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COMBINATIONS RELATED TO CLASSES OF FINITE AND COUNTABLY CATEGORICAL STRUCTURES AND THEIR THEORIES

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Abstract. We consider and characterize classes of finite and countably categorical structures and their theories preserved under E-operators and P-operators. We describe e-spectra and families of finite cardinalities for structures belonging to closures with respect to E-operators and P-operators.

Keywords: finite structure, countably categorical structure, elementary theory, E-operator, P-operator, e-spectrum.

We continue to study structural properties of E-combinations and P-combinations of structures and their theories [14, 15, 16, 17, 18] applying the general context to the classes of finitely categorical and ω-categorical theories.

Approximations of structures by finite ones as well as correspondent approximations of theories were studied in a series of papers, e.g. [3, 19, 20]. We consider these approximations in the context of structural combinations.

We consider and describe e-spectra and families of finite cardinalities for structures belonging to closures with respect to E-operators and P-operators.

The paper is organized as follows. In Section 1 we recall necessary preliminary notions and results. In Section 2 we study closed classes of theories with finite models and of ω-categorical theories. Properties related to approximations of theories with finite and infinite models are described in Section 3. In Section 4 we describe e-spectra for theories with finite models and of ω-categorical theories. Relations

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of (almost) language uniform theories and theories with finite models studies in Section 5. In Section 6 we describe families of finite cardinalities for models of theories in E-closures and P-closures.

1. Preliminaries

Throughout the paper we use the following terminology in [14, 15] as well as in [9, 10].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the P-*union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the P-*operator*. The structure $\mathcal{A}_P$ is called the P-*combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write

$$\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$$

for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being P-*combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$. In general, for non-disjoint case, the theory T_P will be also called a P-*combination* of the theories T_i, but in such a case we will keep in mind that this P-combination is constructed with respect (and depending) to the structure $\mathcal{A}_P$, or, equivalently, with respect to any/some $\mathcal{A}' \equiv \mathcal{A}_P$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the E-*union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the E-*operator*. The structure $\mathcal{A}_E$ is also called the E-*combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being E-*combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the E-*representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}')) \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [15]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 1.1 [15]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Theorem 1.2 [15]. *If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

Definition [15]. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_P(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$ such that $\mathrm{Th}(\mathcal{A}) \in \mathcal{T}$ or $\mathcal{A}$ is a structure of type $p_\infty(x)$ in $\mathcal{A}' \equiv \mathcal{A}_P$, where $\mathcal{A}_P = \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ and $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$ are pairwise distinct. As above, if $\mathcal{T} = \mathrm{Cl}_P(\mathcal{T})$ then $\mathcal{T}$ is said to be *P-closed*.

Using above only disjoint P-combinations $\mathcal{A}_P$ we get the closure $\mathrm{Cl}^d_P(\mathcal{T})$ being a subset of $\mathrm{Cl}_P(\mathcal{T})$.

The closure operator $\mathrm{Cl}^{d,r}_P$ is obtained from Cl^d_P permitting repetitions of structures for predicates P_i.

Replacing E-classes by unary predicates P_i (not necessary disjoint) being universes for structures $\mathcal{A}_i$ and restricting models of $\mathrm{Th}(\mathcal{A}_P)$ to the set of realizations of $p_\infty(x)$ we get the *e-spectrum* e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))$, i. e., the number of pairwise elementary non-equivalent restrictions of $\mathcal{M} \models \mathrm{Th}(\mathcal{A}_P)$ to $p_\infty(x)$ such that these restrictions are not elementary equivalent to the structures $\mathcal{A}_i$.

Definition [9, 10]. A *n-dimensional cube*, or a *n-cube* (where $n \in \omega$) is a graph isomorphic to the graph $\mathcal{Q}_n$ with the universe $\{0,1\}^n$ and such that any two vertices $(\delta_1,\ldots,\delta_n)$ and $(\delta'_1,\ldots,\delta'_n)$ are adjacent if and only if these vertices differ exactly in one coordinate. The described graph $\mathcal{Q}_n$ is called the *canonical representative* for the class of n-cubes.

Let λ be an infinite cardinal. A λ-*dimensional cube*, or a λ-*cube*, is a graph isomorphic to a graph $\Gamma = \langle X; R\rangle$ satisfying the following conditions:

(1) the universe $X \subseteq \{0,1\}^\lambda$ is generated from an arbitrary function $f \in X$ by the operator $\langle f\rangle$ attaching, to the set $\{f\}$, all results of substitutions for any finite tuples $(f(i_1),\ldots,f(i_m))$ by tuples $(1-f(i_1),\ldots,1-f(i_m))$;

(2) the relation R consists of edges connecting functions differing exactly in one coordinate (the *(i-th) coordinate* of function $g \in \{0,1\}^\lambda$ is the value $g(i)$ correspondent to the argument $i < \lambda$).

The described graph $\mathcal{Q} \rightleftharpoons \mathcal{Q}_f$ with the universe $\langle f\rangle$ is a *canonical representative* for the class of λ-cubes.

Note that the canonical representative of the class of n-cubes (as well as the canonical representatives of the class of λ-cubes) are generated by any its function: $\{0,1\}^n = \langle f\rangle$, where $f \in \{0,1\}^n$. Therefore the universes of canonical representatives $\mathcal{Q}_f$ of n-cubes like λ-cubes, is denoted by $\langle f\rangle$.

2. Closed classes of finitely categorical and ω-categorical theories

Remind that a countable complete theory T is *ω-categorical* if T has exactly one countable model up to isomorphisms, i.e. $I(T,\omega) = 1$. A countable theory T is *n-categorical*, for natural $n \geq 1$, if T has exactly one n-element model up to isomorphisms, i.e. $I(T,n) = 1$. A countable theory T is *finitely categorical* if T is n-categorical for some $n \in \omega \setminus \{0\}$.

The classes of all finitely and ω-categorical theories will be denoted by $\overline{\mathcal{T}}_{\mathrm{fin}}$ and $\overline{\mathcal{T}}_{\omega,1}$, respectively.

Let $\mathcal{T}$ be a set (class) of theories in $\overline{\mathcal{T}}_{\mathrm{fin}} \cup \overline{\mathcal{T}}_{\omega,1}$, T be a theory in $\mathcal{T}$. By Ryll-Nardzewski theorem, $S^n(T)$ is finite for any n. Then, for any n, classes $\lceil\varphi(\bar{x})\rceil = \{\varphi'(\bar{x}) \mid \varphi'(\bar{x}) \equiv \varphi(\bar{x})\}$ of T-formulas with n free variables and $\lceil\varphi(\bar{x})\rceil \leq \lceil\psi(\bar{x})\rceil \Leftrightarrow \varphi(\bar{x}) \vdash \psi(\bar{x})$ form a finite Boolean algebra $\mathcal{B}_n(T)$ with 2^{m_n} elements, where m_n is the number of n-types of T.

The algebra $\mathcal{B}_n(T)$ can be interpreted as a m_n-cube $\mathcal{C}_{m_n}(T)$, whose vertices form the universe $B_n(T)$ of $\mathcal{B}_n(T)$, edges $[a,b]$ link vertices a and b such that a $\leqslant$-covers b or b $\leqslant$-covers a, and each vertex a is marked by some $u_a \rightleftharpoons \lceil\varphi(\bar{x})\rceil$, where $a \leqslant b \Leftrightarrow u_a \leq u_b$. The label 0 is used for the vertex corresponding to $\lceil\neg\bar{x} \approx \bar{x}\rceil$ and 1 — for the vertex corresponding to $\lceil\bar{x} \approx \bar{x}\rceil$.

Obviously, the sets $\lceil\varphi(\bar{x})\rceil$ and the relation $\leq$ depend on the theory T but we omit T if the theory is fixed or it is clear by the context.

Clearly, algebras $\mathcal{B}_n(T_1)$ and $\mathcal{B}_n(T_2)$, for $T_1, T_2 \in \mathcal{T}$, may be not coordinated: it is possible $\lceil\varphi(\bar{x})\rceil < \lceil\psi(\bar{x})\rceil$ for T_1 whereas $\lceil\psi(\bar{x})\rceil < \lceil\varphi(\bar{x})\rceil$ for T_2. If $\lceil\varphi(\bar{x})\rceil < \lceil\psi(\bar{x})\rceil$ for T_1 and $\lceil\varphi(\bar{x})\rceil < \lceil\psi(\bar{x})\rceil$ for T_2, we say that T_2 *witnesses* that $\lceil\varphi(\bar{x})\rceil < \lceil\psi(\bar{x})\rceil$ for T_1 (and vice versa).

At the same time, if a countable theory T_0 does not belong to $\overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$ then for some $n \geq 1$, $\mathcal{B}_n(T_0)$ is infinite and therefore there is a formula $\varphi(\bar{x})$, for instance $(\bar{x} \approx \bar{x})$, such that for the label $u = \lceil\varphi(\bar{x})\rceil$ there is an infinite decreasing chain $(u_k)_{k\in\omega}$ of labels: $u_{k+1} < u_k < u$, witnessed by some formulas $\varphi_k(\bar{x})$. In such a case, if $T_0 \in \mathrm{Cl}_E(\mathcal{T})$, then by Proposition 1.1 for any finite sequence $(u_l, \ldots, u_0, u)$ there are infinitely many theories in $\mathcal{T}$ witnessing that $u_l < \ldots < u_0 < u$. In particular, cardinalities m_n for Boolean algebras $\mathcal{B}_n(T)$ and for cubes $\mathcal{C}_{m_n}(T)$ are unbounded for $\mathcal{T}$: distances $\rho_{n,T}(0, u)$ are unbounded for the cubes $\mathcal{C}_{m_n}(T)$, i.e., $\sup\{\rho_{n,T}(0, u) \mid T \in \mathcal{T}\} = \infty$. It is equivalent to take $(\bar{x} \approx \bar{x})$ for $\varphi(\bar{x})$ and to get $\sup\{\rho_{n,T}(0, 1) \mid T \in \mathcal{T}\} = \infty$.

Thus we get the following

Theorem 2.1. *Let $\mathcal{T}$ be a class of theories in $\overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$. The following conditions are equivalent:*

(1) $\mathrm{Cl}_E(\mathcal{T}) \not\subseteq \overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$*;*

(2) *for some natural $n \geq 1$, Boolean algebras $\mathcal{B}_n(T)$, $T \in \mathcal{T}$, have unbounded cardinalities and, moreover, there is an infinite decreasing chain $(u_k)_{k\in\omega}$ of labels for some formulas $\varphi_k(\bar{x})$ such that any finite sequence $(u_l, \ldots, u_0)$ with $u_l < \ldots < u_0$ is witnessed by infinitely many theories in $\mathcal{T}$;*

(3) *the same as in* (2) *with $u_0 = 1$.*

Corollary 2.2. *A class $\mathcal{T} \subseteq \overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$ does not generate, using the E-operator, theories, which are neither finitely categorical and ω-categorical, if and only if for the Boolean algebras $\mathcal{B}_n(T)$, $T \in \mathcal{T}$, there are no infinite decreasing chains $(u_k)_{k\in\omega}$ of labels for some formulas $\varphi_k(\bar{x})$ such that any finite sequence $(u_l, \ldots, u_0)$ with $u_l < \ldots < u_0$ is witnessed by infinitely many theories in $\mathcal{T}$.*

Remark 2.3. Corollary 2.2 together with Proposition 1.1 allow to determine E-closed classes of finitely categorical and ω-categorical theories. Here, since finite sets of theories are E-closed, it suffices to consider infinite sets.

Considering a set $\mathcal{T}$ of theories with disjoint languages, for the E-closeness it suffices to add theories of the empty language describing cardinalities, in $\omega + 1$, of universes if these cardinalities meet infinitely many times in $\mathcal{T}$.

In such a case we obtain relative closures [17] and have the following assertions.

Proposition 2.4. *A class $\mathcal{T}$ of theories of pairwise disjoint languages is E-closed if and only if the following conditions hold:*

(i) *for any $n \in \omega \setminus \{0\}$ whenever $\mathcal{T}$ contains infinitely many theories with n-element models then $\mathcal{T}$ contains the theory T^0_n of the empty language and with n-element models;*

(ii) *if $\mathcal{T}$ contains theories with unbounded finite cardinalities of models, or infinitely many theories with infinite models, then $\mathcal{T}$ contains the theory T^0_∞ of the empty language and with infinite models.*

Corollary 2.5. *A class $\mathcal{T} \subset \overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$ of theories of pairwise disjoint languages is E-closed if and only if the conditions* (i) *and* (ii) *hold.*

Corollary 2.6. *A class $\mathcal{T} \subset \overline{\mathcal{T}}_{\rm fin}$ of theories of pairwise disjoint languages is E-closed if and only if the condition* (i) *holds and there is $N \in \omega$ such $\mathcal{T}$ does not have n-categorical theories for $n > N$.*

Corollary 2.7. *A class $\mathcal{T} \subset \overline{\mathcal{T}}_{\omega,1}$ of theories of pairwise disjoint languages is E-closed if and only if the condition* (ii) *holds.*

Corollary 2.8. *For any class $\mathcal{T} \subset \overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$ of theories of pairwise disjoint languages, $\mathrm{Cl}_E(\mathcal{T})$ is contained in the class $\subset \overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$, moreover,*

$$\mathrm{Cl}_E(\mathcal{T}) \subseteq \mathcal{T} \cup \{T^0_\lambda \mid \lambda \in (\omega \setminus \{0\}) \cup \{\infty\}\}.$$

Remark 2.9. Using relative closures [17] the assertions 2.4–2.8 also hold if languages are disjoint modulo a common sublanguage Σ_0 such that all restrictions of n-categorical theories in $\mathcal{T} \cap \overline{\mathcal{T}}_{\rm fin}$ to Σ_0 have isomorphic (finite) models $\mathcal{M}_n$ and all restrictions of theories in $\mathcal{T} \cap \overline{\mathcal{T}}_{\omega,1}$ have isomorphic countable models $\mathcal{M}_\omega$. In such a case, the theories T^0_n should be replaced by $\mathrm{Th}(\mathcal{M}_n)$ and T^0_∞ — by $\mathrm{Th}(\mathcal{M}_\omega)$.

It is also permitted to have finitely many possibilities for each $\mathcal{M}_n$ and for $\mathcal{M}_\omega$.

The following example shows that (even with pairwise disjoint languages) ω-categorical theories T with unbounded $\rho_{n,T}(0,1)$ do not force theories outside the class of ω-categorical theories.

Example 2.10. Let T_n be a theory of infinitely many disjoint n-cubes with a graph relation $R^{(2)}_n$, $R_m \neq R_n$ for $m \neq n$. For the set $\mathcal{T} = \{T_n \mid n \in \omega\}$ we have $\mathrm{Cl}_E(\mathcal{T}) = \mathcal{T} \cup \{T^0_\infty\}$. All theories in $\mathrm{Cl}_E(\mathcal{T})$ are ω-categorical whereas $\rho_{2,T_n}(0,1) = n+2$ that witnessed by formulas describing distances $d(x,y) \in \omega \cup \{\infty\}$ between elements.

Similarly, taking for each $n \in \omega$ exactly one n-cube with a graph relation $R^{(2)}_n$, we get a set $\mathcal{T}$ of theories such that $\mathrm{Cl}_E(\mathcal{T}) \subset \overline{\mathcal{T}}_{\rm fin} \cup \overline{\mathcal{T}}_{\omega,1}$.

Remark 2.11. Assertions 2.1 – 2.5 and 2.7 – 2.9 hold for the operators Cl^d_P and $\mathrm{Cl}^{d,r}_P$ replacing E-closures by P-closures. As non-isolated types always produce infinite structures, Corollary 2.6 holds only for Cl^d_P with finite sets $\mathcal{T}$ of theories.

3. On approximations of theories with (in)finite models

Definition [3]. An infinite structure $\mathcal{M}$ is *pseudofinite* if every sentence true in $\mathcal{M}$ has a finite model.

Definition (cf. [11]). A consistent formula φ *forces* the infinity if φ does not have finite models.

By the definition, an infinite structure $\mathcal{M}$ is pseudofinite if and only if $\mathcal{M}$ does not satisfy formulas forcing the infinity.

We denote the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fin}$ by $\overline{\mathcal{T}}_{\rm inf}$.

Proposition 3.1. *A theory $T \in \overline{\mathcal{T}}_{\rm inf}$ belongs to some E-closure of theories in $\overline{\mathcal{T}}_{\rm fin}$ if and only if T does not have formulas forcing the infinity.*

Proof. If a formula φ forces the infinity then $\mathcal{T}_\varphi \subset \overline{\mathcal{T}}_{\rm inf}$ for any $\mathcal{T} \subseteq \overline{\mathcal{T}}$. Thus, having such a formula $\varphi \in T$, T can not be approximated by theories in $\overline{\mathcal{T}}_{\rm fin}$ and so T does not belong to E-closures of families $\mathcal{T} \subseteq \overline{\mathcal{T}}_{\rm fin}$.

Conversely, if any formula $\varphi \in T$ does not force the infinity then, since $T \notin \mathcal{T}_{\rm fin}$, $(\overline{\mathcal{T}}_{\rm fin})_\varphi$ is infinite using unbounded finite cardinalities and we can choose infinitely many theories in $(\overline{\mathcal{T}}_{\rm fin})_\varphi$, for each $\varphi \in T$, forming a set $\mathcal{T}_0 \subset \overline{\mathcal{T}}_{\rm fin}$ such that $T \in {\rm Cl}_E(\mathcal{T}_0)$. □

Note that, in view of Proposition 1.1, Proposition 3.1 is a reformulation of Lemma 1 in [19].

Corollary 3.2. *If a theory $T \in \overline{\mathcal{T}}_{\rm inf}$ belongs to some E-closure of theories in $\overline{\mathcal{T}}_{\rm fin}$ then T is not finitely axiomatizable.*

Proof. If T is finitely axiomatizable by some formula φ then $|\mathcal{T}_\varphi| \leq 1$ for any $\mathcal{T} \subseteq \overline{\mathcal{T}}$ and φ forces the infinity. Thus, in view of Proposition 3.1, T can not be approximated by theories in $\overline{\mathcal{T}}_{\rm fin}$, i. e., T does not belong to E-closures of families $\mathcal{T}_0 \subset \overline{\mathcal{T}}_{\rm fin}$. □

In fact, in view of Theorem 1.2, the arguments for Corollary 3.3 show that ${\rm Cl}_E(\mathcal{T})$, for a family $\mathcal{T}$ of finitely axiomatizable theories, has the least generating set $\mathcal{T}$ and does not contain new finitely axiomatizable theories.

Note that Proposition 3.1 admits a reformulation for ${\rm Cl}^d_P$ repeating the proof. At the same time theories in $\overline{\mathcal{T}}_{\rm fin}$ can not be approximated by theories in $\overline{\mathcal{T}}_{\rm inf}$ with respect to ${\rm Cl}_E$ (in view of Proposition 1.1) whereas each theory in $\overline{\mathcal{T}}_{\rm fin}$ can be approximated by theories in $\overline{\mathcal{T}}_{\rm inf}$ with respect to ${\rm Cl}^d_P$:

Proposition 3.3. *For any theory $T \in \overline{\mathcal{T}}_{\rm fin}$ there is a family $\mathcal{T}_0 \subset \overline{\mathcal{T}}_{\rm inf}$ such that T belongs to the $\Sigma(T)$-restriction of ${\rm Cl}^d_P(\mathcal{T}_0)$.*

Proof. It suffices to form $\mathcal{T}_0$ by infinitely many theories of structures $\mathcal{A}_i$, $i \in I$, with infinitely many copies of models $\mathcal{M} \models T$ forming E_i-classes for equivalence relations E_i, where E_j is either equality or complete for $j \neq i$. Considering disjoint unary predicates P_i for $\mathcal{A}_i$ we get the nonprincipal 1-type $p_\infty(x)$ isolated by the set $\{\neg P_i(x) \mid i \in I\}$ which can be realized by the set M with the structure $\mathcal{M}$ witnessing that T belongs to the restriction of ${\rm Cl}^d_P(\mathcal{T}_0)$ removing new relations E_i. □

Remark 3.4. We have a similar effect removing all relations E_j in the structures $\mathcal{A}_i$ and obtaining isomorphic structures $\mathcal{A}'_i$: by compactness the P-combination of structures $\mathcal{A}'_i$ (where disjoint $\mathcal{A}'_i$ form unary predicates P_i) has the theory with a model, whose p_∞-restriction forms a structure isomorphic to $\mathcal{M}$. In this case we have ${\rm Cl}^{d,r}_P(\{{\rm Th}(\mathcal{A}'_i)\})$.

Remark 3.5. As in the proof of Proposition 3.1 theories in $\mathcal{T}_0$ can be chosen consistent modulo cardinalities of their models we can add that e-${\rm Sp}(T) = 1$ for the E-combination T of the theories in $\mathcal{T}_0$.

As the same time e-${\rm Sp}(T')$ is infinite for the P-combination T' of $\mathcal{A}_i$ in the proof of Proposition 3.3, since $p_\infty(x)$ has infinitely many possibilities for finite cardinalities of sets of realizations for $p_\infty(x)$.

4. e-SPECTRA FOR FINITELY CATEGORICAL AND ω-CATEGORICAL THEORIES

We refine the notions of e-spectra e-${\rm Sp}(\mathcal{A}_E)$ and e-${\rm Sp}(T)$ for the theories $T = {\rm Th}(\mathcal{A}_E)$ restricting the class of possible theories to a given class $\mathcal{T}$ in the following way.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ with $\mathrm{Th}(\mathcal{B}) \in \mathcal{T}$, which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the $(e, \mathcal{T})$-*spectrum* of $\mathcal{A}_E$ and denoted by $(e, \mathcal{T})$-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{(e, \mathcal{T})\text{-}\mathrm{Sp}(\mathcal{A}')) \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the $(e, \mathcal{T})$-*spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by $(e, \mathcal{T})$-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

The following properties are obvious.

1. (Monotony) If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then $(e, \mathcal{T}_1)$-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E)) \leq (e, \mathcal{T}_2)$-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$ for any structure $\mathcal{A}_E$.

2. (Additivity) If the class $\overline{\mathcal{T}}$ of all complete elementary theories of relational languages is the disjoint union of subclasses $\overline{\mathcal{T}}_1$ and $\overline{\mathcal{T}}_2$ then for any theory $T = \mathrm{Th}(\mathcal{A}_E)$,

$$e\text{-}\mathrm{Sp}(T) = (e, \overline{\mathcal{T}}_1)\text{-}\mathrm{Sp}(T) + (e, \overline{\mathcal{T}}_2)\text{-}\mathrm{Sp}(T).$$

We divide a class $\mathcal{T}$ of theories into two disjoint subclasses $\mathcal{T}^{\mathrm{fin}}$ and $\mathcal{T}^{\mathrm{inf}}$ having finite and infinite non-empty language relations, respectively. More precisely, for functions $f\colon \omega \to \lambda_f$, where λ_f are cardinalities, we divide $\mathcal{T}$ into subclasses $\mathcal{T}^f$ of theories T such that T has $f(n)$ n-ary predicate symbols for each $n \in \omega$.

For the function f we denote by $\mathrm{Supp}(f)$ its *support*, i.e., the set $\{n \in \omega \mid f(n) > 0\}$.

Clearly, the language of a theory $T \in \mathcal{T}^f$ is finite if and only if $\rho_f \subset \omega$ and $\mathrm{Supp}(f)$ is finite.

Illustrating $(e, \mathcal{T})$-spectra for the class $\mathcal{T}$ of all cubic theories and taking the class $\mathcal{T}_0^{\mathrm{fin}} \subset \mathcal{T}$ of all theories of finite cubes we note that for an E-combination T of theories T_i in $\mathcal{T}_0^{\mathrm{fin}}$, $(e, \mathcal{T})$-$\mathrm{Sp}(T)$ is positive if and only if there are infinitely many T_i. In such a case, $(e, \mathcal{T})$-$\mathrm{Sp}(T) = 1$ and new theory, which does not belong to $\mathcal{T}_0^{\mathrm{fin}}$, is the theory of ω-cube.

The class $\overline{\mathcal{T}}_{\mathrm{fin}}$ is represented as disjoint union of subclasses $\overline{\mathcal{T}}_{\mathrm{fin,n}}$ of theories having n-element models, $n \in \omega \setminus \{0\}$. For $N \in \omega$, the class $\bigcup\limits_{n \leq N} \overline{\mathcal{T}}_{\mathrm{fin},n}$ is denoted by $\overline{\mathcal{T}}_{\mathrm{fin},\leq N}$.

Proposition 4.1. *For any $\mathcal{T} \subset \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}) \setminus \overline{\mathcal{T}}_{\mathrm{fin}} \neq \varnothing$ if and only if for any natural N, $\mathcal{T} \not\subset \overline{\mathcal{T}}_{\mathrm{fin},\leq N}$.*

Proof. If $\mathcal{T}$ contains a theory with infinite models, the assertion is obvious. If $\mathcal{T} \subset \overline{\mathcal{T}}_{\mathrm{fin}}$, then we apply Compactness and Proposition 1.1. □

The following obvious proposition is also based on Proposition 1.1.

Proposition 4.2. *If $\mathcal{T} \subset \overline{\mathcal{T}}_{\mathrm{fin},n}$ (respectively $\mathcal{T} \subset \overline{\mathcal{T}}_{\mathrm{fin},\leq N}$) then $\mathrm{Cl}_E(\mathcal{T}) \subset \overline{\mathcal{T}}_{\mathrm{fin},n}$ ($\mathrm{Cl}_E(\mathcal{T}) \subset \overline{\mathcal{T}}_{\mathrm{fin},\leq N}$). For any theory $T = \mathrm{Th}(\mathcal{A}_E)$, where all E-classes have theories in $\mathcal{T}$, e-$\mathrm{Sp}(T) = (e, \overline{\mathcal{T}}_{\mathrm{fin},n})$-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$ (e-$\mathrm{Sp}(T) = (e, \overline{\mathcal{T}}_{\mathrm{fin},\leq N})$-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$). If, additionally, $\mathcal{T}$ is the set of theories in a finite language then $\mathcal{T}$ is finite (and so E-closed). In particular, for any theory $T = \mathrm{Th}(\mathcal{A}_E)$ in a finite language, where all E-classes have theories in $\mathcal{T}$, e-$\mathrm{Sp}(T) = 0$.*

Remark 4.3. In fact, the conclusions of Proposition 4.2 follow implying the following fact. If all theories in $\mathcal{T}$ contain a formula φ then all theories in $\mathrm{Cl}_E(\mathcal{T})$ contain φ. For (1) we take a formula φ "saying" that models have exactly n elements, and for

(2) — a formula φ "saying" that models have at most N elements. If the language is finite there are only finitely many possibilities for isomorphism types on n-element sets and these possibilities are formula-definable.

Similarly Proposition 4.2 we have

Proposition 4.4. *If* $\mathcal{T}\cap\overline{\mathcal{T}}_{\rm fin}=\varnothing$ *then* $\mathrm{Cl}_E(\mathcal{T})\cap\overline{\mathcal{T}}_{\rm fin}=\varnothing$.

Definition [16]. A theory T in a predicate language Σ is called *language uniform*, or a LU-*theory* if for each arity n any substitution on the set of non-empty n-ary predicates (corresponding to the symbols in Σ) preserves T. The LU-theory T is called IILU-*theory* if it has non-empty predicates and as soon as there is a non-empty n-ary predicate then there are infinitely many non-empty n-ary predicates and there are infinitely many empty n-ary predicates.

Since for any finite cardinality n there are IILU-theories with n-element models, repeating the proof of [16, Proposition 12] and [16, Proposition 13] we get

Proposition 4.5. (1) *For any* $n\in\omega\setminus\{0\}$ *and* $\mu\leq\omega$ *there is an* E*-combination* $T=\mathrm{Th}(\mathcal{A}_E)$ *of* IILU*-theories* $T_i\in\overline{\mathcal{T}}_{{\rm fin},n}$ *in a language* Σ *of the cardinality* ω *such that* T *has an e-least model and* e-$\mathrm{Sp}(T)=\mu$.

(2) *For any uncountable cardinality* λ *there is an* E*-combination* $T=\mathrm{Th}(\mathcal{A}_E)$ *of* IILU*-theories* $T_i\in\overline{\mathcal{T}}_{{\rm fin},n}$ *in a language* Σ *of the cardinality* λ *such that* T *has an e-least model and* e-$\mathrm{Sp}(T)=\lambda$.

Proposition 4.6. *For any* $n\in\omega\setminus\{0\}$ *and infinite cardinality* λ *there is an* E*-combination* $T=\mathrm{Th}(\mathcal{A}_E)$ *of* IILU*-theories* $T_i\in\overline{\mathcal{T}}_{{\rm fin},n}$ *in a language* Σ *of cardinality* λ *such that* T *does not have e-least models and* e-$\mathrm{Sp}(T)\geq\max\{2^\omega,\lambda\}$.

Proposition 4.7. *For any* $n\in\omega\setminus\{0\}$ *and infinite cardinality* λ *there is an* E*-combination* $T=\mathrm{Th}(\mathcal{A}_E)$ *of* LU*-theories* $T_i\in\overline{\mathcal{T}}_{{\rm fin},n}$ *in a language* Σ *of cardinality* λ *such that* T *does not have e-least models and* e-$\mathrm{Sp}(T)=2^\lambda$.

Proof. Let Σ be a language consisting, for some natural m, of m-ary predicate symbols R_i, $i<\lambda$. For any $\Sigma'\subseteq\Sigma$ we take a structure $\mathcal{A}_{\Sigma'}$ of the cardinality n such that $R_i=(A_{\Sigma'})^m$ for $R_i\in\Sigma'$, and $R_i=\varnothing$ for $R_i\in\Sigma\setminus\Sigma'$. Clearly, each structure $\mathcal{A}_{\Sigma'}$ has a LU-theory and $\mathcal{A}_{\Sigma'}\not\equiv\mathcal{A}_{\Sigma''}$ for $\Sigma'\neq\Sigma''$. For the E-combination $\mathcal{A}_E$ of the structures $\mathcal{A}_{\Sigma'}$ we obtain the theory $T=\mathrm{Th}(\mathcal{A}_E)$ having a model of the cardinality λ. At the same time $\mathcal{A}_E$ has 2^λ distinct theories of the E-classes $\mathcal{A}_{\Sigma'}$. Thus, e-$\mathrm{Sp}(T)=2^\lambda$. Finally we note that T does not have e-least models by Theorem 1.2 and arguments for [15, Proposition 9]. $\square$

Remark 4.8. Considering countable LU-theories for the assertions above we can assume that these theories belong to a class $\mathcal{T}^f$, where $f\in\omega^\omega$ and $\mathrm{Supp}(f)$ is infinite. Note also that Propositions 4.5–4.7 hold replacing the classes $\overline{\mathcal{T}}_{{\rm fin},n}$ by $\overline{\mathcal{T}}_{\omega,1}$.

Replacing E-classes by unary predicates P_i (not necessary disjoint) being universes for structures $\mathcal{A}_i$ and restricting models of $\mathrm{Th}(\mathcal{A}_P)$ to the set of realizations of $p_\infty(x)$ we get the $(e,\mathcal{T})$*-spectrum* $(e,\mathcal{T})$-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_P))$, i. e., the number of pairwise elementary non-equivalent restrictions $\mathcal{N}$ of $\mathcal{M}\models\mathrm{Th}(\mathcal{A}_P)$ to $p_\infty(x)$ such that $\mathrm{Th}(\mathcal{N})\in\mathcal{T}$.

Proposition 4.9. *If the structures $\mathcal{A}_i$ have pairwise disjoint languages with disjoint predicates P_i then for any natural $n \geq 1$, $(e, \overline{\mathcal{T}}_{\rm fin,n})$-Sp(Th($\mathcal{A}_P$)) ≤ 1, and $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fin})$-Sp(Th($\mathcal{A}_P$)) ≤ 1.*

Proof. Clearly, if the structures $\mathcal{A}_i$ have pairwise disjoint languages with disjoint predicates P_i then structures for $p_\infty(x)$ do not contain realizations of language predicates, i. e., have theories T^0_λ. Now $(e, \overline{\mathcal{T}}_{\rm fin,n})$-Sp(Th($\mathcal{A}_P$)) ≤ 1 and $(e, \overline{\mathcal{T}}_{\rm fin,n})$-Sp(Th($\mathcal{A}_P$)) $= 1$ if and only if there are infinitely many indexes i and Th($\mathcal{A}_i$) $\neq T^0_n$ for any i. Similarly, $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fin})$-Sp(Th($\mathcal{A}_P$)) ≤ 1 and $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fin})$-Sp(Th($\mathcal{A}_P$)) $= 1$ if and only if there are infinitely many indexes i and Th($\mathcal{A}_i$) $\neq T^0_\infty$ for any i. □

Clearly, approximating structures without non-trivial predicates and applying the proof of Proposition 4.9 we get a family of P-combinations with $(e, \overline{\mathcal{T}}_{\rm fin,n})$-Sp(Th($\mathcal{A}_P$)) $= 1$, for $n \in \omega \setminus \{0\}$, and $(e, \overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fin})$-Sp(Th($\mathcal{A}_P$)) $= 1$.

Comparing approximations in Section 3 and proofs for [14, Propositions 4.12, 4.13] we get

Proposition 4.10. *For any infinite cardinality λ there is a theory $T =$ Th($\mathcal{A}_P$) being a P-combination of theories in $\overline{\mathcal{T}}_{\rm fin}$ and of a language Σ such that $|\Sigma| = \lambda$ and e-Sp(T) $= 2^\lambda$.*

5. Almost language uniform theories

Definition. A theory T in a predicate language Σ is called *almost language uniform*, or a ALU-*theory* if for each arity n with n-ary predicates for Σ there is a partition for all n-ary predicates, corresponding to the symbols in Σ, with finitely many classes K such that any substitution preserving these classes preserves T, too. The ALU-theory T is called IIALU-*theory* if it has non-empty predicates and as soon as there is a non-empty n-ary predicate in a class K then there are infinitely many non-empty n-ary predicates in K and there are infinitely many empty n-ary predicates.

By the definition any LU-theory is an ALU-theory and any IILU-theory is an IIALU-theory as well.

Since any finite structure can have only finitely many distinct predicates for each arity n we get the following

Proposition 5.1. *Any theory $T \in \overline{\mathcal{T}}_{\rm fin}$ is an* ALU*-theory.*

Replacing LU- and IILU- by ALU- and IIALU- and the proofs in Propositions 4.5–4.7 we get analogs for these assertions attracting expansions of arbitrary theories in $\overline{\mathcal{T}}_{\rm fin,n}$. Thus any theory in $\overline{\mathcal{T}}_{\rm fin,n}$ can be used obtaining described e-spectra.

6. Families of cardinalities for models of theories in closures

Let $\mathcal{T}$ be a nonempty family of theories in $\overline{\mathcal{T}}$. We denote by $c_E(\mathcal{T})$ (respectively, $c_P(\mathcal{T})$, $c^d_P(\mathcal{T})$, $c^{d,r}_P(\mathcal{T})$) the set of finite cardinalities for models of theories in $\mathrm{Cl}_E(\mathcal{T})$ ($\mathrm{Cl}_P(\mathcal{T})$, $\mathrm{Cl}^d_P(\mathcal{T})$, $\mathrm{Cl}^{d,r}_P(\mathcal{T})$) and by $\bar{c}_E(\mathcal{T})$ (respectively, $\bar{c}_P(\mathcal{T})$, $\bar{c}^d_P(\mathcal{T})$, $\bar{c}^{d,r}_P(\mathcal{T})$) the set of finite cardinalities for models of theories in $\mathrm{Cl}_E(\mathcal{T})$ ($\mathrm{Cl}_P(\mathcal{T})$, $\mathrm{Cl}^d_P(\mathcal{T})$, $\mathrm{Cl}^{d,r}_P(\mathcal{T})$) which are not cardinalities for models of theories in $\mathcal{T}$. Additionally, for $\mathrm{Cl}_P(\mathcal{T})$, $\mathrm{Cl}^d_P(\mathcal{T})$ and $\mathrm{Cl}^{d,r}_P(\mathcal{T})$ we denote by $\hat{c}_P(\mathcal{T})$, $\hat{c}^d_P(\mathcal{T})$, $\hat{c}^{d,r}_P(\mathcal{T})$, respectively, the set of finite cardinalities for models of theories being restrictions for corresponding P-combinations to sets of realizations of types $p_\infty(x)$.

Remark 6.1. Since E-closures preserve finite cardinalities for models of theories in families in $\mathcal{T}$, i.e., $c_E(\mathcal{T})$ consists of these cardinalities for $\mathcal{T}$, then $\bar{c}_E(\mathcal{T}) \equiv \varnothing$. Thus we can use the notation $c_E(\mathcal{T})$ for the set of finite cardinalities for models of theories in $\mathcal{T}$, or, equivalently, for models of theories in $\mathrm{Cl}_E(\mathcal{T})$.

Remark 6.2. If $\mathcal{T}$ is finite, or corresponding $p_\infty(x)$ is consistent and there are no models with finitely many realizations for $p_\infty(x)$, then $c_P(\mathcal{T}) = c^d_P(\mathcal{T}) = c_E(\mathcal{T})$ and $\bar{c}_P(\mathcal{T}) = \bar{c}^d_P(\mathcal{T}) = \hat{c}_P(\mathcal{T}) = \hat{c}^d_P(\mathcal{T}) = \varnothing$.

Examples of families of theories in the empty language Σ_0 witness that the cardinalities for sets of realizations of $p_\infty(x)$ can vary arbitrarily and for finite $\mathcal{T}$ we have $c^{d,r}_P(\mathcal{T}) = \hat{c}^{d,r}_P(\mathcal{T}) = \mathbb{Z}^+$ and $\bar{c}^{d,r}_P(\mathcal{T}) = \mathbb{Z}^+ \setminus c_E(\mathcal{T})$.

Having an infinite family $\mathcal{T}$ in the language Σ_0, similarly we get $c_P(\mathcal{T}) = c^d_P(\mathcal{T}) = c^{d,r}_P(\mathcal{T}) = \hat{c}_P(\mathcal{T}) = \hat{c}^d_P(\mathcal{T}) = \hat{c}^{d,r}_P(\mathcal{T}) = \mathbb{Z}^+$ and $\bar{c}_P(\mathcal{T}) = \bar{c}^d_P(\mathcal{T}) = \bar{c}^{d,r}_P(\mathcal{T}) = \mathbb{Z}^+ \setminus c_E(\mathcal{T})$. The latter formula shows that $\bar{c}_P(\mathcal{T})$, $\bar{c}^d_P(\mathcal{T})$, and $\bar{c}^{d,r}_P(\mathcal{T})$ can be arbitrary subsets of $\mathbb{Z}^+$ with infinite complements. Thus we have the following

Proposition 6.3. *For any infinite set $Y \subseteq \mathbb{Z}^+$ there is a family $\mathcal{T}$ such that $\bar{c}_P(\mathcal{T}) = \bar{c}^d_P(\mathcal{T}) = \bar{c}^{d,r}_P(\mathcal{T}) = \mathbb{Z}^+ \setminus Y$.*

Example 6.4. If the language Σ consists of the symbol E_k of the equivalence relation whose each class has $k \in \omega$ elements then $p_\infty(x)$ can form an arbitrary structure with k-element equivalence classes and for a finite family $\mathcal{T}_k$ we have $c^{d,r}_P(\mathcal{T}_k) = \hat{c}^{d,r}_P(\mathcal{T}_k) = k\mathbb{Z}^+$ and $\bar{c}^{d,r}_P(\mathcal{T}_k) = k\mathbb{Z}^+ \setminus c_E(\mathcal{T}_k)$. If the family $\mathcal{T}_k$ is infinite then, similarly, $c_P(\mathcal{T}_k) = c^d_P(\mathcal{T}_k) = c^{d,r}_P(\mathcal{T}_k) = \hat{c}_P(\mathcal{T}_k) = \hat{c}^d_P(\mathcal{T}_k) = \hat{c}^{d,r}_P(\mathcal{T}_k) = k\mathbb{Z}^+$ and $\bar{c}_P(\mathcal{T}_k) = \bar{c}^d_P(\mathcal{T}_k) = \bar{c}^{d,r}_P(\mathcal{T}_k) = k\mathbb{Z}^+ \setminus c_E(\mathcal{T})$.

More generally, collecting the families of theories with distinct E_k, $k \in K$, $K \subset \omega$, we obtain nonempty values for c_P, c^d_P, $c^{d,r}_P$, $\hat{c}_P$, $\hat{c}^d_P$, $\hat{c}^{d,r}_P$ as unions $\bigcup\limits_{k\in K} k\mathbb{Z}^+$.

Now we have to show that all possible nonempty values for $\hat{c}^d_P$ and $\hat{c}^{d,r}_P$ are exhausted by the sums $\biguplus\limits_{k\in K} k\mathbb{Z}^+$ (unions with finite sums for numbers in $k\mathbb{Z}^+$) whereas values for $\hat{c}_P$ may differ.

Theorem 6.5. *For any nonempty family $\mathcal{T}$ there is $K \subset \omega$ such that $\hat{c}^{d,r}_P(\mathcal{T}) = \biguplus\limits_{k\in K} k\mathbb{Z}^+$.*

Proof. Recall that for P-combinations with respect to $\mathrm{Cl}^{d,r}_P$ there are no links between disjoint predicates P_i with structures $\mathcal{A}_i$ being models of theories in $\mathcal{T}$. Therefore if $p_\infty(x)$ can produce finite structures then structures $\mathcal{A}_i$ with 1-types approximating $p_\infty(x)$, define (partial) definable equivalence relations with bounded finite classes $E(a)$ and without definable extensions, for the approximations and for $p_\infty(x)$. So there are no links between the classes $E(a)$ and having k elements in $E(a)$ we produce, by compactness, a series of $1, 2, \ldots, n, \ldots$ E-classes for $p_\infty(x)$ since $p_\infty(x)$ is not isolated. Thus we get a series $k\mathbb{Z}^+$ for $\hat{c}^{d,r}_P(\mathcal{T})$. Varying finite cardinalities for the classes $E(a)$ we obtain the required formula $\hat{c}^{d,r}_P(\mathcal{T}) = \biguplus\limits_{k\in K} k\mathbb{Z}^+$ for some set $K \subset \omega$ witnessing these cardinalities. If $p_\infty(x)$ can produce finite structures then we set $K \rightleftharpoons \varnothing$. $\square$

Theorem 6.6. *For any infinite family $\mathcal{T}$ there is $K \subset \omega$ such that $\hat{c}^d_P(\mathcal{T}) = \biguplus\limits_{k\in K} k\mathbb{Z}^+$.*

Proof repeats the proof of Theorem 6.5 using structures $\mathcal{A}_i$ which pairwise are not elementary equivalent. □

Remark 6.7. 1. In Theorems 6.5 and 6.6, if we have minimal K with $|K| > 1$ then the type $p_\infty(x)$ is not complete. Indeed, taking, for sets of realizations of $p_\infty(x)$, maximal definable equivalence relations E_1 and E_2 for $k_1 \neq k_2 \in K$ we can not move, by automorphisms, elements of E_1-classes to elements of E_2-classes.

2. Clearly, having E_1-classes and E_2-classes of same cardinalities with non-isomorphic structures we again can not connect elements of these classes by automorphisms. Thus, $|K| = 1$ is a necessary but not sufficient condition for the completeness of $p_\infty(x)$.

3. The least cardinality $|K|$, with positive $\hat{c}^{d,r}_P$ or $\hat{c}^d_P$, gives a lower bound for independent equivalence relations with respect to their realizability/omitting for restrictions of models to sets of realizations of $p_\infty(x)$.

Remark 6.8. Finite structures $\mathcal{A}_\infty$ for maximal definable equivalence relations for $p_\infty(x)$ with respect to Cl^d_P and to $\mathrm{Cl}^{d,r}_P$ can be isomorphic if and only if they are represented in some $\mathcal{A}_i$ for $\mathrm{Cl}^{d,r}_P$ and infinitely many times for Cl^d_P, or approximated both for $\mathrm{Cl}^{d,r}_P$ and for Cl^d_P. Hence, for any infinite family $\mathcal{T}$, $\hat{c}^d_P(\mathcal{T}) = \hat{c}^{d,r}_P(\mathcal{T})$ if and only if each n-element class for maximal definable equivalence relations for $p_\infty(x)$ with respect to $\mathrm{Cl}^{d,r}_P$ has n-element classes for correspondent definable equivalence relations in infinitely many pairwise elementary non-equivalent structures $\mathcal{A}_i$, with respect to Cl^d_P.

Definition [2]. Let $\mathcal{M}$ be a model of a theory T, $\bar{a}$ and $\bar{b}$ tuples in $\mathcal{M}$, A a subset of M. The tuple $\bar{a}$ *semi-isolates* the tuple $\bar{b}$ over the set A if there exists a formula $\varphi(\bar{a}, \bar{y}) \in \mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a}, \bar{y}) \vdash \mathrm{tp}(\bar{b}/A)$ holds. In this case we say that the formula $\varphi(\bar{x}, \bar{y})$ (with parameters in A) *witnesses that $\bar{b}$ is semi-isolated over $\bar{a}$ with respect to A.*

If $p \in S(T)$ and $\mathcal{M} \models T$ then $\mathrm{SI}^{\mathcal{M}}_p$ denotes the relation of semi-isolation (over $\varnothing$) on the set of all realizations of p:

$$\mathrm{SI}^{\mathcal{M}}_p \rightleftharpoons \{(\bar{a}, \bar{b}) \mid \mathcal{M} \models p(\bar{a}) \land p(\bar{b}) \text{ and } \bar{a} \text{ semi-isolates } \bar{b}\}.$$

The following definition generalizes the previous one for a family of 1-types, in particular, for incomplete $p_\infty(x)$.

Definition [8]. Let T be a complete theory, $\mathcal{M} \models T$. We consider *closed* nonempty sets (under the natural topology) sets $\mathbf{p}(x) \subseteq S^1(\varnothing)$, i. e., sets $\mathbf{p}(x)$ such that $\mathbf{p}(x) = \bigcap\limits_{i\in I} [\varphi_{\mathbf{p},i}(x)]$, where $[\varphi_{\mathbf{p},i}(x)] \rightleftharpoons \{p(x) \in S^1(\varnothing) \mid \varphi_{\mathbf{p},i}(x) \in p(x)\}$ for some formulas $\varphi_{\mathbf{p},i}(x)$ of T.

For closed sets $\mathbf{p}(x), \mathbf{q}(y) \subseteq S(\varnothing)$ of types, realized in $\mathcal{M}$, we consider $(\mathbf{p}, \mathbf{q})$-*preserving* $(\mathbf{p}, \mathbf{q})$-*semi-isolating*, $(\mathbf{p} \to \mathbf{q})$-, or $(\mathbf{q} \leftarrow \mathbf{p})$-*formulas* $\varphi(x, y)$ of T, i. e., formulas for which if $a \in M$ realizes a type in $\mathbf{p}(x)$ then every solution of $\varphi(a, y)$ realizes a type in $\mathbf{q}(y)$.

If $\mathbf{p}(x) = \mathbf{q}(y)$ then $(\mathbf{p}, \mathbf{q})$-preserving formulas are called $\mathbf{p}$-*preserving* or $\mathbf{p}$-*semi-isolating* and we define, similarly to $\mathrm{SI}^{\mathcal{M}}_p$, the *generalized* relation $\mathrm{SI}^{\mathcal{M}}_{\mathbf{p}}$ of semi-isolation for the set of realizations of types in $\mathbf{p}(x)$:

$$\mathrm{SI}^{\mathcal{M}}_{\mathbf{p}} \rightleftharpoons \{(a,b) \mid \mathcal{M} \models p(a) \wedge p'(b) \wedge \varphi(a,b)$$
$$\text{for } p, p' \in \mathbf{p} \text{ and a } \mathbf{p}\text{-preserving formula } \varphi(x,y)\}.$$

If $(a,b) \in \mathrm{SI}^{\mathcal{M}}_{\mathbf{p}}$ we say that *a semi-isolates b with respect to* $\mathbf{p}$.

Thus, a semi-isolates b (in sense of [2]) if and only if a semi-isolates b with respect to $\{\mathrm{tp}(a), \mathrm{tp}(b)\}$.

Remark 6.9. Since there are no links between structures $\mathcal{A}_i$ with respect to Cl^d_P and $\mathrm{Cl}^{d,r}_P$, the set $\mathbf{p}_\infty$ of all completions $q(x)$ of $p_\infty(x)$ has symmetric $\mathrm{SI}^{\mathcal{M}}_{\mathbf{p}_\infty}$. Thus, the relations $\mathrm{SI}^{\mathcal{M}}_{\mathbf{p}_\infty}$ form equivalence relations. Positive values for $\hat{c}^d_P$ and $\hat{c}^{d,r}_P$ imply that these equivalence relations have finite classes. Cardinalities of these classes define formulas in Theorems 6.5 and 6.6.

Now we consider the general case, with the operator Cl_P. One can hardly expect productive descriptions considering arbitrary links of structures with respect to arbitrary links of predicates P_i, in contrast to the disjoint predicates when, obviously, there are no links between the structures. So we will fix a P-combination $\mathcal{A}_P$ (and its theory $T = \mathrm{Th}(\mathcal{A}_P)$) and consider the set $\hat{c}_P(T)$ of values of finite cardinalities for $p_\infty(x)$ with respect to given P-combination T, instead of the set $\hat{c}_P(\mathcal{T})$ of values for all finite values for all possible P-combinations. In other words we argue to describe sets of finite cardinalities for sets of realizations of a nonprincipal, not necessary complete, 1-type $p_\infty(x)$.

We note the following obvious observations.

Remark 6.10. 1. If any $n \in \omega$ realizations of a type $p_\infty(x)$ force infinitely many realizations of $p_\infty(x)$ then it is true for any $m > n$.

2. If a and b are realizations of a type $p_\infty(x)$ and a does not semi-isolate b with respect to $\mathbf{p}_\infty$ then there are no formulas $\varphi(x,y)$ with $\models \varphi(a,b)$ and forcing finitely or infinitely many realizations for the type $q = \mathrm{tp}(b/a)$, i. e., the set of realizations of q can be empty and infinite, depending on a model.

The first observation shows that having n which forces infinity, we get $\hat{c}_P(T) \subset n$. The second one implies that realizations of $\mathbf{p}_\infty$, which are not connected by the relation of semi-isolation, contribute to $\hat{c}_P(T)$ independently on the binary level. Moreover, these contributions by realizations a and b can generate distinct series, as in Theorems 6.5 and 6.5, only if $\mathrm{tp}(a) \neq \mathrm{tp}(b)$.

The following example shows that there is a theory T with $\hat{c}_P(T) = \{1\}$ clarifying that contributions above on the binary level deny by the ternary level.

Example 6.11. Consider a coloring Col: $M \to \omega \cup \{\infty\}$ of an infinite set M such that each color $\lambda \in \omega \cup \{\infty\}$ has infinitely many elements in M, i. e., each $\mathrm{Col}_n = \{a \in M \mid \mathrm{Col}(a) = n\}$ is infinite as well as there are infinitely many elements of the infinite color. We put $P_i = M \setminus \bigcup\limits_{j<i} \mathrm{Col}_j$ and $p_\infty(x) = \{\neg P_n(x) \mid n \in \omega\}$. Now we define, using a generic construction with free amalgams [12, 7], a ternary relation R such that for the definable relation $Q(x,y) \equiv \exists z R(x,y,z)$ we have the following properties:

1) the Q-structure has unique 1-type and, moreover, its automorphism group is transitive;

2) $R(x,y,z) \equiv Q(x,z) \wedge Q(y,z)$;

3) Col is an inessential coloring which is not neither Q-ordered nor Q^{-1}-ordered [12, 6], moreover, for any element $a \in M$ the sets of solutions for $Q(a, y)$ and $Q(x, a)$ have infinitely many elements of each color;

4) for any $a \neq b \in M$ the set of solutions for $R(a, b, z)$ is infinite for each color $n \geq \min\{\mathrm{Col}(a), \mathrm{Col}(b)\}$ and does not have have elements of colors $< \min\{\mathrm{Col}(a), \mathrm{Col}(b)\}$, hence, $R(a, b, z) \vdash p_\infty(z)$ if $\models p_\infty(a)$ and $\models p_\infty(b)$.

Taking the generic structure $\mathcal{M}$ in the language $\langle P_n^{(1)}, Q^{(2)}, R^{(3)}\rangle_{n\in\omega}$ and its theory $T = \mathrm{Th}(\mathcal{M})$, being a P-combination, we have $\hat{c}_P(T) = \{1\}$ since the nonisolated type $p_\infty(x)$ can have, in a model of T, 0, 1, or infinitely many realizations: one realization of $p_\infty(x)$ does not force new ones and two distinct realizations a and b of $p_\infty(x)$ force infinitely many ones by the formula $R(a, b, z)$.

Example 6.12. We modify Example 6.11 replacing elements by E_k-classes, where each class contains k elements, and repeat the generic construction satisfying the following conditions:

2) if aE_ka' then $\mathrm{Col}(a) = \mathrm{Col}(a')$;

2) if $(a, b) \in Q$, aE_ka', bE_kb', then $(a', b') \in Q$.

The theory T_k of resulting generic structure $\mathcal{M}_k$ satisfies $\hat{c}_P(T_k) = \{k\}$ since each realization a of $p_\infty(x)$ forces k realizations of $p_\infty(x)$ consisting of $E_k(a)$ and any two realizations of $p_\infty(x)$ belonging to distinct E_k-classes forces infinitely many E_k-classes with elements satisfying $p_\infty(x)$.

Combining structures $\mathcal{M}_k$ with distinct k we obtain a generic structure whose theory T satisfies $\hat{c}_P(T) = K$ for a given set $K \subseteq \mathbb{Z}^+$. Here sets of realizations of $p_\infty(x)$ are divided into E_k-classes for $k \in K$.

Thus we have the following theorem asserting that values $\hat{c}_P(T)$ can be arbitrary.

Theorem 6.13. *For any set $K \subseteq \mathbb{Z}^+$ there is a P-combination T such that $\hat{c}_P(T) = K$.*

Now we argue to modify the generic construction and Theorem 6.13 using transitive arrangements of algebraic systems similar to [9, 5], and obtaining a similar result with complete $p_\infty(x)$ describing possibilities for $\hat{c}_P(T)$.

For this aim we fix a nonempty set $K \subseteq \mathbb{Z}^+$ claiming for $\hat{c}_P(T) = K$ with some P-combination T. Note that if $1 \notin K$ then either any realization a of $p_\infty(x)$ forces infinitely many realizations or a belongs to the maximal finite definable E-class with some $k_0 > 1$ elements. At first case, by completeness of $p_\infty(x)$, any finite set of realizations of $p_\infty(x)$ forces that infinity and therefore $K = \varnothing$ contradicting the condition $K \neq \varnothing$. At second case, again by completeness of $p_\infty(x)$, we have $K \subseteq k_0\mathbb{Z}^+$. Replacing elements by their E-classes we reduce the problem of construction of T with $\hat{c}_P(T) = K$ to the case $1 \in K$.

Example 6.11 witnesses the possibility for $\hat{c}_P(T) = \{1\}$. So below we assume that $1 \in K$ and $|K| \geq 2$.

Now for each $k \in K \setminus \{1\}$ we introduce a ternary relation R_k defining a free (acyclic) precise pseudoplane $\mathcal{P}_k$ [9] with infinitely many lines containing any fixed point and exactly k points belonging to any fixed line such that $\mathcal{P}_k$ has infinitely many connected components. Then we combine these free pseudoplanes $\mathcal{P}_k$ allowing that each point belongs to each pseudoplane $\mathcal{P}_k$ and the union of sets of lines does not form cycles. We embed copies of that combination $\mathcal{P}$ of the pseudoplanes into unary predicates Col_n as well as to the structure of $p_\infty(x)$.

Modifying Example 6.11 we introduce a binary predicate Q such that;

1) if $(a,b)\in Q$ and $(a,c)\in Q$ then a, b, c belong to pairwise distinct connected components of $\mathcal{P}$, the same is satisfied for Q^{-1} (as in Example described in [12, Section 1.3] and in [4]);

2) elements $a_1,\ldots,a_m$, $m>1$, realizing $p_\infty(x)$ and belonging to a common line l force all elements of l and do not force elements outside l;

3) if a and b are realizations of $p_\infty(x)$ which do not have a common line then a and b force infinitely many realizations of $p_\infty(x)$ by the formula $Q(a,y)\wedge Q(b,y)$.

The resulted generic structure $\mathcal{M}$ of the language $\langle \mathrm{Col}_n, Q, R_k\rangle_{n\in\omega,k\in K\setminus\{1\}}$ and its theory T satisfy the following properties:

i) any realization of $p_\infty(x)$ does not force new realizations of $p_\infty(x)$ witnessing $1\in\hat{c}_P(T)$;

ii) any at least two distinct realizations of $p_\infty(x)$ in a line l belonging to $\mathcal{P}_k$ force exactly the set l witnessing $k\in\hat{c}_P(T)$ for $k\in K$;

iii) any two distinct realizations of $p_\infty(x)$ which do not have common lines force infinitely many realizations of $p_\infty(x)$ witnessing $k'\notin\hat{c}_P(T)$ for $k'\notin K$.

Thus we get $\hat{c}_P(T)=K$.

Collecting the arguments above we have the following

Theorem 6.14. (1) *If T is a P-combination with a type $p_\infty(x)$ isolating a complete 1-type then $\hat{c}_P(T)$ is either empty or contains k_0 such that $\hat{c}_P(T)\subseteq k_0\mathbb{Z}^+$.*

(2) *For any set $K\subseteq k_0\mathbb{Z}^+$, being empty or containing k_0, there is a P-combination T with a type $p_\infty(x)$ isolating a complete 1-type such that $\hat{c}_P(T)=K$.*

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