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GROUP STRUCTURES OF A FUNCTION SPACES WITH THE SET-OPEN TOPOLOGY

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Abstract. In this paper, we find at the properties of the family λ which imply that the space $C(X, \mathbb{R}^\alpha)$ — the set of all continuous mappings on a Tychonoff space X to the space $\mathbb{R}^\alpha$ with the λ-open topology is a semitopological group (paratopological group, topological group, topological vector space and other algebraic structures) under the usual operations of addition and multiplication (and multiplication by scalars). For example, if $X = [0, \omega_1)$ and λ is a family of C-compact subsets of X, then $C_\lambda(X, \mathbb{R}^\omega)$ is a semitopological group (locally convex topological vector space, topological algebra), but $C_\lambda(X, \mathbb{R}^{\omega_1})$ is not semitopological group.

Keywords: set-open topology, topological group, C-compact subset, semitopological group, paratopological group, topological vector space, C_α-compact subset, topological algebra.

1. Introduction

In articles [9, 10], we investigated the topological-algebraic properties of $C_\lambda(X)$ where $C_\lambda(X)$ the space of all real-valued continuous functions on X with set-open topology. Recall that a subset A of a space X is called *C-compact subset X* (or $\mathbb{R}$-compact) if, for any real-valued function f continuous on X, the set $f(A)$ is compact in $\mathbb{R}$. Given a family λ of non-empty subsets of X, let $\lambda(C)$ be the set of all elements $A \in \lambda$ such that for every C-compact subset B of the space X with $B \subset A$, the set $[B, U]$ is open in $C_\lambda(X)$ for any open set U of the space $\mathbb{R}$.

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Theorem 1 (Theorem 3.3. in [10]). *For a space X, the following statements are equivalent.*

(1) $C_\lambda(X) = C_{\lambda,u}(X)$.
(2) $C_\lambda(X)$ *is a topological group.*
(3) $C_\lambda(X)$ *is a topological vector space.*
(4) $C_\lambda(X)$ *is a locally convex topological vector space.*
(5) λ *is a family of C-compact sets and* $\lambda = \lambda(C)$.

In this paper, we find at the properties of the family λ which imply that the space $C_\lambda(X, \mathbb{R}^\alpha)$ — the set of all continuous mappings of the space X to the space $\mathbb{R}^\alpha$ with the λ-open topology is a semitopological group (paratopological group, topological group, topological vector space and other algebraic structures) under the usual operations of addition and multiplication (and multiplication by scalars).

2. Main definitions and notation

Let X be a Tychonoff space and α be a cardinal number. We shall denote by $C(X, \mathbb{R}^\alpha)$ the set of all continuous mappings of the space X to the space $\mathbb{R}^\alpha$. We say that a subset B of a space X is C_α-compact in X if for every continuous function $f : X \mapsto \mathbb{R}^\alpha$, $f(B)$ is a compact subset of $\mathbb{R}^\alpha$ ([5], [6]). This concept generalizes the notion of α-pseudocompactness introduced by J.F. Kennison ([8]): a space X is α-pseudocompact if X is C_α-compact in itself. If $\alpha \leq \omega_0$, then we say C-compact instead of C_α-compact and α-pseudocompactness agrees with pseudocompactness.

A family λ of C_α-compact subsets of X is said to be closed under (hereditary with respect to) C_α-compact subsets if it satisfies the following condition: whenever $A \in \lambda$ and B is a C_α-compact (in X) subset of A, then $B \in \lambda$ also.

For a Hausdorff space X, a family λ of subsets of X and a uniform space $(Y, \mathcal{U})$ we shall denote by $\mathcal{U}|\lambda$ the uniformity on $C(X, Y)$ generated by the base consisting of all finite intersections of the sets of the form

$\hat{V}|A = \{(f, g) : (f(x), g(x)) \in V$ for every $x \in A\}$, where $V \in \mathcal{U}$, $A \in \lambda$.

The uniformity $\mathcal{U}|\lambda$ will be called the uniformity of uniform convergence on family λ induced by $\mathcal{U}$.

Recall that all sets of the form

$[F, U] := \{f \in C(X, Y) : f(F) \subseteq U\}$, where $F \in \lambda$ and U is an open subset of Y, form a subbase of the set-open (λ-open) topology on $C(X, Y)$.

We use the following notations for various topological spaces on the set $C(X, \mathbb{R}^\alpha)$:

$C_{\lambda,u}(X, \mathbb{R}^\alpha)$ for the topology induced by the uniformity $\mathcal{U}|\lambda$,

$C_\lambda(X, \mathbb{R}^\alpha)$ for the λ-open topology.

Given a family λ of non-empty subsets of X, let $\lambda(C_\alpha)$ be the set of elements $A \in \lambda$ such that for every C_α-compact subset B of the space X with $B \subset A$, the set $[B, U]$ is open in $C_\lambda(X, \mathbb{R}^\alpha)$ for any open set U of the space $\mathbb{R}^\alpha$.

Let λ_m denote the maximal with respect to inclusion family, provided that $C_{\lambda_m}(X, \mathbb{R}^\alpha) = C_\lambda(X, \mathbb{R}^\alpha)$. Note that a family λ_m is unique for each family λ.

A subset A of X is said to be a Y*-zero-set* provided $A = f^{-1}(Z)$ for some zero-set Z of Y and $f \in C(X, Y)$. For example, if Y is the real numbers space $\mathbb{R}$ then any zero-set subset of X is a $\mathbb{R}$-zero-set of X.

Definition 1 (for π-network see in [11]). *A family λ of subsets of X, hereditary with respect to the $\mathbb{R}^\alpha$-zero-subsets of X will be called a saturation family.*

Definition 2 (Definition 2.5 in [7]). *A family λ of subsets of X is called a functional refinement if for every $A \in \lambda$, every finite sequence $U_1, ..., U_n$ of open subsets of Y, and every $f \in C(X, Y)$ such that $A \subseteq \bigcup\limits_{i=1}^{n} f^{-1}(U_i)$, there exists a finite sequence $A_1, ..., A_m$ of members of λ which refines $f^{-1}(U_1), ..., f^{-1}(U_n)$ and whose union contains A.*

• A group G with a topology τ is a semitopological (paratopological, respectively) group if the multiplication is separately continuous (jointly continuous, respectively).

• If G is a semitopological and the inverse operation $x \to x^{-1}$ is continuous, then G is said to be a quasitopological group.

Clearly, that if G is a paratopological group and the inverse operation $x \to x^{-1}$ is continuous, then G is a topological group.

• A topological algebra over a topological field $\mathbb{K}$ is a topological vector space together with a continuous bilinear multiplication.

In [3, Ch. 3], N.Bourbaki noticed that if Y is a topological ring then $C(X, Y)$ is a ring under the usual operations of pointwise addition and pointwise multiplication.

3. Main results

Theorem 2. *For a Tychonoff space X and a cardinal number α, the following statements are equivalent.*

(1) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a semitopological group.*
(2) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a paratopological group.*
(3) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a quasitopological group.*
(4) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a topological group.*
(5) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a topological vector space.*
(6) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a locally convex topological vector space.*
(7) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a topological ring.*
(8) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a topological algebra.*
(9) *λ is a family of C_α-compact sets and $\lambda = \lambda(C_\alpha)$.*
(10) *λ_m is a saturation family of C_α-compact subsets of X.*
(11) *$C_\lambda(X, \mathbb{R}^\alpha) = C_{\lambda,u}(X, \mathbb{R}^\alpha)$.*
(12) *λ_m is a functional refinement family of C_α-compact subsets of X.*

Proof. The following implications are the scheme of the proof of the theorem

$$\begin{array}{ccccccccccc}
 & & & & (8) & \Rightarrow (7) \Rightarrow & (4) & \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \\
 & & & & \Uparrow & & \Uparrow & \\
(1) & \Rightarrow & (9) & \Rightarrow & (11) & \Rightarrow (6) \Rightarrow & (5) & \\
 & & \Downarrow & & \Uparrow & & & \\
 & & (10) & \Rightarrow & (12) & & &
\end{array}$$

$(1) \Rightarrow (9)$. Let $C_\lambda(X, \mathbb{R}^\alpha)$ be a semitopological group. Suppose that there exists $A \in \lambda$ such that A is not a C_α-compact subset of X. Then there is $f \in C_\lambda(X, \mathbb{R}^\alpha)$ such that $f(A)$ is not a compact subset of $\mathbb{R}^\alpha$. By Theorem 1.2 in [5], we can assume that $f(A)$ is not a closed subset of $\mathbb{R}^\alpha$. Let us consider a point $a \in \overline{f(A)} \setminus f(A)$ and the subbasic open set $O(f) := [A, \mathbb{R}^\alpha \setminus \{a\}]$ which contains the point f. Since $C_\lambda(X, \mathbb{R}^\alpha)$ is a semitopological group, there is a basis neighborhood $[B, W]$ of a point $h \equiv \mathbf{0}$ such that $f+[B,W] \subseteq O(f)$. Choose $x_0 \in A$ such that $f(x_0) \in (a+W)$. Let $g \equiv a - f(x_0)$ be a constant function. It is clear that $g \in C_\lambda(X, \mathbb{R}^\alpha)$ and $g \in [B,W]$, but $(f+g) \notin O(f)$, because $f(x_0)+g(x_0) = a \notin \mathbb{R}^\alpha \setminus \{a\}$. This contradicts our assumption that $f+[B,W] \subseteq O(f)$. It follows that A is a C_α-compact subset of X.

Suppose that $A \in \lambda$, $B \subset A$ and B is a C_α-compact subset of X. We claim that $[B,U]$ is an open set in the space $C_\lambda(X,\mathbb{R}^\alpha)$ for each open set U in $\mathbb{R}^\alpha$. Let $f \in [B,U]$ and $\mathcal{U}$ a uniformity on the space $\mathbb{R}^\alpha$. By Lemma 8.2.5 in [4], there exists a $V \in \mathcal{U}$ such that $B(f(B),V) \subset U$ where $B(f(B),V) := \bigcup\limits_{z \in f(B)} \{y \in \mathbb{R}^\alpha, (z,y) \in V\}$. The set $W = f + [A, IntB(0,V)]$ is the open set in $C_\lambda(X,\mathbb{R}^\alpha)$. It remains to prove that $W \subset [B,U]$. For $g \in W$ and $x \in B$ we have $g(x) = f(x) + h(x)$ where $h \in [A, IntB(0,V)]$. It follows that $g(x) \in U$ and $W \subset [B,U]$.

Recall that a space Y be called cub-space if for any $x \in Y \times Y$ there are a continuous map f from $Y \times Y$ to Y and a point $y \in Y$ such that $f^{-1}(y) = x$ (see [11]). Since $\mathbb{R}^\alpha \times \mathbb{R}^\alpha$ is homeomorphic to $\mathbb{R}^\alpha$ for $\alpha \geq \aleph_0$ we have that $\mathbb{R}^\alpha$ is cub-space for $\alpha \geq \omega_0$.

One can see that if Y is a Tychonoff space with countable pseudocharacter contains a nontrivial path then Y is a cub- space. Really, if Y is a Tychonoff space with countable pseudocharacter then each point in $Y \times Y$ is a zero-set. For a point $(x,y) \in Y \times Y$ there is a continuous function $f : Y \times Y \mapsto \mathbb{I} = [0,1]$ such that $f^{-1}(0) = (x,y)$. Since Y contains a nontrivial path, $\mathbb{I}$ is homeomorphic to a subspace of Y. It follows that $\mathbb{R}^\alpha$ is cub-space for $\alpha \in \omega_0$.

(9) $\Rightarrow$ (11). Let λ is a family of C_α-compact sets and $\lambda = \lambda(C_\alpha)$. Without loss of generality we can assume that if B is C_α-compact sets and $B \subset A$ for some $A \in \lambda$ then $B \in \lambda$.

By Propositions 2.2 and 2.3 in [11], the family λ is hereditary with respect to the $\mathbb{R}^\alpha$-zero-subsets of X (i.e. any nonempty $A \bigcap B \in \lambda$ where $A \in \lambda$ and B is a $\mathbb{R}^\alpha$-zero-set of X). By Theorem 3.3 in [11], the topology on $C(X,\mathbb{R}^\alpha)$ induced by the uniformity $\mathcal{U}|\lambda$ of uniform convergence on the family λ coincides with the λ-open topology on $C(X,\mathbb{R}^\alpha)$. It follows that $C_\lambda(X,\mathbb{R}^\alpha) = C_{\lambda,u}(X,\mathbb{R}^\alpha)$.

(11) $\Rightarrow$ (6). Now for each $A \in \lambda$ and $\beta \in \alpha$, define the pseudo-seminorm $p_{A,\beta}$ on $C(X,\mathbb{R}^\alpha)$ by $p_{A,\beta}(f) = \min\{1, sup\{|pr_\beta \circ f(x)| : x \in A\}\}$ where $pr_\beta : \mathbb{R}^\alpha = \prod\limits_{\gamma<\alpha} \mathbb{R}_\gamma \mapsto \mathbb{R}_\beta$ is the β-th projection from $\mathbb{R}^\alpha$ onto $\mathbb{R}_\beta$.

Also for each $A \in \lambda$ and $\epsilon > 0$, let $V_{A,\beta,\epsilon} = \{f \in C(X,\mathbb{R}^\alpha) : p_{A,\beta}(f) < \epsilon\}$.

Let $\mathcal{V} = \{V_{A,\beta_1,\epsilon} \bigcap ... \bigcap V_{A,\beta_s,\epsilon} : A \in \lambda, s \in \omega, \epsilon > 0\}$.

It can be easily shown that for each $f \in C(X,\mathbb{R}^\alpha)$, $f+\mathcal{V} := \{f+V : V \in \mathcal{V}\}$ form a neighborhood base at f. We say that this topology is generated by the collection of pseudo-seminorms $\{p_{A,\beta} : A \in \lambda, \beta \in \alpha\}$. Note that if we choose $\epsilon \in (0,1)$, then for each $f \in C(X,\mathbb{R}^\alpha)$ and $A \in \lambda$, we have $f + V \subseteq < f, A, \epsilon >$ for some $V \in \mathcal{V}$ and $< f, A, \frac{\epsilon}{2} > \subseteq f + V$. This shows that the topology of uniform convergence on λ is the same as the topology generated by the collection of pseudo-seminorms $\{p_{A,\beta} : A \in \lambda, \beta \in \alpha\}$. We see from this point of view that $C_{\lambda,u}(X,\mathbb{R}^\alpha)$ is a topological group with respect to addition. By $C_\lambda(X,\mathbb{R}^\alpha) = C_{\lambda,u}(X,\mathbb{R}^\alpha)$ and the fact that $C_\lambda(X,\mathbb{R}^\alpha)$ is a topological (semitopological) group, we have that λ is a family of C_α-compact subsets of X (see the implication (1) $\Rightarrow$ (9)). It follows that the topology is generated by the collection of seminorms $\{p_{A,\beta} : A \in \lambda, \beta \in \alpha\}$. Consequently, $C_\lambda(X,\mathbb{R}^\alpha)$ is a locally convex topological vector space.

(6) $\Rightarrow$ (5). It is immediate.

(11) $\Rightarrow$ (8). By the implication (11) $\Rightarrow$ (6), $C_\lambda(X,\mathbb{R}^\alpha)$ is a locally convex topological vector space. It remains to prove the continuity of the operation of multiplication. Let $W=[A,V]$ be a base neighborhood of the point $\mathbf{0}$ where $A\in\lambda$ and V is an open set of $\mathbb{R}^\alpha$. Since $\mathbb{R}^\alpha$ is a topological algebra, there is an open set V^1 of $\mathbb{R}^\alpha$ such that $V^1 * V^1\subseteq V$. Let $W^1=[A,V^1]$. Then $W^1*W^1\subseteq W$. Really, $W^1*W^1=\{f*g: f\in W^1, g\in W^1\}=\{f*g: f(A)\subseteq V^1 \text{ and } g(A)\subseteq V^1\}$. Clearly, that $f(x)*g(x)\in V^1*V^1$ for each $x\in A$. It follows that $(f*g)(A)\subseteq V$ and $W^1*W^1\subseteq W$. We prove that if $W=[A,V]$ be a base neighborhood of the point $\mathbf{0}$ and $f\in C(X,\mathbb{R}^\alpha)$ then there is an open set $V^1\ni 0$ such that $f(A)*V^1\subseteq V$ and $V^1*f(A)\subseteq V$. Indeed, $g=f*h$ and $g^1=h^1*f$ where $h,h^1\in W^1$. Then $g(x)=f(x)*h(x)\in f(A)*V^1$ and $g^1(x)=h^1(x)*f(x)\in V^1*f(A)$ for each $x\in A$. Note that $g(A)\subseteq V$ and $g^1(A)\subseteq V$.

By definitions of algebraic structures we have the next implications:

(8) $\Rightarrow$ (7) $\Rightarrow$ (4) and (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(9) $\Rightarrow$ (10). Let λ is a family of C_α-compact sets and $\lambda=\lambda(C_\alpha)$. Without loss of generality we can assume that if B is C_α-compact sets and $B\subset A$ for some $A\in\lambda$ then $B\in\lambda$.

Note that for any finite collection $A_1,...,A_k\in\lambda$ and an open set W of $\mathbb{R}^\alpha$ the set $[A_1\bigcup...\bigcup A_k, W]$ is an open set of $C_\lambda(X,\mathbb{R}^\alpha)$. Hence, without loss of generality we can assume that λ is closed under finite unions.

Let $\mu:=\{B: B \text{ is a } C_\alpha\text{-compact subset of } X \text{ and } B\subseteq\overline{A} \text{ for } A\in\lambda\}$. We prove that $\lambda_m=\mu$. Note that if $A\in\lambda$, then $\overline{A}\in\lambda_m$. Really, $[A,W]=[\overline{A},W]$ for any open set W of $\mathbb{R}^\alpha$ and C_α-compact subset A of X.

1. $\mu\subseteq\lambda_m$. Let B be a C_α-compact subset of X and $B\subseteq\overline{A}$ for some $A\in\lambda$. Consider a set $[B,V]$ for an open set V of $\mathbb{R}^\alpha$. Let $f\in[B,V]$. Since $f(B)$ is a compact set there is a zero-set Z of $\mathbb{R}^\alpha$ such that $f(B)\subseteq Z\subseteq V$. Consider a zero-set $f^{-1}(Z)\bigcap\overline{A}$. Since A a C_α-compact subsets of X, $D=f^{-1}(Z)\bigcap A\neq\emptyset$, and, by condition (9), $[D,V]$ is an open set in $C_\lambda(X,\mathbb{R}^\alpha)$. It follows that $[\overline{D},V]$ is an open set in $C_\lambda(X,\mathbb{R}^\alpha)$, too. Note that $f\in[\overline{D},V]\subseteq[B,V]$. Hence $[B,V]$ is an open set in $C_\lambda(X,\mathbb{R}^\alpha)$ for any open set V in $\mathbb{R}^\alpha$ and $B\in\lambda_m$.

2. $\lambda_m\subseteq\mu$. Suppose that $B\in\lambda_m$ and $B\notin\mu$. Let W be an open set in $\mathbb{R}^\alpha$ such that $W\neq\mathbb{R}^\alpha$. Then $[B,W]$ is an open set in $C_\lambda(X,\mathbb{R}^\alpha)$ and, hence, for $f\in[B,W]$ there is a base neighborhood $\bigcap\limits_{i=1}^{k}[A_i,W_i]$ of the point f where $A_i=\overline{A_i}$ and $A_i\in\mu$ for $i=\overline{1,k}$ such that $\bigcap\limits_{i=1}^{k}[A_i,W_i]\subseteq[B,W]$. Since $B\notin\mu$, $B\setminus\bigcup\limits_{i=1}^{k}A_i\neq\emptyset$. Let $h\in C(X,\mathbb{R}^\alpha)$ such that $h\restriction\bigcup\limits_{i=1}^{k}A_i=f$ and $h(z)\notin W$ for some $z\in B\setminus\bigcup\limits_{i=1}^{k}A_i$. Then $h\in\bigcap\limits_{i=1}^{k}[A_i,W_i]$, but $h\notin[B,W]$, a contradiction.

(10) $\Rightarrow$ (12). Let $A\in\lambda_m$, $U_1,...,U_n$ be a finite sequence of open subsets of $\mathbb{R}^\alpha$ and $f\in C(X,\mathbb{R}^\alpha)$ such that $A\subseteq\bigcup\limits_{i=1}^{n}f^{-1}(U_i)$. For $y\in f(A)\bigcap U_i$ we choose a zero set $F(y)$ such that $y\in IntF(y)\subseteq F(y)\subseteq U_i$. Then $\{IntF(y): y\in f(A)\bigcap U_i$ and $i=1,...,n\}$ is an open cover of the compact set $f(A)$. There exists a finite sequence $F(y_1),...,F(y_m)$ such that $f(A)\subset\bigcup_{j=1}^{m}IntF(y_j)$. Then $A_j=f^{-1}(F(y_j))$ is zero-set of X for $j=1,...m$. Since λ_m is a saturation family, $B_j=A\bigcap A_j\in\lambda_m$.

Since $A \subset \bigcup\limits_{j=1}^{m} A_j$, then $A = \bigcup\limits_{j=1}^{m} B_j$, and the finite sequence $B_1, ..., B_m$ refines $f^{-1}(U_1), ..., f^{-1}(U_n)$.

(12) $\Rightarrow$ (11). By Theorem 2 in [2], replacing an admissible family by a functional refinement family. $\square$

Note that a space $C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff if and only if λ is a π-network of X.

Corollary 1. *For a Tychonoff space X and a cardinal number α, the following statements are equivalent.*

(1) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff semitopological group.*
(2) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff paratopological group.*
(3) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff quasitopological group.*
(4) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff topological group.*
(5) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff topological vector space.*
(6) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff locally convex topological vector space.*
(7) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff topological ring.*
(8) *$C_\lambda(X, \mathbb{R}^\alpha)$ is a Hausdorff topological algebra.*
(9) *λ is a π-network of X consisting of C_α-compact sets and $\lambda = \lambda(C_\alpha)$.*
(10) *λ_m is a π-network of X and is a saturation family of C_α-compact sets.*
(11) *λ_m is a π-network of X and is a functional refinement family of C_α-compact sets of X.*

4. Example

The following results were obtained in [5].

Theorem 3 (Theorem 2.6 in [5]). *Let α be a cardinal with $cf(\alpha) > \omega$. Then $[0, \alpha)$ is γ-pseudocompact for all $\omega \leq \gamma < cf(\alpha)$ and it is not $cf(\alpha)$-pseudocompact.*

Corollary 2 (Corollary 2.8 in [5]). *If γ and α are cardinals with $\gamma < \alpha$, then $[0, \gamma^+)$ is γ-pseudocompact and is not α-pseudocompact.*

By combining Theorem 3 and Theorem 2, we obtain the following.

Theorem 4. *Let α be a cardinal with $cf(\alpha) > \omega$, $\omega \leq \gamma < cf(\alpha)$ and λ be a family of C_γ-compact subsets of $X = [0, \alpha)$. Then $C_\lambda(X, \mathbb{R}^\gamma)$ is a semitopological group (locally convex topological vector space, topological algebra), but $C_\lambda(X, \mathbb{R}^{cf(\alpha)})$ is not semitopological group.*

Proof. By Theorem 3, X is γ-pseudocompact and is not $cf(\alpha)$-pseudocompact. By Theorem 2, $C_\lambda(X, \mathbb{R}^\gamma)$ is a semitopological group (locally convex topological vector space, topological algebra), but $C_\lambda(X, \mathbb{R}^{cf(\alpha)})$ is not semitopological group. $\square$

For example, if $X = [0, \omega_1)$ and λ is a family of C-compact subsets of X, then $C_\lambda(X, \mathbb{R}^\omega)$ is a semitopological group (locally convex topological vector space, topological algebra), but $C_\lambda(X, \mathbb{R}^{\omega_1})$ is not semitopological group.

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