\text{fin}}$, i. e., T does not belong to E -closures of families $\mathcal{T}_0 \subset \overline{\mathcal{T}}_{\text{fin}}$. \square

In fact, in view of Theorem 1.2, the arguments for Corollary 3.3 show that $\text{Cl}_E(\mathcal{T})$, for a family \mathcal{T} of finitely axiomatizable theories, has the least generating set \mathcal{T} and does not contain new finitely axiomatizable theories.

Note that Proposition 3.1 admits a reformulation for Cl_P^d repeating the proof. At the same time theories in $\overline{\mathcal{T}}_{\text{fin}}$ can not be approximated by theories in $\overline{\mathcal{T}}_{\text{inf}}$ with respect to Cl_E (in view of Proposition 1.1) whereas each theory in $\overline{\mathcal{T}}_{\text{fin}}$ can be approximated by theories in $\overline{\mathcal{T}}_{\text{inf}}$ with respect to Cl_P^d :

Proposition 3.3. *For any theory $T \in \overline{\mathcal{T}}_{\text{fin}}$ there is a family $\mathcal{T}_0 \subset \overline{\mathcal{T}}_{\text{inf}}$ such that T belongs to the $\Sigma(T)$ -restriction of $\text{Cl}_P^d(\mathcal{T}_0)$.*

Proof. It suffices to form \mathcal{T}_0 by infinitely many theories of structures \mathcal{A}_i , $i \in I$, with infinitely many copies of models $\mathcal{M} \models T$ forming E_i -classes for equivalence relations E_i , where E_j is either equality or complete for $j \neq i$. Considering disjoint unary predicates P_i for \mathcal{A}_i we get the nonprincipal 1-type $p_{\infty}(x)$ isolated by the set $\{\neg P_i(x) \mid i \in I\}$ which can be realized by the set M with the structure \mathcal{M} witnessing that T belongs to the restriction of $\text{Cl}_P^d(\mathcal{T}_0)$ removing new relations E_i . \square

Remark 3.4. We have a similar effect removing all relations E_j in the structures \mathcal{A}_i and obtaining isomorphic structures \mathcal{A}'_i : by compactness the P -combination of structures \mathcal{A}'_i (where disjoint \mathcal{A}'_i form unary predicates P_i) has the theory with a model, whose p_{∞} -restriction forms a structure isomorphic to \mathcal{M} . In this case we have $\text{Cl}_P^{d,r}(\{\text{Th}(\mathcal{A}'_i)\})$.

Remark 3.5. As in the proof of Proposition 3.1 theories in \mathcal{T}_0 can be chosen consistent modulo cardinalities of their models we can add that $e\text{-Sp}(T) = 1$ for the E -combination T of the theories in \mathcal{T}_0 .

As the same time $e\text{-Sp}(T')$ is infinite for the P -combination T' of \mathcal{A}_i in the proof of Proposition 3.3, since $p_{\infty}(x)$ has infinitely many possibilities for finite cardinalities of sets of realizations for $p_{\infty}(x)$.

4. e -SPECTRA FOR FINITELY CATEGORICAL AND ω -CATEGORICAL THEORIES

We refine the notions of e -spectra $e\text{-Sp}(\mathcal{A}_E)$ and $e\text{-Sp}(T)$ for the theories $T = \text{Th}(\mathcal{A}_E)$ restricting the class of possible theories to a given class \mathcal{T} in the following way.

For a structure \mathcal{A}_E the number of *new* structures with respect to the structures \mathcal{A}_i , i. e., of the structures \mathcal{B} with $\text{Th}(\mathcal{B}) \in \mathcal{T}$, which are pairwise elementary non-equivalent and elementary non-equivalent to the structures \mathcal{A}_i , is called the (e, \mathcal{T}) -*spectrum* of \mathcal{A}_E and denoted by $(e, \mathcal{T})\text{-Sp}(\mathcal{A}_E)$. The value $\sup\{(e, \mathcal{T})\text{-Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the (e, \mathcal{T}) -*spectrum* of the theory $\text{Th}(\mathcal{A}_E)$ and denoted by $(e, \mathcal{T})\text{-Sp}(\text{Th}(\mathcal{A}_E))$.

The following properties are obvious.

1. (Monotony) If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then $(e, \mathcal{T}_1)\text{-Sp}(\text{Th}(\mathcal{A}_E)) \leq (e, \mathcal{T}_2)\text{-Sp}(\text{Th}(\mathcal{A}_E))$ for any structure \mathcal{A}_E .

2. (Additivity) If the class $\overline{\mathcal{T}}$ of all complete elementary theories of relational languages is the disjoint union of subclasses $\overline{\mathcal{T}}_1$ and $\overline{\mathcal{T}}_2$ then for any theory $T = \text{Th}(\mathcal{A}_E)$,

$$e\text{-Sp}(T) = (e, \overline{\mathcal{T}}_1)\text{-Sp}(T) + (e, \overline{\mathcal{T}}_2)\text{-Sp}(T).$$

We divide a class \mathcal{T} of theories into two disjoint subclasses \mathcal{T}^{fin} and \mathcal{T}^{inf} having finite and infinite non-empty language relations, respectively. More precisely, for functions $f: \omega \rightarrow \lambda_f$, where λ_f are cardinalities, we divide \mathcal{T} into subclasses \mathcal{T}^f of theories T such that T has $f(n)$ n -ary predicate symbols for each $n \in \omega$.

For the function f we denote by $\text{Supp}(f)$ its *support*, i.e., the set $\{n \in \omega \mid f(n) > 0\}$.

Clearly, the language of a theory $T \in \mathcal{T}^f$ is finite if and only if $\rho_f \subset \omega$ and $\text{Supp}(f)$ is finite.

Illustrating (e, \mathcal{T}) -spectra for the class \mathcal{T} of all cubic theories and taking the class $\mathcal{T}_0^{\text{fin}} \subset \mathcal{T}$ of all theories of finite cubes we note that for an E -combination T of theories T_i in $\mathcal{T}_0^{\text{fin}}$, $(e, \mathcal{T})\text{-Sp}(T)$ is positive if and only if there are infinitely many T_i . In such a case, $(e, \mathcal{T})\text{-Sp}(T) = 1$ and new theory, which does not belong to $\mathcal{T}_0^{\text{fin}}$, is the theory of ω -cube.

The class $\overline{\mathcal{T}}_{\text{fin}}$ is represented as disjoint union of subclasses $\overline{\mathcal{T}}_{\text{fin},n}$ of theories having n -element models, $n \in \omega \setminus \{0\}$. For $N \in \omega$, the class $\bigcup_{n \leq N} \overline{\mathcal{T}}_{\text{fin},n}$ is denoted by $\overline{\mathcal{T}}_{\text{fin}, \leq N}$.

Proposition 4.1. *For any $\mathcal{T} \subset \overline{\mathcal{T}}$, $\text{Cl}_E(\mathcal{T}) \setminus \overline{\mathcal{T}}_{\text{fin}} \neq \emptyset$ if and only if for any natural N , $\mathcal{T} \not\subset \overline{\mathcal{T}}_{\text{fin}, \leq N}$.*

Proof. If \mathcal{T} contains a theory with infinite models, the assertion is obvious. If $\mathcal{T} \subset \overline{\mathcal{T}}_{\text{fin}}$, then we apply Compactness and Proposition 1.1. \square

The following obvious proposition is also based on Proposition 1.1.

Proposition 4.2. *If $\mathcal{T} \subset \overline{\mathcal{T}}_{\text{fin},n}$ (respectively $\mathcal{T} \subset \overline{\mathcal{T}}_{\text{fin}, \leq N}$) then $\text{Cl}_E(\mathcal{T}) \subset \overline{\mathcal{T}}_{\text{fin},n}$ ($\text{Cl}_E(\mathcal{T}) \subset \overline{\mathcal{T}}_{\text{fin}, \leq N}$). For any theory $T = \text{Th}(\mathcal{A}_E)$, where all E -classes have theories in \mathcal{T} , $e\text{-Sp}(T) = (e, \overline{\mathcal{T}}_{\text{fin},n})\text{-Sp}(\text{Th}(\mathcal{A}_E))$ ($e\text{-Sp}(T) = (e, \overline{\mathcal{T}}_{\text{fin}, \leq N})\text{-Sp}(\text{Th}(\mathcal{A}_E))$). If, additionally, \mathcal{T} is the set of theories in a finite language then \mathcal{T} is finite (and so E -closed). In particular, for any theory $T = \text{Th}(\mathcal{A}_E)$ in a finite language, where all E -classes have theories in \mathcal{T} , $e\text{-Sp}(T) = 0$.*

Remark 4.3. In fact, the conclusions of Proposition 4.2 follow implying the following fact. If all theories in \mathcal{T} contain a formula φ then all theories in $\text{Cl}_E(\mathcal{T})$ contain φ . For (1) we take a formula φ “saying” that models have exactly n elements, and for

(2) — a formula φ “saying” that models have at most N elements. If the language is finite there are only finitely many possibilities for isomorphism types on n -element sets and these possibilities are formula-definable.

Similarly Proposition 4.2 we have

Proposition 4.4. *If $\mathcal{T} \cap \overline{\mathcal{T}}_{\text{fin}} = \emptyset$ then $\text{Cl}_E(\mathcal{T}) \cap \overline{\mathcal{T}}_{\text{fin}} = \emptyset$.*

Definition [16]. A theory T in a predicate language Σ is called *language uniform*, or a *LU-theory* if for each arity n any substitution on the set of non-empty n -ary predicates (corresponding to the symbols in Σ) preserves T . The LU-theory T is called *IILU-theory* if it has non-empty predicates and as soon as there is a non-empty n -ary predicate then there are infinitely many non-empty n -ary predicates and there are infinitely many empty n -ary predicates.

Since for any finite cardinality n there are IILU-theories with n -element models, repeating the proof of [16, Proposition 12] and [16, Proposition 13] we get

Proposition 4.5. (1) *For any $n \in \omega \setminus \{0\}$ and $\mu \leq \omega$ there is an E -combination $T = \text{Th}(\mathcal{A}_E)$ of IILU-theories $T_i \in \overline{\mathcal{T}}_{\text{fin},n}$ in a language Σ of the cardinality ω such that T has an e -least model and $e\text{-Sp}(T) = \mu$.*

(2) *For any uncountable cardinality λ there is an E -combination $T = \text{Th}(\mathcal{A}_E)$ of IILU-theories $T_i \in \overline{\mathcal{T}}_{\text{fin},n}$ in a language Σ of the cardinality λ such that T has an e -least model and $e\text{-Sp}(T) = \lambda$.*

Proposition 4.6. *For any $n \in \omega \setminus \{0\}$ and infinite cardinality λ there is an E -combination $T = \text{Th}(\mathcal{A}_E)$ of IILU-theories $T_i \in \overline{\mathcal{T}}_{\text{fin},n}$ in a language Σ of cardinality λ such that T does not have e -least models and $e\text{-Sp}(T) \geq \max\{2^\omega, \lambda\}$.*

Proposition 4.7. *For any $n \in \omega \setminus \{0\}$ and infinite cardinality λ there is an E -combination $T = \text{Th}(\mathcal{A}_E)$ of LU-theories $T_i \in \overline{\mathcal{T}}_{\text{fin},n}$ in a language Σ of cardinality λ such that T does not have e -least models and $e\text{-Sp}(T) = 2^\lambda$.*

Proof. Let Σ be a language consisting, for some natural m , of m -ary predicate symbols R_i , $i < \lambda$. For any $\Sigma' \subseteq \Sigma$ we take a structure $\mathcal{A}_{\Sigma'}$ of the cardinality n such that $R_i = (A_{\Sigma'})^m$ for $R_i \in \Sigma'$, and $R_i = \emptyset$ for $R_i \in \Sigma \setminus \Sigma'$. Clearly, each structure $\mathcal{A}_{\Sigma'}$ has a LU-theory and $\mathcal{A}_{\Sigma'} \not\cong \mathcal{A}_{\Sigma''}$ for $\Sigma' \neq \Sigma''$. For the E -combination \mathcal{A}_E of the structures $\mathcal{A}_{\Sigma'}$ we obtain the theory $T = \text{Th}(\mathcal{A}_E)$ having a model of the cardinality λ . At the same time \mathcal{A}_E has 2^λ distinct theories of the E -classes $\mathcal{A}_{\Sigma'}$. Thus, $e\text{-Sp}(T) = 2^\lambda$. Finally we note that T does not have e -least models by Theorem 1.2 and arguments for [15, Proposition 9]. \square

Remark 4.8. Considering countable LU-theories for the assertions above we can assume that these theories belong to a class \mathcal{T}^f , where $f \in \omega^\omega$ and $\text{Supp}(f)$ is infinite. Note also that Propositions 4.5–4.7 hold replacing the classes $\overline{\mathcal{T}}_{\text{fin},n}$ by $\overline{\mathcal{T}}_{\omega,1}$.

Replacing E -classes by unary predicates P_i (not necessary disjoint) being universes for structures \mathcal{A}_i and restricting models of $\text{Th}(\mathcal{A}_P)$ to the set of realizations of $p_\infty(x)$ we get the (e, \mathcal{T}) -spectrum $(e, \mathcal{T})\text{-Sp}(\text{Th}(\mathcal{A}_P))$, i. e., the number of pairwise elementary non-equivalent restrictions \mathcal{N} of $\mathcal{M} \models \text{Th}(\mathcal{A}_P)$ to $p_\infty(x)$ such that $\text{Th}(\mathcal{N}) \in \mathcal{T}$.

Proposition 4.9. *If the structures \mathcal{A}_i have pairwise disjoint languages with disjoint predicates P_i then for any natural $n \geq 1$, $(e, \overline{\mathcal{T}}_{\text{fin},n})\text{-Sp}(\text{Th}(\mathcal{A}_P)) \leq 1$, and $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\text{fin}})\text{-Sp}(\text{Th}(\mathcal{A}_P)) \leq 1$.*

Proof. Clearly, if the structures \mathcal{A}_i have pairwise disjoint languages with disjoint predicates P_i then structures for $p_\infty(x)$ do not contain realizations of language predicates, i. e., have theories T_λ^0 . Now $(e, \overline{\mathcal{T}}_{\text{fin},n})\text{-Sp}(\text{Th}(\mathcal{A}_P)) \leq 1$ and $(e, \overline{\mathcal{T}}_{\text{fin},n})\text{-Sp}(\text{Th}(\mathcal{A}_P)) = 1$ if and only if there are infinitely many indexes i and $\text{Th}(\mathcal{A}_i) \neq T_n^0$ for any i . Similarly, $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\text{fin}})\text{-Sp}(\text{Th}(\mathcal{A}_P)) \leq 1$ and $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\text{fin}})\text{-Sp}(\text{Th}(\mathcal{A}_P)) = 1$ if and only if there are infinitely many indexes i and $\text{Th}(\mathcal{A}_i) \neq T_\infty^0$ for any i . \square

Clearly, approximating structures without non-trivial predicates and applying the proof of Proposition 4.9 we get a family of P -combinations with $(e, \overline{\mathcal{T}}_{\text{fin},n})\text{-Sp}(\text{Th}(\mathcal{A}_P)) = 1$, for $n \in \omega \setminus \{0\}$, and $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\text{fin}})\text{-Sp}(\text{Th}(\mathcal{A}_P)) = 1$.

Comparing approximations in Section 3 and proofs for [14, Propositions 4.12, 4.13] we get

Proposition 4.10. *For any infinite cardinality λ there is a theory $T = \text{Th}(\mathcal{A}_P)$ being a P -combination of theories in $\overline{\mathcal{T}}_{\text{fin}}$ and of a language Σ such that $|\Sigma| = \lambda$ and $e\text{-Sp}(T) = 2^\lambda$.*

5. ALMOST LANGUAGE UNIFORM THEORIES

Definition. A theory T in a predicate language Σ is called *almost language uniform*, or a *ALU-theory* if for each arity n with n -ary predicates for Σ there is a partition for all n -ary predicates, corresponding to the symbols in Σ , with finitely many classes K such that any substitution preserving these classes preserves T , too. The ALU-theory T is called *IILU-theory* if it has non-empty predicates and as soon as there is a non-empty n -ary predicate in a class K then there are infinitely many non-empty n -ary predicates in K and there are infinitely many empty n -ary predicates.

By the definition any LU-theory is an ALU-theory and any IILU-theory is an IIALU-theory as well.

Since any finite structure can have only finitely many distinct predicates for each arity n we get the following

Proposition 5.1. *Any theory $T \in \overline{\mathcal{T}}_{\text{fin}}$ is an ALU-theory.*

Replacing LU- and IILU- by ALU- and IIALU- and the proofs in Propositions 4.5–4.7 we get analogs for these assertions attracting expansions of arbitrary theories in $\overline{\mathcal{T}}_{\text{fin},n}$. Thus any theory in $\overline{\mathcal{T}}_{\text{fin},n}$ can be used obtaining described e -spectra.

6. FAMILIES OF CARDINALITIES FOR MODELS OF THEORIES IN CLOSURES

Let \mathcal{T} be a nonempty family of theories in $\overline{\mathcal{T}}$. We denote by $c_E(\mathcal{T})$ (respectively, $c_P(\mathcal{T})$, $c_P^d(\mathcal{T})$, $c_P^{d,r}(\mathcal{T})$) the set of finite cardinalities for models of theories in $\text{Cl}_E(\mathcal{T})$ ($\text{Cl}_P(\mathcal{T})$, $\text{Cl}_P^d(\mathcal{T})$, $\text{Cl}_P^{d,r}(\mathcal{T})$) and by $\bar{c}_E(\mathcal{T})$ (respectively, $\bar{c}_P(\mathcal{T})$, $\bar{c}_P^d(\mathcal{T})$, $\bar{c}_P^{d,r}(\mathcal{T})$) the set of finite cardinalities for models of theories in $\text{Cl}_E(\mathcal{T})$ ($\text{Cl}_P(\mathcal{T})$, $\text{Cl}_P^d(\mathcal{T})$, $\text{Cl}_P^{d,r}(\mathcal{T})$) which are not cardinalities for models of theories in \mathcal{T} . Additionally, for $\text{Cl}_P(\mathcal{T})$, $\text{Cl}_P^d(\mathcal{T})$ and $\text{Cl}_P^{d,r}(\mathcal{T})$ we denote by $\hat{c}_P(\mathcal{T})$, $\hat{c}_P^d(\mathcal{T})$, $\hat{c}_P^{d,r}(\mathcal{T})$, respectively, the set of finite cardinalities for models of theories being restrictions for corresponding P -combinations to sets of realizations of types $p_\infty(x)$.

Remark 6.1. Since E -closures preserve finite cardinalities for models of theories in families in \mathcal{T} , i.e., $c_E(\mathcal{T})$ consists of these cardinalities for \mathcal{T} , then $\bar{c}_E(\mathcal{T}) \equiv \emptyset$. Thus we can use the notation $c_E(\mathcal{T})$ for the set of finite cardinalities for models of theories in \mathcal{T} , or, equivalently, for models of theories in $\text{Cl}_E(\mathcal{T})$.

Remark 6.2. If \mathcal{T} is finite, or corresponding $p_\infty(x)$ is consistent and there are no models with finitely many realizations for $p_\infty(x)$, then $c_P(\mathcal{T}) = c_P^d(\mathcal{T}) = c_E(\mathcal{T})$ and $\bar{c}_P(\mathcal{T}) = \bar{c}_P^d(\mathcal{T}) = \hat{c}_P(\mathcal{T}) = \hat{c}_P^d(\mathcal{T}) = \emptyset$.

Examples of families of theories in the empty language Σ_0 witness that the cardinalities for sets of realizations of $p_\infty(x)$ can vary arbitrarily and for finite \mathcal{T} we have $c_P^{d,r}(\mathcal{T}) = \hat{c}_P^{d,r}(\mathcal{T}) = \mathbb{Z}^+$ and $\bar{c}_P^{d,r}(\mathcal{T}) = \mathbb{Z}^+ \setminus c_E(\mathcal{T})$.

Having an infinite family \mathcal{T} in the language Σ_0 , similarly we get $c_P(\mathcal{T}) = c_P^d(\mathcal{T}) = c_P^{d,r}(\mathcal{T}) = \hat{c}_P(\mathcal{T}) = \hat{c}_P^d(\mathcal{T}) = \hat{c}_P^{d,r}(\mathcal{T}) = \mathbb{Z}^+$ and $\bar{c}_P(\mathcal{T}) = \bar{c}_P^d(\mathcal{T}) = \bar{c}_P^{d,r}(\mathcal{T}) = \mathbb{Z}^+ \setminus c_E(\mathcal{T})$. The latter formula shows that $\bar{c}_P(\mathcal{T})$, $\bar{c}_P^d(\mathcal{T})$, and $\bar{c}_P^{d,r}(\mathcal{T})$ can be arbitrary subsets of \mathbb{Z}^+ with infinite complements. Thus we have the following

Proposition 6.3. *For any infinite set $Y \subseteq \mathbb{Z}^+$ there is a family \mathcal{T} such that $\bar{c}_P(\mathcal{T}) = \bar{c}_P^d(\mathcal{T}) = \bar{c}_P^{d,r}(\mathcal{T}) = \mathbb{Z}^+ \setminus Y$.*

Example 6.4. If the language Σ consists of the symbol E_k of the equivalence relation whose each class has $k \in \omega$ elements then $p_\infty(x)$ can form an arbitrary structure with k -element equivalence classes and for a finite family \mathcal{T}_k we have $c_P^{d,r}(\mathcal{T}_k) = \hat{c}_P^{d,r}(\mathcal{T}_k) = k\mathbb{Z}^+$ and $\bar{c}_P^{d,r}(\mathcal{T}_k) = k\mathbb{Z}^+ \setminus c_E(\mathcal{T}_k)$. If the family \mathcal{T}_k is infinite then, similarly, $c_P(\mathcal{T}_k) = c_P^d(\mathcal{T}_k) = c_P^{d,r}(\mathcal{T}_k) = \hat{c}_P(\mathcal{T}_k) = \hat{c}_P^d(\mathcal{T}_k) = \hat{c}_P^{d,r}(\mathcal{T}_k) = k\mathbb{Z}^+$ and $\bar{c}_P(\mathcal{T}_k) = \bar{c}_P^d(\mathcal{T}_k) = \bar{c}_P^{d,r}(\mathcal{T}_k) = k\mathbb{Z}^+ \setminus c_E(\mathcal{T}_k)$.

More generally, collecting the families of theories with distinct E_k , $k \in K$, $K \subset \omega$, we obtain nonempty values for c_P , c_P^d , $c_P^{d,r}$, \hat{c}_P , \hat{c}_P^d , $\hat{c}_P^{d,r}$ as unions $\bigcup_{k \in K} k\mathbb{Z}^+$.

Now we have to show that all possible nonempty values for \hat{c}_P^d and $\hat{c}_P^{d,r}$ are exhausted by the sums $\biguplus_{k \in K} k\mathbb{Z}^+$ (unions with finite sums for numbers in $k\mathbb{Z}^+$) whereas values for \hat{c}_P may differ.

Theorem 6.5. *For any nonempty family \mathcal{T} there is $K \subset \omega$ such that $\hat{c}_P^{d,r}(\mathcal{T}) = \biguplus_{k \in K} k\mathbb{Z}^+$.*

Proof. Recall that for P -combinations with respect to $\text{Cl}_P^{d,r}$ there are no links between disjoint predicates P_i with structures \mathcal{A}_i being models of theories in \mathcal{T} . Therefore if $p_\infty(x)$ can produce finite structures then structures \mathcal{A}_i with 1-types approximating $p_\infty(x)$, define (partial) definable equivalence relations with bounded finite classes $E(a)$ and without definable extensions, for the approximations and for $p_\infty(x)$. So there are no links between the classes $E(a)$ and having k elements in $E(a)$ we produce, by compactness, a series of $1, 2, \dots, n, \dots$ E -classes for $p_\infty(x)$ since $p_\infty(x)$ is not isolated. Thus we get a series $k\mathbb{Z}^+$ for $\hat{c}_P^{d,r}(\mathcal{T})$. Varying finite cardinalities for the classes $E(a)$ we obtain the required formula $\hat{c}_P^{d,r}(\mathcal{T}) = \biguplus_{k \in K} k\mathbb{Z}^+$ for some set $K \subset \omega$ witnessing these cardinalities. If $p_\infty(x)$ can produce finite structures then we set $K \equiv \emptyset$. \square

Theorem 6.6. *For any infinite family \mathcal{T} there is $K \subset \omega$ such that $\hat{c}_P^d(\mathcal{T}) = \biguplus_{k \in K} k\mathbb{Z}^+$.*

Proof repeats the proof of Theorem 6.5 using structures \mathcal{A}_i which pairwise are not elementary equivalent. \square

Remark 6.7. 1. In Theorems 6.5 and 6.6, if we have minimal K with $|K| > 1$ then the type $p_\infty(x)$ is not complete. Indeed, taking, for sets of realizations of $p_\infty(x)$, maximal definable equivalence relations E_1 and E_2 for $k_1 \neq k_2 \in K$ we can not move, by automorphisms, elements of E_1 -classes to elements of E_2 -classes.

2. Clearly, having E_1 -classes and E_2 -classes of same cardinalities with non-isomorphic structures we again can not connect elements of these classes by automorphisms. Thus, $|K| = 1$ is a necessary but not sufficient condition for the completeness of $p_\infty(x)$.

3. The least cardinality $|K|$, with positive $\hat{c}_P^{d,r}$ or \hat{c}_P^d , gives a lower bound for independent equivalence relations with respect to their realizability/omitting for restrictions of models to sets of realizations of $p_\infty(x)$.

Remark 6.8. Finite structures \mathcal{A}_∞ for maximal definable equivalence relations for $p_\infty(x)$ with respect to Cl_P^d and to $\text{Cl}_P^{d,r}$ can be isomorphic if and only if they are represented in some \mathcal{A}_i for $\text{Cl}_P^{d,r}$ and infinitely many times for Cl_P^d , or approximated both for $\text{Cl}_P^{d,r}$ and for Cl_P^d . Hence, for any infinite family \mathcal{T} , $\hat{c}_P^d(\mathcal{T}) = \hat{c}_P^{d,r}(\mathcal{T})$ if and only if each n -element class for maximal definable equivalence relations for $p_\infty(x)$ with respect to $\text{Cl}_P^{d,r}$ has n -element classes for correspondent definable equivalence relations in infinitely many pairwise elementary non-equivalent structures \mathcal{A}_i , with respect to Cl_P^d .

Definition [2]. Let \mathcal{M} be a model of a theory T , \bar{a} and \bar{b} tuples in \mathcal{M} , A a subset of M . The tuple \bar{a} *semi-isolates* the tuple \bar{b} over the set A if there exists a formula $\varphi(\bar{a}, \bar{y}) \in \text{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a}, \bar{y}) \vdash \text{tp}(\bar{b}/A)$ holds. In this case we say that the formula $\varphi(\bar{x}, \bar{y})$ (with parameters in A) *witnesses that \bar{b} is semi-isolated over \bar{a} with respect to A* .

If $p \in S(T)$ and $\mathcal{M} \models T$ then $\text{SI}_p^{\mathcal{M}}$ denotes the relation of semi-isolation (over \emptyset) on the set of all realizations of p :

$$\text{SI}_p^{\mathcal{M}} = \{(\bar{a}, \bar{b}) \mid \mathcal{M} \models p(\bar{a}) \wedge p(\bar{b}) \text{ and } \bar{a} \text{ semi-isolates } \bar{b}\}.$$

The following definition generalizes the previous one for a family of 1-types, in particular, for incomplete $p_\infty(x)$.

Definition [8]. Let T be a complete theory, $\mathcal{M} \models T$. We consider *closed* nonempty sets (under the natural topology) sets $\mathbf{p}(x) \subseteq S^1(\emptyset)$, i. e., sets $\mathbf{p}(x)$ such that $\mathbf{p}(x) = \bigcap_{i \in I} [\varphi_{\mathbf{p},i}(x)]$, where $[\varphi_{\mathbf{p},i}(x)] = \{p(x) \in S^1(\emptyset) \mid \varphi_{\mathbf{p},i}(x) \in p(x)\}$ for some formulas $\varphi_{\mathbf{p},i}(x)$ of T .

For closed sets $\mathbf{p}(x), \mathbf{q}(y) \subseteq S(\emptyset)$ of types, realized in \mathcal{M} , we consider (\mathbf{p}, \mathbf{q}) -*preserving* (\mathbf{p}, \mathbf{q})-*semi-isolating*, $(\mathbf{p} \rightarrow \mathbf{q})$ -, or $(\mathbf{q} \leftarrow \mathbf{p})$ -*formulas* $\varphi(x, y)$ of T , i. e., formulas for which if $a \in M$ realizes a type in $\mathbf{p}(x)$ then every solution of $\varphi(a, y)$ realizes a type in $\mathbf{q}(y)$.

If $\mathbf{p}(x) = \mathbf{q}(y)$ then (\mathbf{p}, \mathbf{q}) -preserving formulas are called \mathbf{p} -*preserving* or \mathbf{p} -*semi-isolating* and we define, similarly to $\text{SI}_p^{\mathcal{M}}$, the *generalized* relation $\text{SI}_{\mathbf{p}}^{\mathcal{M}}$ of semi-isolation for the set of realizations of types in $\mathbf{p}(x)$:

$$\text{SI}_{\mathbf{p}}^{\mathcal{M}} \equiv \{(a, b) \mid \mathcal{M} \models p(a) \wedge p'(b) \wedge \varphi(a, b)\}$$

for $p, p' \in \mathbf{p}$ and a \mathbf{p} -preserving formula $\varphi(x, y)$.

If $(a, b) \in \text{SI}_{\mathbf{p}}^{\mathcal{M}}$ we say that a semi-isolates b with respect to \mathbf{p} .

Thus, a semi-isolates b (in sense of [2]) if and only if a semi-isolates b with respect to $\{\text{tp}(a), \text{tp}(b)\}$.

Remark 6.9. Since there are no links between structures \mathcal{A}_i with respect to Cl_P^d and $\text{Cl}_P^{d,r}$, the set \mathbf{p}_{∞} of all completions $q(x)$ of $p_{\infty}(x)$ has symmetric $\text{SI}_{\mathbf{p}_{\infty}}^{\mathcal{M}}$. Thus, the relations $\text{SI}_{\mathbf{p}_{\infty}}^{\mathcal{M}}$ form equivalence relations. Positive values for \hat{c}_P^d and $\hat{c}_P^{d,r}$ imply that these equivalence relations have finite classes. Cardinalities of these classes define formulas in Theorems 6.5 and 6.6.

Now we consider the general case, with the operator Cl_P . One can hardly expect productive descriptions considering arbitrary links of structures with respect to arbitrary links of predicates P_i , in contrast to the disjoint predicates when, obviously, there are no links between the structures. So we will fix a P -combination \mathcal{A}_P (and its theory $T = \text{Th}(\mathcal{A}_P)$) and consider the set $\hat{c}_P(T)$ of values of finite cardinalities for $p_{\infty}(x)$ with respect to given P -combination T , instead of the set $\hat{c}_P(T)$ of values for all finite values for all possible P -combinations. In other words we argue to describe sets of finite cardinalities for sets of realizations of a nonprincipal, not necessary complete, 1-type $p_{\infty}(x)$.

We note the following obvious observations.

Remark 6.10. 1. If any $n \in \omega$ realizations of a type $p_{\infty}(x)$ force infinitely many realizations of $p_{\infty}(x)$ then it is true for any $m > n$.

2. If a and b are realizations of a type $p_{\infty}(x)$ and a does not semi-isolate b with respect to \mathbf{p}_{∞} then there are no formulas $\varphi(x, y)$ with $\models \varphi(a, b)$ and forcing finitely or infinitely many realizations for the type $q = \text{tp}(b/a)$, i. e., the set of realizations of q can be empty and infinite, depending on a model.

The first observation shows that having n which forces infinity, we get $\hat{c}_P(T) \subset n$. The second one implies that realizations of \mathbf{p}_{∞} , which are not connected by the relation of semi-isolation, contribute to $\hat{c}_P(T)$ independently on the binary level. Moreover, these contributions by realizations a and b can generate distinct series, as in Theorems 6.5 and 6.5, only if $\text{tp}(a) \neq \text{tp}(b)$.

The following example shows that there is a theory T with $\hat{c}_P(T) = \{1\}$ clarifying that contributions above on the binary level deny by the ternary level.

Example 6.11. Consider a coloring $\text{Col}: M \rightarrow \omega \cup \{\infty\}$ of an infinite set M such that each color $\lambda \in \omega \cup \{\infty\}$ has infinitely many elements in M , i. e., each $\text{Col}_n = \{a \in M \mid \text{Col}(a) = n\}$ is infinite as well as there are infinitely many elements of the infinite color. We put $P_i = M \setminus \bigcup_{j < i} \text{Col}_j$ and $p_{\infty}(x) = \{\neg P_n(x) \mid n \in \omega\}$.

Now we define, using a generic construction with free amalgams [12, 7], a ternary relation R such that for the definable relation $Q(x, y) \equiv \exists z R(x, y, z)$ we have the following properties:

- 1) the Q -structure has unique 1-type and, moreover, its automorphism group is transitive;
- 2) $R(x, y, z) \equiv Q(x, z) \wedge Q(y, z)$;

3) Col is an inessential coloring which is not neither Q -ordered nor Q^{-1} -ordered [12, 6], moreover, for any element $a \in M$ the sets of solutions for $Q(a, y)$ and $Q(x, a)$ have infinitely many elements of each color;

4) for any $a \neq b \in M$ the set of solutions for $R(a, b, z)$ is infinite for each color $n \geq \min\{\text{Col}(a), \text{Col}(b)\}$ and does not have elements of colors $< \min\{\text{Col}(a), \text{Col}(b)\}$, hence, $R(a, b, z) \vdash p_\infty(z)$ if $\models p_\infty(a)$ and $\models p_\infty(b)$.

Taking the generic structure \mathcal{M} in the language $\langle P_n^{(1)}, Q^{(2)}, R^{(3)} \rangle_{n \in \omega}$ and its theory $T = \text{Th}(\mathcal{M})$, being a P -combination, we have $\hat{c}_P(T) = \{1\}$ since the nonisolated type $p_\infty(x)$ can have, in a model of T , 0, 1, or infinitely many realizations: one realization of $p_\infty(x)$ does not force new ones and two distinct realizations a and b of $p_\infty(x)$ force infinitely many ones by the formula $R(a, b, z)$.

Example 6.12. We modify Example 6.11 replacing elements by E_k -classes, where each class contains k elements, and repeat the generic construction satisfying the following conditions:

- 2) if $aE_k a'$ then $\text{Col}(a) = \text{Col}(a')$;
- 2) if $(a, b) \in Q$, $aE_k a'$, $bE_k b'$, then $(a', b') \in Q$.

The theory T_k of resulting generic structure \mathcal{M}_k satisfies $\hat{c}_P(T_k) = \{k\}$ since each realization a of $p_\infty(x)$ forces k realizations of $p_\infty(x)$ consisting of $E_k(a)$ and any two realizations of $p_\infty(x)$ belonging to distinct E_k -classes forces infinitely many E_k -classes with elements satisfying $p_\infty(x)$.

Combining structures \mathcal{M}_k with distinct k we obtain a generic structure whose theory T satisfies $\hat{c}_P(T) = K$ for a given set $K \subseteq \mathbb{Z}^+$. Here sets of realizations of $p_\infty(x)$ are divided into E_k -classes for $k \in K$.

Thus we have the following theorem asserting that values $\hat{c}_P(T)$ can be arbitrary.

Theorem 6.13. *For any set $K \subseteq \mathbb{Z}^+$ there is a P -combination T such that $\hat{c}_P(T) = K$.*

Now we argue to modify the generic construction and Theorem 6.13 using transitive arrangements of algebraic systems similar to [9, 5], and obtaining a similar result with complete $p_\infty(x)$ describing possibilities for $\hat{c}_P(T)$.

For this aim we fix a nonempty set $K \subseteq \mathbb{Z}^+$ claiming for $\hat{c}_P(T) = K$ with some P -combination T . Note that if $1 \notin K$ then either any realization a of $p_\infty(x)$ forces infinitely many realizations or a belongs to the maximal finite definable E -class with some $k_0 > 1$ elements. At first case, by completeness of $p_\infty(x)$, any finite set of realizations of $p_\infty(x)$ forces that infinity and therefore $K = \emptyset$ contradicting the condition $K \neq \emptyset$. At second case, again by completeness of $p_\infty(x)$, we have $K \subseteq k_0 \mathbb{Z}^+$. Replacing elements by their E -classes we reduce the problem of construction of T with $\hat{c}_P(T) = K$ to the case $1 \in K$.

Example 6.11 witnesses the possibility for $\hat{c}_P(T) = \{1\}$. So below we assume that $1 \in K$ and $|K| \geq 2$.

Now for each $k \in K \setminus \{1\}$ we introduce a ternary relation R_k defining a free (acyclic) precise pseudoplane \mathcal{P}_k [9] with infinitely many lines containing any fixed point and exactly k points belonging to any fixed line such that \mathcal{P}_k has infinitely many connected components. Then we combine these free pseudoplanes \mathcal{P}_k allowing that each point belongs to each pseudoplane \mathcal{P}_k and the union of sets of lines does not form cycles. We embed copies of that combination \mathcal{P} of the pseudoplanes into unary predicates Col_n as well as to the structure of $p_\infty(x)$.

Modifying Example 6.11 we introduce a binary predicate Q such that;

1) if $(a, b) \in Q$ and $(a, c) \in Q$ then a, b, c belong to pairwise distinct connected components of \mathcal{P} , the same is satisfied for Q^{-1} (as in Example described in [12, Section 1.3] and in [4]);

2) elements a_1, \dots, a_m , $m > 1$, realizing $p_\infty(x)$ and belonging to a common line l force all elements of l and do not force elements outside l ;

3) if a and b are realizations of $p_\infty(x)$ which do not have a common line then a and b force infinitely many realizations of $p_\infty(x)$ by the formula $Q(a, y) \wedge Q(b, y)$.

The resulted generic structure \mathcal{M} of the language $\langle \text{Col}_n, Q, R_k \rangle_{n \in \omega, k \in K \setminus \{1\}}$ and its theory T satisfy the following properties:

i) any realization of $p_\infty(x)$ does not force new realizations of $p_\infty(x)$ witnessing $1 \in \hat{c}_P(T)$;

ii) any at least two distinct realizations of $p_\infty(x)$ in a line l belonging to \mathcal{P}_k force exactly the set l witnessing $k \in \hat{c}_P(T)$ for $k \in K$;

iii) any two distinct realizations of $p_\infty(x)$ which do not have common lines force infinitely many realizations of $p_\infty(x)$ witnessing $k' \notin \hat{c}_P(T)$ for $k' \notin K$.

Thus we get $\hat{c}_P(T) = K$.

Collecting the arguments above we have the following

Theorem 6.14. (1) *If T is a P -combination with a type $p_\infty(x)$ isolating a complete 1-type then $\hat{c}_P(T)$ is either empty or contains k_0 such that $\hat{c}_P(T) \subseteq k_0\mathbb{Z}^+$.*

(2) *For any set $K \subseteq k_0\mathbb{Z}^+$, being empty or containing k_0 , there is a P -combination T with a type $p_\infty(x)$ isolating a complete 1-type such that $\hat{c}_P(T) = K$.*

REFERENCES

- [1] B. S. Baizhanov, S. V. Sudoplatov, V. V. Verbovskiy, *Conditions for non-symmetric relations of semi-isolation*, Siberian Electronic Mathematical Reports, **9** (2012), 161–184. MR2954703
- [2] A. Pillay, *Countable models of stable theories*, Proc. Amer. Math. Soc., **89** (1983), 666–672. MR0718994
- [3] E. Rosen, *Some Aspects of Model Theory and Finite Structures*, The Bulletin of Symbolic Logic, **8** (2002), 380–403. MR1931349
- [4] S. V. Sudoplatov, *Type reduction and powerful types*, Siberian Math. J., **33** 1992, 125–133. MR1165687
- [5] S. V. Sudoplatov, *Transitive arrangements of algebraic systems*, Siberian Math. J., **40** (1999), 1347–1351. MR1741088
- [6] S. V. Sudoplatov, *Inessential combinations and colorings of models*, Siberian Math. J., **44** (2003), 883–890. MR2019567
- [7] S. V. Sudoplatov, *Syntactic approach to constructions of generic models*, Algebra and Logic, **46** (2007), 134–146. MR2356525
- [8] S. V. Sudoplatov, *Algebras of distributions of formulas with respect to generalized semi-isolation*, Algebra and Model Theory 9. Collection of papers, NSTU, Novosibirsk, 2013, 67–100. Zbl 1299.03033
- [9] S. V. Sudoplatov, *Group polygonometries*, NSTU, Novosibirsk, 2013.
- [10] S. V. Sudoplatov, *Models of cubic theories*, Bulletin of the Section of Logic, **43** (2014), 19–34. MR3240014
- [11] S. V. Sudoplatov, *Forcing of infinity and algebras of distributions of binary semi-isolating formulas for strongly minimal theories*, Mathematics and Statistics, **2** (2014), 183–187.
- [12] S. V. Sudoplatov, *Classification of countable models of complete theories*, NSTU, Novosibirsk, 2014.
- [13] S. V. Sudoplatov, P. Tanović, *Semi-isolation and the strict order property*, Notre Dame J. of Formal Logic, **56** (2015), 555–572. MR3403091
- [14] S. V. Sudoplatov, *Combinations of structures*, arXiv:1601.00041v1 [math.LO], (2016), 19 pp.

- [15] S. V. Sudoplatov, *Closures and generating sets related to combinations of structures*, Reports of Irkutsk State University. Series “Mathematics”, **16** (2016), 131–144. Zbl 06648831
- [16] S. V. Sudoplatov, *Families of language uniform theories and their generating sets*, Reports of Irkutsk State University. Series “Mathematics”, **16** (2016), 62–76.
- [17] S. V. Sudoplatov, *Relative e -spectra and relative closures for families of theories*, arXiv:1701.00206v1 [math.LO], (2017), 15 pp.
- [18] S. V. Sudoplatov, *On semilattices and lattices for families of theories*, arXiv:1701.00208v1 [math.LO], (2017), 8 pp.
- [19] J. Väänänen, *Pseudo-finite model theory*, *Matematica Contemporanea*, **24** (2003), 169–183. MR2007169
- [20] B. Zil’ber, *Structural approximation*, Oxford University, Oxford, 2010, 27 pp. (Preprint)

SERGEY VLADIMIROVICH SUDOPLATOV
SOBOLEV INSTITUTE OF MATHEMATICS,
ACADEMICIAN KOPTYUG AVENUE, 4
630090, NOVOSIBIRSK, RUSSIA.
NOVOSIBIRSK STATE TECHNICAL UNIVERSITY,
K. MARX AVENUE, 20
630073, NOVOSIBIRSK, RUSSIA.
NOVOSIBIRSK STATE UNIVERSITY,
PIROGOVA STREET, 1
630090, NOVOSIBIRSK, RUSSIA.
INSTITUTE OF MATHEMATICS AND MATHEMATICAL MODELING,
PUSHKINA STREET, 125
050010, ALMATY, KAZAKHSTAN.
E-mail address: sudoplat@math.nsc.ru