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ALTERNATIVE AND JORDAN ALGEBRAS ADMITTING TERNARY DERIVATIONS WITH INVERTIBLE VALUES

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Abstract. In this paper we prove analogues of H. Komatsu and A. Nakajima theorems (see [1]) for alternative and Jordan algebras. In particular, we give a description of alternative and Jordan algebras which have ternary derivations with invertible values.

Keywords: Alternative algebras, Jordan algebras, Cayley—Dickson algebra, Albert algebra, derivation, generalized derivation, ternary derivation.

1. Introduction

In [2] J. Bergen, I.N. Herstein, and C. Lanski gave a description of associative rings admitting derivations with invertible values. Various generalizations of this result in the class of associative rings can be found in [3, 4, 5, 6]. H. Komatsu and A. Nakajima [1] extended this result for generalized derivations of associative rings.

The notion of a generalized derivation of associative rings was introduced by M. Bresar in [7]. Namely, an additive mapping $D : A \mapsto A$ of an associative ring A containing 1 is referred to as a generalized derivation if the following equality holds:

$$D(xy) = D(x)y + xD(y) - xD(1)y \text{ for all } x, y \in A.$$

It is well known (see [8]) that for Cayley—Dickson algebras we have

Principle of triality. *Let $\mathcal{C}$ be a Cayley—Dickson algebra of characteristic other than $2, 3$ with the norm $n(x)$, and let $o(8, n)$ be the orthogonal Lie algebra of linear*

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transformations of the algebra $\mathcal{C}$ *which are skew relative to* $n(x)$*. Then for all* $W \in o(8,n)$ *there exist unique* $W', W'' \in o(8,n)$ *such that the following holds:*

$$W(xy) = W'(x)y + xW''(y) \text{ for all } x, y \in \mathcal{C}.$$

A. Elduque [9] proved the principle of triality for a Cayley—Dickson algebra of characteristic 3.

The principle of triality gives rise the notion of ternary derivation, which was introduced by C. Jimenez-Gestal and J.M. Perez-Izquierdo in [10]. Ternary derivations are an important tool for the study of non-associative algebras [11, 12, 13].

Let A be an arbitrary algebra over a field F. Then the triplet (D, D_1, D_2) of linear transformations of the vector space A is called a ternary derivation if the following equality holds:

$$D(xy) = D_1(x)y + xD_2(y).$$

Clearly, for a generalized derivation D of an associative algebra, the triplet $(D, D - R_{D(1)}, D)$ is a ternary derivation, where R_a stands for the operator of right multiplication by $a \in A$. Vice versa, if A is a unital associative algebra and (D, D_1, D_2) is a ternary derivation of A then D is a generalized derivation. Indeed, for every $x \in A$ we have

$$D(x) = D(x \cdot 1) = D_1(x) \cdot 1 + xD_2(1) = D_1(x) + xD_2(1).$$

Similarly, the equality $D(x) = D_1(1)x + D_2(x)$ holds. Therefore, we obtain

$$D(xy) = D_1(x)y + xD_2(y) = (D(x) - xD_2(1))y + x(D(y) - D_1(1)y) =$$

$$D(x)y + xD(y) - xD(1)y \text{ for all } x, y \in A.$$

Denote the associator of elements $x, y, z \in A$ by $(x, y, z) = (xy)z - x(yz)$, and the commutator of elements $x, y \in A$ by $[x, y] = xy - yx$.

Recall that for an arbitrary algebra A its associative center $N(A)$, commutative center $K(A)$, and center $Z(A)$ are defined as follows:

$$N(A) = \{n \in A \mid (n, x, y) = (x, y, n) = (x, n, y) = 0 \text{ for all } x, y \in A\},$$

$$K(A) = \{k \in A \mid [k, x] = 0 \text{ for all } x \in A\},$$

$$Z(A) = N(A) \cap K(A).$$

Hereinafter we assume that considered algebras are containing the unit element 1.

Lemma 1. *Let* A *be an algebra over a field* F *and* (D, D_1, D_2) *be a ternary derivation of the algebra* A*. Then for all* $x \in A$ *we have*

$$D(x) = D_1(x) + xD_2(1), D(x) = D_1(1)x + D_2(x).$$

If the field F *is of characteristic other than 2 then for all* $x, y \in A$ *the following holds:*

$$D(xy) = D(x)y + xD(y) - \frac{1}{2}((xD(1))y + x(D(1)y)) + \left(x, \frac{D_1(1) - D_2(1)}{2}, y\right).$$

Proof. By definition of ternary derivation, we have

$$D(x) = D(x \cdot 1) = D_1(x) + xD_2(1).$$

Similarly,

$$D(y) = D_1(1)y + D_2(y).$$

Therefore,

$$\begin{gathered} D(xy) = D_1(x)y + xD_2(y) = D(x)y - (xD_2(1))y + xD(y) - x(D_1(1)y) = \\ D(x)y - (x, D_2(1), y) - x(D_2(1)y) + xD(y) - x(D_1(1)y) = \\ D(x)y - (x, D_2(1), y) + xD(y) - x((D_1(1) + D_2(1))y) = \\ D(x)y - (x, D_2(1), y) + xD(y) - x(D(1)y). \end{gathered}$$

Similarly,

$$D(xy) = D(x)y - (xD(1))y + xD(y) + (x, D_1(1), y).$$

Hence we obtain that

$$2D(xy) = 2D(x)y - (xD(1))y + 2xD(y) - x(D(1)y) + (x, D_1(1) - D_2(1), y).$$

If the field F is of characteristic other than 2 then

$$D(xy) = D(x)y + xD(y) - \frac{1}{2}((xD(1))y + x(D(1)y)) + \left(x, \frac{D_1(1) - D_2(1)}{2}, y\right).$$

□

Lemma 2. *Let A be an algebra over a field F of characteristic $\neq 2$, (D, D_1, D_2) be a ternary derivation of A, and I be an ideal of A. Then $2D(x)^2 \in D^2(x^2) + I$ for all $x \in I$.*

Proof. Note that

$$D(I^2) \subseteq D_1(I)I + ID_2(I) \subseteq I.$$

According to Lemma 1, for $x \in I$ we obtain

$$D(x^2) \in D(x)x + xD(x) + I^2.$$

Let $x \in I$. By Lemma 1, we have

$$\begin{aligned} D(D(x)x) = {} & D^2(x)x + D(x)D(x) \\ & - \frac{1}{2}((D(x)D(1))x + D(x)(D(1)x)) + \left(D(x), \frac{D_1(1) - D_2(1)}{2}, x\right), \end{aligned}$$

$$\begin{aligned} D(xD(x)) = {} & D(x)D(x) + xD^2(x) \\ & - \frac{1}{2}((xD(1))D(x) + x(D(1)D(x))) + \left(x, \frac{D_1(1) - D_2(1)}{2}, D(x)\right). \end{aligned}$$

By adding these equations we obtain

$$D^2(x^2) = D^2(x)x + 2D(x)D(x) + xD^2(x) + r,$$

where $r \in I$. Therefore, $2D(x)D(x) \in D^2(x^2) + I$. □

Assume that $U \subseteq A \setminus \{0\}$ is the set of all such elements $a \in A$ that $ab = ca = 1$ for some $b, c \in A$.

Ternary derivation (D, D_1, D_2) of the algebra A is referred to as a ternary derivation with *invertible values* if $D(A) \subseteq U \cup \{0\}$ and $D(A) \neq 0$.

Lemma 3. *Let (D, D_1, D_2) be a ternary derivation of the algebra A and I be an ideal (right ideal) of A. Then $I + D(I)$ is an ideal (right ideal). If D is a ternary derivation with invertible values and $I \neq A$ then $I \cap D(I) = 0$.*

Proof. Let I be an ideal and $a \in A, r \in I$. By Lemma 1 we have

$$D(r)a = D_1(r)a + (rD_2(1))a = D(ra) - rD_2(a) + (rD_2(1))a \in I + D(I).$$

Therefore, $I + D(I)$ is a right ideal of the algebra A. Similarly, $I + D(I)$ is a left ideal of A.

Let $I \neq A$. By condition, $D(I) \subseteq U \cup \{0\}$. Since $U \cap I = \{0\}$, then $I \cap D(I) = \{0\}$. □

Lemma 4. *Let (D, D_1, D_2) be a ternary derivation with invertible values. Suppose $I \subseteq \ker D$ implies $I = 0$ for every ideal (right ideal) I of the algebra A. Then $I + D(I) = A$ for every non-zero ideal (right ideal) I. Moreover, the ideal (right ideal) I is an ideal of A that is both minimal and maximal.*

Proof. Let I be a non-zero ideal of the algebra A. By Lemma 3 $I + D(I)$ is an ideal of A. Since $D(I) \neq 0$, then $I + D(I)$ contains an invertible element. Therefore, $I + D(I) = A$.

Let I' be an ideal of the algebra A containing I. Then for every $r' \in I'$ we obtain $r' \in r + D(I)$, where $r \in I$. Hence we obtain that $r' - r \in D(I) \subseteq D(I')$. Thus, either $I' = I$ or $I' = A$. Therefore, I is a maximal ideal. Similarly, I is a minimal ideal. □

2. The case of alternative algebras

Algebra A is called alternative if it satisfies the identities

$$x^2y = x(xy),\ yx^2 = (yx)x.$$

For every alternative algebra (see [14]) the following identities hold:

$$[xy, z] = x[y, z] + [x, z]y + 3(x, y, z), \tag{1}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 6(x, y, z), \tag{2}$$

$$(xy, z, t) = x(y, z, t) + (x, z, t)y - (x, y, [z, t]). \tag{3}$$

Denote by L_a, R_a the operators of left and right multiplication by an element $a \in A$ and let $T_a = L_a + R_a$. Then for an alternative algebra A the following operator equations hold:

$$L_xL_y + L_yL_x = L_{xy+yx}, \tag{4}$$

$$R_xR_y + R_yR_x = R_{xy+yx}. \tag{5}$$

Hence we obtain (see [10]) that $(L_a, T_a, -L_a)$ and $(R_a, -R_a, T_a)$ are ternary derivations of A.

Let the triplet (D, D_1, D_2) be a ternary derivation of an alternative algebra A with 1. By Lemma 1 we have $D_1(x) = D(x) - xD_2(1)$. Then, since (4) and (5), the following holds:

$$D_1(xy) = D(xy) - (xy)D_2(1) = D_1(x)y + xD_2(y) + (xD_2(1))y - x(yD_2(1) + D_2(1)y) =$$

$$(D_1 + R_{D_2(1)})(x) \cdot y + x \cdot (D_2 - T_{D_2(1)})(y).$$

Therefore, the triplet $D_1, D_1+R_{D_2(1)}, D_2-T_{D_2(1)})$ is a ternary derivation. Similarly, the triplet $(D_2, D_1 - T_{D_1(1)}, D_2 + L_{D_1(1)})$ is a ternary derivation of the alternative algebra A.

If the characteristic of the field is not equal to 3 and $a_1 = D_1(1)$, $a_2 = D_2(1)$ then every component of a ternary derivation

$$(D, D_1, D_2) - \frac{1}{3}(L_{2a_1+a_2}, T_{2a_1+a_2}, -L_{2a_1+a_2}) - \frac{1}{3}(R_{2a_2+a_1}, -R_{2a_2+a_1}, T_{2a_2+a_1})$$

annihilates 1. Hence we obtain that

(6)

$$(D, D_1, D_2) - \frac{1}{3}(L_{2a_1+a_2}, T_{2a_1+a_2}, -L_{2a_1+a_2}) - \frac{1}{3}(R_{2a_2+a_1}, -R_{2a_2+a_1}, T_{2a_2+a_1}) = (d, d, d)$$

is a derivation (see [10]), i.e. $d(xy) = d(x)y + xd(y)$ for all $x, y \in A$.

Lemma 5. *Let A be an alternative algebra, (D, D_1, D_2) be a ternary derivation of A with invertible values, and I be a proper ideal of A. We assume that $\ker D$ does not contain ideals of A. Then $I^2 = 0$. Particularly, the quasi-regular radical $\mathcal{J}(A)$ of the algebra A is the largest ideal with zero multiplication. Moreover, $\mathcal{J}(A) = tD(\mathcal{J}(A))$, where $t \in \mathcal{J}(A)$, t belongs to the commutative center of the algebra A and $\mathcal{J}(A) \cap \ker D = 0$.*

Proof. Since $D(I^2) \subseteq D_1(I)I + ID_2(I) \subseteq I$, then $D(I^2) = 0$. For an alternative algebra, I^2 is again an ideal. Therefore, $I^2 = 0$. Hence, $\mathcal{J}(A)$ is the largest ideal and $\mathcal{J}(A)^2 = 0$.

By condition of the statement we have $\mathcal{J}(A) \not\subseteq \ker D$. Therefore, by Lemma 4 $A = \mathcal{J}(A) + D(\mathcal{J}(A))$. Then $1 = r + D(t)$, where $r, t \in \mathcal{J}(A)$. Let $y \in \mathcal{J}(A)$. Since $\mathcal{J}(A)^2 = 0$, then the following holds:

$$0 = D(ty) = D_1(t)y+tD_2(y) = (D(t)-tD_2(1))y+t(D(y)-D_1(1)y) = D(t)y+tD(y).$$

Hence, $D(t)y = -tD(y)$. Therefore,

$$y = (r + D(t))y = D(t)y = -tD(y).$$

Similarly, $y = y(r + D(t)) = yD(t) = -D(y)t$. Hence we obtain that $\mathcal{J}(A) = tD(\mathcal{J}(A))$ and $tD(y) = D(y)t$ for every $y \in \mathcal{J}(A)$. Therefore, t belongs to the commutative center of the algebra A. Since $y = D(t)y = -tD(y)$ for every $y \in \mathcal{J}(A)$, then $\mathcal{J}(A) \cap \ker D = 0$. □

Theorem 1. *Let A be a unital alternative algebra over a field F and (D, D_1, D_2) be a ternary derivation with invertible values. If $\ker D$ does not contain non-zero ideals of the algebra A then one of the following propositions is true:*

1. A is an associative algebra and D is a generalized derivation,

2. A is a Cayley—Dickson algebra,

3. The field F is of characteristic 2, $A = \mathcal{C} + t\mathcal{C}$, where $\mathcal{C}$ is a Cayley—Dickson division algebra, $t \in Z(A)$, $Z(A) = Z(\mathcal{C}) + tZ(\mathcal{C})$, and $t^2 = 0$. Besides, for all $a, b \in \mathcal{C}$ we have

$$D(a+tb) = b+c_2a+ac_1+t(\alpha b+(\alpha+c_2)(b+c_2a)+(b+ac_1)(\alpha+c_1)+(a,c_1,c_2)),$$

$$D_1(a + tb) = D(a + tb) - (a + tb)c_2 - ta(\alpha c_2 + c_2^2 + \beta + [c_1, c_2]),$$

$$D_2(a + tb) = D(a + tb) - c_1(a + tb) - t(\alpha c_1 + c_1^2 + \beta + [c_1, c_2])a,$$

where $\alpha, \beta \in Z(\mathcal{C})$, $c_1, c_2 \in \mathcal{C}$.

Proof. Assume that $\ker D$ does not contain non-zero ideals of the algebra A. By Lemma 5 $\mathcal{J}(A)^2 = 0$ and $\mathcal{J}(A)$ is the largest ideal. If $\mathcal{J}(A) = 0$ then A is simple and therefore it is either associative or a Cayley—Dickson algebra.

Assume that F is of characteristic $\neq 2$. Then, by Lemma 2, for $x \in \mathcal{J}(A)$ we obtain that $2D(x)D(x) \in \mathcal{J}(A)$. If $D(\mathcal{J}(A)) \neq \{0\}$ then $D(x)$ is invertible for some $x \in \mathcal{J}(A)$. Therefore, $\mathcal{J}(A) = 0$ and by the above-mentioned reasonings the algebra A is simple.

Assume that $\mathcal{J}(A) \neq 0$. Then field F is of characteristic 2. In that case due to (6) for $a_1 = D_1(1), a_2 = D_2(1)$ we obtain that

$$(d, d, d) = (D, D_1, D_2) - (L_{a_2}, T_{a_2}, -L_{a_2}) - (R_{a_1}, -R_{a_1}, T_{a_1})$$

is a derivation of the algebra A. Since A does not contain proper ideals which are invariant with respect to D, then A does not contain proper ideals invariant under the derivation d. Hence, A is differentiably simple with respect to the derivation d. Due to [15] either the algebra A is associative or $A/\mathcal{J}(A)$ is a Cayley—Dickson algebra.

Assume that $A/\mathcal{J}(A)$ is a Cayley—Dickson algebra. Let $\mathcal{C} = d^{-1}(\mathcal{J}(A))$. Then $\mathcal{C}$ is a subalgebra of A. Since $A = \mathcal{J}(A) \oplus D(\mathcal{J}(A))$, then by Lemma 5 we have $A = \mathcal{J}(A) \oplus d(\mathcal{J}(A))$. Hence we obtain that $A = \mathcal{J}(A) \oplus \mathcal{C}$. Therefore, $\mathcal{C}$ is a Cayley—Dickson algebra over the field $K = Z(\mathcal{C})$. By Lemma 5 $\mathcal{J}(A) = tD(\mathcal{J}(A))$, where $t \in \mathcal{J}(A)$ and t belongs to the commutative center of the algebra A. Then due to (1) we have $t \in Z(A)$. Since

$$tA = t(\mathcal{J}(A) \oplus D(\mathcal{J}(A))) = tD(\mathcal{J}(A)) = \mathcal{J}(A),$$

then

$$\mathcal{J}(A) = tA = t(\mathcal{J}(A) \oplus \mathcal{C}) = t\mathcal{C}.$$

Therefore, $A = \mathcal{C} \oplus t\mathcal{C}$, $\mathcal{J}(A) \cap Z(A) = tZ(A) = tK$, and $Z(A) = K + tK$.

Since $A = \mathcal{C} + \mathcal{J}(A)$, then for $x \in \mathcal{J}(A)$ we obtain $d(x) = \phi(x) + \psi(x)$, where $\phi(x) \in \mathcal{C}$, $\psi(x) \in \mathcal{J}(A)$. For all $a \in \mathcal{C}$ and $x \in \mathcal{J}(A)$ we have

$$d(ax) = d(a)x + ad(x) = ad(x), \quad d(xa) = d(x)a + xd(a) = d(x)a,$$

as $d(a) \in \mathcal{J}(A)$ and $\mathcal{J}(A)^2 = 0$. Hence we obtain that mappings

$$\phi : \mathcal{J}(A) \mapsto \mathcal{C}, \quad \psi : \mathcal{J}(A) \mapsto \mathcal{J}(A)$$

are $\mathcal{C}$-bimodule homomorphisms. Since $d(\mathcal{J}(A)) \nsubseteq \mathcal{J}(A)$, then $\phi \neq 0$. Clearly, $\ker \phi$ is an ideal of the algebra A. By Lemma 4 $\ker \phi = \{0\}$. The image $\phi(\mathcal{J}(A))$ is an ideal of $\mathcal{C}$. Therefore, $\phi(\mathcal{J}(A)) = \mathcal{C}$. Hence we obtain that ϕ is an isomorphism. Since $\mathcal{J}(A) = t\mathcal{C}$, then $\phi(ta) = 1$ for some $a \in \mathcal{C}$. Then for every $b \in \mathcal{C}$ we have

$$\phi(tab - tba) = \phi(tab) - \phi(bta) = \phi(ta)b - b\phi(ta) = 0.$$

Hence, $t(ab - ba) = 0$. Therefore, $a \in K$. Replacing element t with ta we can assume that $\phi(t) = 1$.

Let $D_1(1) = a_1 = c_1 + tc_1'$, $D_2(1) = a_2 = c_2 + tc_2'$, $\psi(t) = t\alpha$, where $c_1, c_2, c_1', c_2', \alpha \in \mathcal{C}$. Similarly to the above-mentioned we have $\alpha \in K$. Then for all $a, b \in \mathcal{C}$ the following holds:

$$D(a + tb) = d(a + tb) + a_2(a + tb) + (a + tb)a_1 =$$

$$d(a) + \phi(t)b + \psi(t)b + (c_2 + tc_2')(a + tb) + (a + tb)(c_1 + tc_1') =$$

$$b + c_2a + ac_1 + (d(a) + t(\alpha b + c_2b + c_2'a + bc_1 + ac_1')).$$

Note that the fourth term belongs to $\mathcal{J}(A)$. Assuming $b = -c_2a - ac_1$ we obtain that $D(a + tb) \in \mathcal{J}(A)$. Therefore, $D(a - t(c_2a + ac_1)) = 0$. Hence we obtain that

$$d(a) = t(\alpha c_2a + \alpha ac_1 + c_2^2a + c_2(ac_1) + c_2'a + (c_2a)c_1 + ac_1^2 + ac_1') =$$

$$t(\alpha c_2a + \alpha ac_1 + c_2^2a + c_2'a + ac_1^2 + ac_1' + (c_2, a, c_1)) =$$

$$t(\alpha c_2 + \alpha c_1 + c_2^2 + c_2' + c_1^2 + c_1')a + t([a, \alpha c_1 + c_1^2 + c_1'] + (c_2, a, c_1)).$$

Assuming $a = 1$, we obtain

$$t(\alpha c_2 + \alpha c_1 + c_2^2 + c_2' + c_1^2 + c_1') = 0.$$

Therefore, $\alpha c_2 + \alpha c_1 + c_2^2 + c_2' + c_1^2 + c_1' = 0$ and

$$d(a) = t([a, \alpha c_1 + c_1^2 + c_1'] + (a, c_2, c_1)).$$

Since d is a derivation, then the mapping

$$a \mapsto [a, \alpha c_1 + c_1^2 + c_1'] + (a, c_2, c_1)$$

is a derivation of the algebra $\mathcal{C}$. Due to (1) and (3) we have

$$(a, b, \alpha c_1 + c_1^2 + c_1' + [c_2, c_1]) = 0$$

for all $a, b \in \mathcal{C}$. Therefore, $\alpha c_1 + c_1^2 + c_1' + [c_2, c_1] \in K$. Similarly, $\alpha c_2 + c_2^2 + c_2' + [c_2, c_1] \in K$. Hence,

$$d(a) = t([a, [c_2, c_1]] + (a, c_2, c_1)).$$

Therefore, we obtain that for all $a, b \in \mathcal{C}$ the following stands:

$$D(a + tb) = d(a + tb) + a_2(a + tb) + (a + tb)a_1 =$$

$$b + c_2a + ac_1 + t([a, [c_2, c_1]] + (a, c_2, c_1) + \alpha b + c_2b + c_2'a + bc_1 + ac_1') =$$

$$b + c_2a + ac_1 + t(\alpha b + c_2b + \alpha c_2a + c_2^2a + bc_1 + a\alpha c_1 + ac_1^2 + (a, c_2, c_1)) =$$

$$b + c_2a + ac_1 + t(\alpha b + (\alpha + c_2)(b + c_2a) + (b + ac_1)(\alpha + c_1) + (a, c_1, c_2)).$$

By Lemma 1

$$D_1(a+tb) = D(a+tb) - (a+tb)a_2 = D(a+tb) - (a+tb)c_2 - ta(\alpha c_2 + c_2^2 + \beta + [c_1, c_2]),$$

where $\beta = \alpha c_2 + c_2^2 + c_2' + [c_1, c_2]$. Similarly,

$$D_2(a + tb) = D(a + tb) - c_1(a + tb) - t(\alpha c_1 + c_1^2 + \beta + [c_1, c_2])a.$$

□

Now let D be a derivation of an alternative algebra A with invertible values, and let $D(x) = 0$. Then for $a \in A$ we have

$$D(ax) = D(a)x + aD(x) = D(a)x.$$

Therefore, either $x = 0$ or x is invertible in A (also see [2]). Hence, $\ker D$ does not contain non-zero ideals of the algebra A. The case of derivations was studied in [16].

Now let the field F be of characteristic other than 2. An algebra J is referred to as a Jordan algebra if the following identities are true:

$$xy = yx,\ (x^2y)x = x^2(yx).$$

For a Jordan algebra (see [14]) the following equation holds:

$$(xy, z, t) + (xt, z, y) + (ty, z, x) = 0. \qquad (7)$$

For every ternary derivation (D, D_1, D_2) of the algebra J the following identity is true (see [17])

$$D(xy) = D(x)y + xD(y) - \frac{1}{2}((xD(1))y + x(D(1)y)). \tag{8}$$

Hence, if $D(1) = 0$ then D is a derivation.

An element a of a Jordan algebra J is called invertible if $ab = 1$ and $a^2 b = a$ for some $b \in J$. Let $U(J)$ be a set of invertible elements of the algebra J.

Lemma 6. *Let J be a Jordan algebra and (D, D_1, D_2) be a ternary derivation of J with invertible values. We assume that I is an ideal in J and $I \neq J$. Then $I \subseteq \ker D$. Furthermore, if $D(1) = 0$ then $I^2 = 0$.*

Proof. Since $D(I^2) \subseteq D_1(I)I + ID_2(I) \subseteq I$ and $D(I^2) \subseteq U \cup \{0\}$, then $I^2 \subseteq \ker D$. By Lemma 2 for every $x \in I$ we have

$$2D(x)^2 \in D^2(x^2) + I \subseteq I.$$

Since $D(I) \subseteq U(J) \cup \{0\}$, then $I \subseteq \ker D$.

Let $D(1) = 0$. Then due to (8) D is a derivation. Therefore, for all $a \in J, x \in I$ we obtain

$$0 = D(ax) = D(a)x + aD(x) = D(a)x.$$

Since $bD(a) = 1$ for some $a \in J$ and $b \in J$, then for all $x, y \in I$ according to the identity (7) we obtain

$$xy = (xy)(bD(a)) = -(xy, b, D(a)) = (xD(a), b, y) + (D(a)y, b, x) = 0.$$

Therefore, $I^2 = 0$. □

Let A be an alternative algebra. Define a new multiplication $\circ$ on the vector space A by the following formula:

$$a \circ b = \frac{1}{2}(ab + ba).$$

Then we obtain a Jordan algebra which we denote as $A^{(+)}$.

Theorem 2. *Let A be a unital alternative algebra over a field F of characteristic other than 2. Let (D, D_1, D_2) be a ternary derivation of A with invertible values. If $\ker D$ contains non-zero ideals of the algebra A then the quasi-regular radical $\mathcal{J}(A)$ of the algebra A is the largest ideal, $\mathcal{J}(A) \subseteq \ker D$, $\mathcal{J}(A)$ is a nil-ideal of nil-index 2, and $\mathcal{J}(A)^4 = 0$. Furthermore, one of the following propositions is true:*

1. *The quotient algebra $A/\mathcal{J}(A)$ is an associative division algebra;*
2. *The quotient algebra $A/\mathcal{J}(A)$ is an algebra of all 2×2 matrices with elements in an associative division algebra;*
3. *The quotient algebra $A/\mathcal{J}(A)$ is a Cayley—Dickson algebra.*

Proof. First, we prove that $\mathcal{J}(A)$ is the largest ideal, $\mathcal{J}(A) \subseteq \ker D$, and $\mathcal{J}(A)^4 = 0$. Let I be an ideal of the algebra A, different from A. Then

$$D(I^2) \subseteq D_1(I)I + ID_2(I) \subseteq I.$$

Hence we have $I^2 \subseteq \ker D$. Let $x \in I$. By Lemma 2 $2D(x)^2 \in D^2(x^2) + I \subseteq I$. Therefore, $D(x) = 0$, otherwise, $D(x)$ is invertible and $I = A$. Hence, $I \subseteq \ker D$.

Let us show that $I^4 = 0$. For every $x \in I$ by Lemma 1 we have

$$0 = D(x^2) = D(x)x + xD(x) - xD(1)x = -xD(1)x.$$

If $D(1) \neq 0$ then it is known that

$$(D(1)^{-1}x)D(1)(D(1)^{-1}x) = D(1)^{-1}x^2.$$

Since $(D(1)^{-1}x)D(1)(D(1)^{-1}x) = 0$, then $D(1)^{-1}x^2 = 0$. Hence, $x^2 = 0$ for every $x \in I$.

If $D(1) = 0$ then for every $a \in A$ we obtain

$$D(a^2) = D(a)a + aD(a).$$

Hence we have that D is a derivation of the Jordan algebra $A^{(+)}$. Clearly I is an ideal of $A^{(+)}$. Then by Lemma 6 $x^2 = 0$ for every $x \in I$. Hence, $I^4 = 0$ (see [14]).

Thus, $\mathcal{J}(A)$ is the largest ideal, $\mathcal{J}(A) \subseteq \ker D$, $x^2 = 0$ for $x \in \mathcal{J}(A)$, and $\mathcal{J}(A)^4 = 0$.

Therefore, we obtain that the quotient algebra $\overline{A} = A/\mathcal{J}(A)$ is simple. Thus, either $\overline{A}$ is an associative algebra or $\overline{A}$ is a Cayley—Dickson algebra.

Let $\overline{A}$ be an associative algebra. Clearly, D induces a generalized derivation with invertible values on $\overline{A}$. Again, denote it by D. If $\ker D$ contains no non-zero right ideals of the algebra $\overline{A}$ then by Theorem 2.1 of [1] for the algebra $\overline{A}$ the propositions 1 and 2 are true.

Let I be a right ideal of $\overline{A}$ contained in $\ker D$. Then for $r \in I$, $a \in \overline{A}$

$$0 = D(ra) = D(r)a + rD(a) - rD(1)a = r(D(a) - D(1)a).$$

Hence, $D(a) - D(1)a$ belongs to the right annihilator of the right ideal I. Since the algebra $\overline{A}$ is simple, then $I = 0$ or $D(a) - D(1)a = 0$ for all $a \in \overline{A}$.

Assume that $D(a) - D(1)a = 0$ for all $a \in \overline{A}$. Then $D(1) \neq 0$, and for every non-zero $a \in \overline{A}$ we obtain that element a is invertible in $\overline{A}$. Therefore, $\overline{A}$ is an associative division algebra. Therefore, we obtain that $I = 0$. Let $a \in \ker D$. Hence, $\ker D$ contains no right ideals of algebra $\overline{A}$. □

Now, in the conditions of Theorem 2 and assuming that $A/\mathcal{J}(A)$ is a Cayley—Dickson algebra, let us show that $\mathcal{J}(A)^2 = 0$.

Lemma 7. *Let A be an alternative algebra over a field F of characteristic other than 2, 3, B be a subalgebra, and V be a vector subspace in A, containing $1 \in A$. Assume $z^4 \in B$ for every $z \in V$. Then $V \subseteq B$.*

Proof. Let $z \in V$. Then

$$z + z^3 = \frac{1}{8}((1+z)^4 - (1-z)^4) \in B$$

for every $z \in V$. Hence, we obtain that $-6z = 2z + (2z)^3 - 8(z + z^3) \in B$. Therefore, $z \in B$. □

Lemma 8. *A be an alternative algebra over a field of characteristic $\neq$ 2, 3. Assume that $A/\mathcal{J}(A)$ is a Cayley–Dickson algebra. Then $N(A)/\mathcal{J}(A) = Z(A/\mathcal{J}(A))$. If $\mathcal{J}(A)$ is a non-zero nil-ideal then $N(A) \cap \mathcal{J}(A) \neq 0$.*

Proof. Let $\overline{A} = A/\mathcal{J}(A)$ be a Cayley—Dickson algebra. Let us show that $Z(\overline{A}) = N(A)/\mathcal{J}(A)$. Let $\overline{a}$ be an image of $a \in A$ in the algebra $\overline{A}$. Choose elements $\overline{a}, \overline{b} \in \overline{A}$ such that the commutator $[\overline{a}, \overline{b}]$ is invertible. Then for $z \in Z(\overline{A})$ we obtain $[z\overline{a}, \overline{b}]^4 = z^4[\overline{a}, \overline{b}]^4$. Hence, $z^4 = [z\overline{a}, \overline{b}]^4([\overline{a}, \overline{b}]^4)^{-1} \in N(A)/\mathcal{J}(A)$. By Lemma 7 we obtain that $Z(\overline{A}) \subseteq N(A)/\mathcal{J}(A)$. Since $N(A)/\mathcal{J}(A) \subseteq Z(\overline{A})$, then $N(A)/\mathcal{J}(A) = Z(\overline{A})$.

Let $\mathcal{J}(A)$ be a nil-ideal and $\mathcal{J}(A) \neq 0$. Assume that $N(A) \cap \mathcal{J}(A) = 0$. Then $N(A) = Z(A)$ and $Z(A)$ is a field. Since $Z(A)/\mathcal{J}(A) = Z(\overline{A})$, then for every $a \in A$ exist $\alpha, \beta \in Z(A)$ such that $a^2 - \alpha a + \beta \in \mathcal{J}(A)$. Consequently, we obtain that the algebra A is algebraic over the field $Z(A)$. Since $Z(A) = N(A)$, then for the algebra A the identity $[[x, y]^4, z] = 0$ is true. By the Shirshov Theorem (see [14]) the algebra A is locally finite-dimensional over the field $Z(A)$.

Consider $\overline{A}$ as an algebra over $Z(A)$. The dimension of $\overline{A}$ over the field $Z(A)$ is equal to 8. Let $\overline{e_0}, \dots, \overline{e_7}$ be a basis for $\overline{A}$ over $Z(A)$. Consider a $Z(A)$-subalgebra B in A generated by the elements $e_0, \dots, e_7$. Then B is finite-dimensional over the field $Z(A)$ and $B/(B \cap \mathcal{J}(A)) \cong \overline{A}$. By the Wedderburn Theorem for alternative algebras (see [8]) we have $B = \mathcal{C} + B \cap \mathcal{J}(A)$, where $\mathcal{C}$ is a Cayley—Dickson algebra over the field $Z(A)$, i.e. $Z(\mathcal{C}) = Z(A)$. The identity elements of algebras A and $\mathcal{C}$ coincide. Therefore, by virtue of [18] we have $A \cong \mathcal{C} \otimes_{Z(A)} Z(A)$. We have come to a contradiction, therefore, $N(A) \cap \mathcal{J}(A) \neq 0$. □

Lemma 9. *Let A be an alternative algebra over a field of characteristic $\neq 2, 3$. Assume that $\mathcal{J}(A)$ is nilpotent, $A/\mathcal{J}(A)$ is a Cayley—Dickson algebra, and $A = B \oplus \mathcal{J}(A)$, where B is a Jordan subalgebra of $A^{(+)}$. Then $A = \mathcal{C} \oplus \mathcal{J}(A)$, where $\mathcal{C}$ is a Cayley—Dickson algebra. In particular, $B \cong \mathcal{C}^{(+)}$.*

Proof. Let $\overline{A} = A/\mathcal{J}(A)$. Then $B \cong \overline{A}^{(+)}$. Hence, B is a simple Jordan algebra that is finite-dimensional over its center $Z(B)$. Clearly, $Z(B) \cong Z(\overline{A})$. There are such elements a, b in the algebra B that the commutator $[a, b]$ is invertible in A. Since $[a, b]^4$ is a Jordan polynomial in a, b, then $[a, b]^4 \in B$ and $[a, b]^4$ is invertible in B. For every $x, y, w \in A$ the following equation is true (see [14]):

$$4(x, y, w)^+ = 2(x, w, y) + [y, [x, w]], \tag{9}$$

where $(x, y, w)^+$ is an associator of the elements x, y, w of the algebra $A^{(+)}$. Hence, $[a, b]^4 \in Z(B)$.

Let $z \in Z(B)$. Then $\overline{z^4 \circ [a, b]^4 - [z \circ a, b]^4} = 0$, where $\circ$ is the multiplication in the algebra $A^{(+)}$. Therefore, $z^4 = [z \circ a, b]^4 \circ ([a, b]^4)^{-1} \in N(A)$. Denote by N_1 the subalgebra of $N(A)$ generated by the elements $[z \circ a, b]^4 \circ ([a, b]^4)^{-1}$, where $z \in Z(B)$. Then by Lemma 7 $Z(B) \subseteq N_1 \subseteq N(A)$. By virtue of (9) $[z, [x, y]] = [x, [z, y]] = 0$ for all $x, y \in B$, $z \in Z(B)$. Since

$$[z, [x, y]^2] = [z, [x, y]] \circ [x, y] = 0,$$

then $[Z(B), N_1] = 0$. Hence, $[Z(B), Z(B)] = 0$. Due to (2) and (9) we have $[[z, x], [y, w]] = [z, [x, [y, w]] = 0$ for all $x, y, w \in B$, $z \in Z(B)$. It is known that $[Z(B), (B, B, B)] \subseteq [N(A), (B, B, B)] = 0$. Therefore, by virtue of (9),

$$[Z(B), (B, B, B)^+] = 0.$$

Hence, we obtain $[Z(B), B] \subseteq [Z(B), Z(B) + (B, B, B)^+] = 0$.

Now let A_1 be a subalgebra of A generated by the vector space B. It is easy to see that $Z(B) \subseteq Z(A_1)$. Consider A_1 as an algebra over the field $Z(B)$. Clearly, $A_1/(A_1 \cap \mathcal{J}(A)) \cong A/\mathcal{J}(A)$. Note that for $a \in A_1$ there are such $\alpha, \beta \in Z(B)$ that $a^2 - \alpha a + \beta b \in A_1 \cap \mathcal{J}(A)$. Hence we obtain the algebra A_1 is algebraic over $Z(B)$ and its algebraic index is bounded. By the Shirshov Theorem (see [14]) the algebra A_1 is finite-dimensional over the field $Z(B)$. Therefore, A_1 contains a Cayley—Dickson subalgebra $\mathcal{C}$ over the field $Z(B)$ such that $A_1 = \mathcal{C} \oplus A_1 \cap \mathcal{J}(A)$. Hence, $A = \mathcal{C} \oplus \mathcal{J}(A)$. □

Theorem 3. *Let A be a unital alternative algebra over the field of characteristic $\neq 2, 3$ and (D, D_1, D_2) be a ternary derivation of A with invertible values. Assume that $\mathcal{J}(A) \neq 0$ and $A/\mathcal{J}(A)$ is a Cayley—Dickson algebra. Then* $\ker D = \mathcal{J}(A)$ *and* $\mathcal{J}(A)^2 = 0$. *Moreover, $B = \{a \in A \mid D(a) = D(1)a\}$ is a Jordan division subalgebra in $A^{(+)}$, $A = B \oplus \mathcal{J}(A)$, and A contains a Cayley—Dickson division algebra $\mathcal{C}$ such that $\mathcal{C}^{(+)} \cong B$.*

Proof. Let $\overline{A} = A/\mathcal{J}(A)$ be a Cayley—Dickson algebra. Then by Lemma 8 $R = N(A) \cap \mathcal{J}(A) \neq 0$. Since $[A, N(A)] \subseteq N(A)$, then $[A, R] \subseteq R$ and RA is an ideal of the algebra A. By Theorem 1 and Theorem 2 $RA \subseteq \ker D$. Note that $D(1) \neq 0$. Indeed, if $D(1) = 0$ then for every $r \in R$ and $a \in A$ we obtain

$$0 = D(ra) = D(r)a + rD(a) = rD(a).$$

Since $D(a)$ is invertible for some $a \in A$, then $r = 0$. Hence, $D(1) \neq 0$.

Let $r(X) = [a \in A \mid xa = 0 \, \forall x \in X\}$ be a right annihilator of a subset $X \subseteq A$. Then $r(R) = r(RA)$. Clearly, a right annihilator of an ideal of algebra A is again an ideal.

Let us show that $\ker D = \mathcal{J}(A)$ and $D(a) - aD(1) \in \ker D$ for every $a \in A$. Let $r \in R$, $x \in \ker D$. Then the following equalities hold:

$$0 = D(rx) = D(r)x + rD(x) - rD(1)x = -rD(1)x.$$

Therefore, $D(1)x \in r(RA)$. Since $D(1)$ is invertible, then $x \in r(RA)$, i.e. $\ker D \subseteq r(RA)$. As $r(RA) \subseteq \mathcal{J}(A) \subseteq \ker D$ we have $\ker D = \mathcal{J}(A) = r(RA)$ and $\ker D$ is an ideal of the algebra A.

Let $r \in R$, $a \in A$. Then

$$0 = D(ra) = D(r)a + rD(a) - rD(1)a = r(D(a) - D(1)a).$$

Therefore, $D(a) - D(1)a \in r(RA) = \ker D$ for every $a \in A$. Similarly,

$$0 = D(ar) = D(a)r + aD(r) - aD(1)r = (D(a) - aD(1))r.$$

Replacing the right annihilator with the left one we obtain that $D(a) - aD(1) \in \ker D$.

By Lemma 1

$$D_1(\mathcal{J}(A)) \subseteq D(\mathcal{J}(A)) + \mathcal{J}(A)D_2(1) \subseteq \mathcal{J}(A).$$

Similarly, $D_2(\mathcal{J}(A)) \subseteq \mathcal{J}(A)$. Therefore, (D, D_1, D_2) induces ternary derivation $(\overline{D}, \overline{D_1}, \overline{D_2})$ with invertible values on the algebra $\overline{A}$. Specifically,

$$\overline{D}(\overline{a}) = \overline{D(a)}, \quad \overline{D_1}(\overline{a}) = \overline{D_1(a)}, \quad \overline{D_2}(\overline{a}) = \overline{D_2(a)},$$

where $\overline{x}$ is an image $x \in A$ in a factor algebra $\overline{A}$. By the above-mentioned for every $a \in A$ we obtain

$$\overline{D}(\overline{a}) = \overline{a} \cdot \overline{D(1)} = \overline{D(1)} \cdot \overline{a}.$$

Hence, $A/\mathcal{J}(A)$ is an division algebra and $\overline{D(1)} \in Z(\overline{A})$. Therefore, by Lemma 1

$$\frac{1}{2}(\overline{a}, \overline{D_1(1) - D_2(1)}, \overline{b}) = \overline{D}(\overline{ab}) - \overline{D}(\overline{a}) \cdot \overline{b} - \overline{a} \cdot \overline{D}(\overline{b}) + \overline{a} \cdot \overline{D(1)} \cdot \overline{b} = 0$$

for all $\overline{a}, \overline{b} \in \overline{A}$. Hence, $\overline{D_1(1) - D_2(1)} \in Z(\overline{A})$. Hence, by Lemma 8 $D_1(1) - D_2(1) \in N(A) + \mathcal{J}(A)$ and $D(1) \in N(A) + \mathcal{J}(A)$.

Let us show that $\mathcal{J}(A)^2 = 0$. Let $x \in \mathcal{J}(A)^2$, $y \in \mathcal{J}(A)$. By Theorem 2 $\mathcal{J}(A)^4 = 0$. Then

$$(x, D(1), y) = 0, \quad (x, D_1(1) - D_2(1), y) = 0$$

and by Lemma 1

$$0 = D(xy) = D(x)y + xD(y) - xD(1)y = -xD(1)y$$

for every $x \in \mathcal{J}(A)^2, y \in \mathcal{J}(A)$. Hence,

$$xy = x(D(1)D(1)^{-1}y) = xD(1)(D(1)^{-1}y) = 0.$$

Thus, $\mathcal{J}(A)^3 = 0$.

Let $x, y \in \mathcal{J}(A)$. Then

$$0 = D(xy) = D(x)y + xD(y) - xD(1)y = -xD(1)y.$$

Hence we obtain $xy = 0$. Therefore, $\mathcal{J}(A)^2 = 0$.

Now, let $d = D - L_{D(1)}$. As shown above $d(A) \subseteq \mathcal{J}(A)$. Since $D(\mathcal{J}(A)) = 0$, then

$$\mathcal{J}(A) = d(D(1)^{-1}\mathcal{J}(A)) \subseteq d(\mathcal{J}(A)).$$

Therefore, $d(A) \subseteq \mathcal{J}(A) \subseteq d(\mathcal{J}(A)) \subseteq d^2(A) \subseteq d(A)$. Hence we obtain

$$A = \ker d + d(A) = \ker d + \mathcal{J}(A).$$

Since every non-zero element in $\ker d$ has an inverse element, then $A = \ker d \oplus \mathcal{J}(A)$. Note that d is a derivation of the Jordan algebra $A^{(+)}$. Indeed, by Lemma 1

$$d(a^2) = D(a^2) - D(1)a^2 = D(a)a + aD(a) - aD(1)a - D(1)a^2 = d(a)a + ad(a).$$

Therefore, $B = \ker d$ is a subalgebra of $A^{(+)}$ and $B \cong \overline{A}^{(+)}$. Then by Lemma 9 we obtain that A contains a Cayley—Dickson algebra $\mathcal{C}$ and $A = \mathcal{C} \oplus \mathcal{J}(A)$. Moreover, $\mathcal{C}^{(+)} \cong B$. Hence we obtain that $\mathcal{C}$ is a division algebra. □

Theorem 4. *Let A be a unital alternative algebra over a field of characteristic other than 2, 3 and let (D, D_1, D_2) be a ternary derivation of A with invertible values. Assume that $\mathcal{J}(A) \neq 0$ and $A/\mathcal{J}(A)$ is an associative algebra. Then one of the following propositions is true:*

1. A is an associative algebra and D is a generalized derivation with invertible values,

2. $A/\mathcal{J}(A)$ is an algebra of generalized quaternions,

3. $A/\mathcal{J}(A)$ is a field.

Proof. Assume that A is non-associative. Denote by $AsId(A) = (A, A, A)A$ the associator ideal of A, and let $ZN(A) = [N(A), A]A$. Then $AsId(A) \subseteq \mathcal{J}(A) \subseteq \ker D$ and the quotient algebra $\overline{A} = A/AsId(A)$ is associative. The ternary derivation (D, D_1, D_2) induces a ternary derivation $(\overline{D}, \overline{D_1}, \overline{D_2})$ on $\overline{A}$ with invertible values. Then $\overline{D}$ is a generalized derivation of the algebra $\overline{A}$. By virtue of [1], either $\overline{A}$ is simple or $\ker \overline{D} = \mathcal{J}(\overline{A})$. Hence, we obtain either $AsId(A) = \mathcal{J}(A)$ or $AsId(A) \neq \mathcal{J}(A)$ and $\ker D = \mathcal{J}(A)$.

Assume that $ZN(A) = A$. Hence, $(A, A, A) \subseteq N(A)$ (see [14]). Repeating the reasoning from Theorem 3 we obtain that

$$r((A, A, A)) = r(AssId(A)) = \ker D = \mathcal{J}(A)$$

and $D(a) - D(1)a, D(a) - aD(1) \in \ker D$ for every $a \in A$. Then $A/\mathcal{J}(A)$ is a division algebra. Let $x, y, z \in A$ be non-associating elements, i.e., $(x, y, z) \neq 0$. Then

$$(x, y, z)[N(A), z] \subseteq (x, y, z[N(A), z]) \subseteq$$

$$(x, y, [z, [N(A), z]]) + (x, y, [N(A), z] \circ z) \subseteq (x, y, [N(A), z^2]) = 0.$$

Hence, $[N(A), z] \subseteq \mathcal{J}(A) = r(AsId(A))$. Let $u \in A, n \in N(A)$. Then

$$(x, y, z)[n, u] = (x, y, z[n, u]) = (x, y, [n, zu]) - (x, y, [n, z]u)$$
$$= -(x, y, u[n, z]) = -(x, y, u)[n, z] = 0.$$

Hence, we obtain $(x, y, z) = 0$. Hence, $ZN(A) \neq A$.

Let $ZN(A) \neq 0$. By Theorem 2 $ZN(A) \subseteq \mathcal{J}(A)$. Repeating the reasoning from Theorem 3 we obtain that

$$r([N(A), A]) = r(ZN(A)) = \ker D = \mathcal{J}(A)$$

and $D(a) - D(1)a, D(a) - aD(1) \in r(ZN(A))$ for every $a \in A$. The image of the ideal $ZN(A)$ on $A/\mathcal{J}(A)$ is equal to zero. Therefore, $A/\mathcal{J}(A)$ is a division algebra and the following identity is true: $[[x, y]^4, z] = 0$. Hence by the Kaplansky Theorem $A/\mathcal{J}(A)$ is a finite-dimensional division ring over its center $Z(A/\mathcal{J}(A))$.

Assume that $A/\mathcal{J}(A)$ is a non-commutative algebra. Let K be a splitting field of the algebra $A/\mathcal{J}(A)$. Then $A/\mathcal{J}(A) \otimes_{Z(A/\mathcal{J}(A))} K = K_n$. Hence in K_n the following identity holds: $[[x, y]^4, z] = 0$. Thus we obtain $n = 2$ and $\dim_{Z(A/\mathcal{J}(A))} A/\mathcal{J}(A) = 4$. Therefore, $A/\mathcal{J}(A)$ is a division algebra of generalized quaternions.

Let $ZN(A) = 0$. Then in A the following equality is true $[[x, y]^4, z] = 0$. Thus, the quotient algebra $A/\mathcal{J}(A)$ is either an algebra of generalized quaternions or a field. □

3. The case of the matrix Cayley—Dickson algebra

In this section we describe ternary derivations with invertible values of matrix Cayley—Dickson algebras over a field F of characteristic $\neq 2, 3, 5$.

Let F_2 stand for the algebra of all 2×2 matrices over a field F, and let $\mathcal{C} = F_2 + vF_2$ be a matrix Cayley—Dickson algebra over F. Denote by $t(x)$ and $n(x)$ the trace and the norm of an element $x \in \mathcal{C}$, respectively, and let $f(x, y)$ be the bilinear form associated with the norm $n(x)$. Algebra $\mathcal{C}$ contains a vector subspace of dimension four over F which consists of elements with zero norm. For example, $B = Fe_{11} + Fe_{12} + Fve_{11} + Fve_{12}$.

Lemma 10. *Let $\mathcal{C} = F_2 + vF_2$ be a matrix Cayley—Dickson algebra over a field F of characteristic $\neq 2$, and let (D, D_1, D_2) be a ternary derivation of the ring $\mathcal{C}$ with invertible values. Let B be a vector subspace of dimension four over F which consists of elements with zero norm. Then D is either a linear mapping over F, or $\mathcal{C} = B \oplus \ker D$ is the direct sum of Abelian groups. Furthermore, if $D(1) = 0$, then D is a linear mapping over F.*

Proof. According to Lemma 1, the following equality holds in the algebra $\mathcal{C}$:

$$D(x^2) = D(x)x + xD(x) - xD(1)x.$$

Let $d(x) = D(x) - D(1)x$. Then

$$d(x^2) = D(x^2) - D(1)x^2 = D(x)x + xD(x) - xD(1)x - D(1)x^2 =$$
$$(D(x) - D(1)x)x + x(D(x) - D(1)x) = d(x)x - xd(x).$$

Therefore, d is a derivation of the Jordan ring $\mathcal{C}^{(+)}$. Furthermore, by Lemma 1, we have

$$D(zx) = D(z)x + zD(x) - zD(1)x = d(z)x + zD(x) = D(z)x + d(x)z$$

for all $x \in \mathcal{C}$ and $z \in F$. Therefore, D is a linear mapping over F if and only if $d(z) = 0$ for every $z \in F$.

Clearly, $d(F) \subseteq F$. In the algebra $\mathcal{C}$ we have

$$x^2 - t(x)x + n(x) = 0.$$

Then

$$d(x)x + xd(x) - d(t(x))x - t(x)d(x) + d(n(x)) = 0.$$

On the other hand,

$$d(x)x + xd(x) = t(d(x))x + t(x)d(x) - f(d(x), x).$$

Hence we have

$$(d(t(x)) - t(d(x)))x - d(n(x)) + f(d(x), x) = 0.$$

Therefore, for every $x \in \mathcal{C}$

$$d(t(x)) = t(d(x)), d(n(x)) = f(d(x), x). \tag{10}$$

Let $x \in \ker D$, $x \neq 0$, $z \in F$. Then

$$D(zx) = d(z)x + zD(x) = d(z)x.$$

Hence, either $d(z) = 0$ for all $z \in F$ and thus D is a linear mapping over F, or $n(x) \neq 0$.

Suppose $n(x) \neq 0$ for all nonzero elements $x \in \ker D$. Since $n(b) = 0$ for every $b \in B$, we have $B \cap \ker D = 0$.

Let $V = D(B) + B$. Then

$$FV \subseteq FD(B) + FB \subseteq D(FB) + d(F)B + B \subseteq V.$$

Hence, V is a vector subspace in $\mathcal{C}$. Since $B \cap \ker D = 0$ and nonzero elements from $D(B)$ are invertible, i.e., their norm is nonzero, then $D(B) \cap B = 0$.

Let b_1, b_2, b_3, b_4 be a basis of the vector space B. Then $D(b_1), D(b_2), D(b_3), D(b_4)$ and $b \in B$ are linearly independent over F. Indeed, assume

$$z_1D(b_1) + z_2D(b_2) + z_3D(b_3) + z_4D(b_4) + z_5b = 0,$$

where $z_1, \dots, z_5 \in F$. Then

$$D(z_1b+z_2b_2+z_3b_3+z_4b_4) = d(z_1)b_1+d(z_2)b_2+d(z_3)b_3+d(z_4)b_4-z_5b \in B\cap D(B) = 0.$$

Hence we have

$$z_1b + z_2b_2 + z_3b_3 + z_4b_4 \in \ker D \cap B = 0.$$

Therefore, $z_i = 0$, $i = 1, \dots, 4$ and

$$b_1, b_2, b_3, b_4, D(b_1), D(b_2), D(b_3), D(b_4)$$

is a basis of the vector space V. Thus, $\mathcal{C} = B + D(B)$.

Let $x \in \mathcal{C}$. Then $D(x) = b + D(y)$, where $b, y \in B$. Therefore, $D(x - y) = b \in B$. Hence we have $D(x - y) = 0$, i.e., $x \in B + \ker D$. Thus, $\mathcal{C} = B \oplus \ker D$ is the direct sum of Abelian groups.

Let $z \in F$. Then $z = b + c$, where $b \in B$, $c \in \ker D$. Since $n(b) = 0$ and $f(xu, yu) = f(x, y)n(u)$, then, according to (10),

$$\begin{aligned}0 = d(n(b)) = f(b, d(b)) = f(b, D(b) - D(1)b)&\\ = f(b, D(b)) - f(1, D(1))n(b) = f(b, D(b)).&\end{aligned}$$

Hence we have $f(b, D(z)) = f(b, D(b)) = 0$.

Let $D(1) = 0$. Then $D = d$. Therefore,

$$0 = f(b, d(z)) = d(z)f(b, 1) = d(z)t(b).$$

If $t(b) \neq 0$, then $d(z) = 0$. If $t(b) = 0$, then $2z = t(z) = t(c)$, and, according to (10),

$$2d(z) = d(2z) = d(t(c)) = t(d(c)) = t(D(c)) = 0.$$

Therefore, $d(z) = 0$. Hence D is a linear mapping over F. □

Lemma 11. *Let F be a field of characteristic different from $2, 3, 5$, $\mathcal{C} = F_2 + vF_2$ be a matrix Cayley–Dickson algebra over F, and let (D, D_1, D_2) be a ternary derivation of the ring $\mathcal{C}$ with invertible values. Then D is a linear mapping over F.*

Proof. Let $d = D - L_{D(1)}$. Then d is a derivation of the field F. Let $F_0 = \ker d \cap F$. Then F_0 is a subfield of the field F. Since for $b \in \ker D$ and $z \in F_0$

$$D(zb) = d(z)b + zD(b) = 0,$$

the $\ker D$ a vector space over F_0.

According to Lemma 10 we can suppose $\mathcal{C} = B + \ker D$, where B is a vector subspace of dimension four over F which consists of elements with zero norm. Let $\mathrm{Lin}\,(\ker D)$ be the linear span of $\ker D$. Then $\mathcal{C} = B + \mathrm{Lin}\,(\ker D)$.

Let us show that the elements $b_1, \ldots, b_k \in \ker D$ are linearly independent over F_0 if and only if $b_1, \ldots, b_k$ are linearly independent over F. Assume $b_1, \ldots, b_k$ are linearly dependent over F. Let $b_1, \ldots, b_l$ be a maximal system of linearly independent elements over F in $\{b_1, \ldots, b_k\}$. Then

$$b_{l+1} = z_1 b_1 + \ldots + z_l b_l,$$

where $z_1, \ldots, z_l \in F$. Hence we obtain that

$$0 = D(b_{l+1}) = d(z_1)b_1 + \ldots + d(z_l)b_l,$$

Consequently, $d(z_1) = \ldots = d(z_l) = 0$, i.e. $z_1, \ldots, z_l \in F_0$. Therefore, $b_1, \ldots, b_k$ are linearly independent over F.

Let $b_1, \ldots, b_k$ be a basis of the vector space $\ker D$ over the field F_0. Then $4 \leq k \leq 8$. Since $\mathcal{C} = B + \mathrm{Lin}\,(\ker D)$, we have $\dim_F (B \cap \mathrm{Lin}\,(\ker D)) = k - 4$.

Let us show that the field F is an algebraic extension of the field F_0 and the degree of algebraicity does not exceed 5. We consider several cases:

Case 1: $k = 4$. Then $\dim_F (B \cap \mathrm{Lin}\,(\ker D)) = 0$ and, according to Lemma 10,

$$zb_1 = v + z_1 b_1 + \ldots + z_k b_k$$

for $z \in F$, where $v \in B$, $z_1, \ldots, z_k \in F_0$. Hence we obtain that $v \in B \cap \mathrm{Lin}\,(\ker D) = 0$. Therefore, $z = z_1$ and $F = F_0$.

Case 2: $k = 5$. Then $\dim_F (B \cap \mathrm{Lin}\,(\ker D)) = 1$ and

$$zb_1 = v_1 + z_{11} b_1 + \ldots + z_{1k} b_k,$$
$$zb_2 = v_2 + z_{21} b_1 + \ldots + z_{2k} b_k$$

for $z \in F \setminus F_0$, where $v_1, v_2 \in B$, $z_{i1}, \ldots, z_{ik} \in F_0$, $i = 1, 2$. Hence we obtain that $v_1, v_2 \in B \cap \mathrm{Lin}\,(\ker D)$. Since $\dim_F (B \cap \mathrm{Lin}\,(\ker D)) = 1$, then the rank of the system $\{v_1, v_2\}$ does not exceed 1. Therefore,

$$\det \begin{pmatrix} z_{11} - z & z_{12} \\ z_{21} & z_{22} - z \end{pmatrix} = 0.$$

Consequently, z is a root of an equation of degree 2.

Case 3: $k = 6$. Then $\dim_F(B \cap \mathrm{Lin}\,(\ker D)) = 2$ and

$$\begin{aligned} zb_1 &= v_1 + z_{11}b_1 + \ldots + z_{1k}b_k, \\ zb_2 &= v_2 + z_{21}b_1 + \ldots + z_{2k}b_k, \\ zb_3 &= v_3 + z_{31}b_1 + \ldots + z_{3k}b_k \end{aligned}$$

for $z \in F \setminus F_0$, where $v_1, v_2, v_3 \in B$, $z_{i1}, \ldots, z_{ik} \in F_0$, $i = 1, 2, 3$. Hence we obtain that $v_1, v_2, v_3 \in B \cap \mathrm{Lin}\,(\ker D)$. Since $\dim_F(B \cap \mathrm{Lin}\,(\ker D)) = 2$, then the rank of the system $\{v_1, v_2, v_3\}$ is not higher than 2. Therefore,

$$\det \begin{pmatrix} z_{11} - z & z_{12} & z_{13} \\ z_{21} & z_{22} - z & z_{23} \\ z_{31} & z_{32} & z_{33} - z \end{pmatrix} = 0.$$

Consequently, z is a root of an equation of degree 3.

In a similar way, we obtain that for $k = 7$ or $k = 8$ an element $z \in F \setminus F_0$ is a root of an equation of degree 4 or 5, respectively.

Now let $z \in F \setminus F_0$ and let $f(x) \in F_0[x]$ be a polynomial of minimal degree such that $f(z) = 0$. Then $\deg f(x) \leq 5$. Therefore, $f'(x) \neq 0$ and $f'(z) \neq 0$. On the other hand, $0 = d(f(z)) = f'(z)d(z)$. Consequently, $d(z) = 0$ for every $z \in F$. Since for $a \in \mathcal{C}$

$$D(za) = d(z)a + zD(a) = zD(a),$$

then D is a linear mapping over F. □

In every alternative algebra the mapping $D_{x,y}$, defined by the rule

$$D_{x,y}(z) = [[x, y], z] - 3(z, x, y)$$

is a derivation due to (1) and to (3). More generally, the mappings $\sum_i D_{x_i,y_i}$ are derivations. Derivations of this form are called *inner*. We denote by $\mathrm{InDer}(A)$ the set all inner derivation of an alternative algebra A.

Now, Lemmas 10 and 11, Theorem 1.1, Theorem 1.3 from [10], and Corollary 3.29 from [8] imply

Theorem 5. *Let $\mathcal{C} = F_2 + vF_2$ be a matrix Cayley–Dickson algebra over field F of characteristic $\neq 2$, and let (D, D_1, D_2) be a ternary derivation of the ring $\mathcal{C}$ with invertible values. Suppose that either $D(1) = 0$ or F is a field of characteristic $\neq 3, 5$. Then either*

$$(D, D_1, D_2) \in \mathrm{InDer}(\mathcal{C}) + \mathrm{Lin}\,\{(L_a, T_a, -L_a), (R_a, -R_a, T_a) \mid a \in \mathcal{C}\},$$

if F is a field of characteristic $\neq 3$, or

$$(D, D_1, D_2) \in alg_{Lie}\langle (L_a, T_a, -L_a), (R_a, -R_a, T_a) \mid a \in \mathcal{C}\rangle + \mathrm{Lin}\,\{(Id, 0, Id)\},$$

if F is a field of characteristic 3.

4. The case of Jordan algebras

The study of ternary derivations of Jordan algebras was initiated in [17, 19], where the description of ternary derivations of finite-dimensional separable Jordan algebras and of simple Jordan superalgebras was obtained.

For an ideal I of a Jordan algebra J define (see [14]) a chain of ideals, considering $I^{[1]} = I^3, I^{[i+1]} = (I^{[i]})^3$. Define one more chain of vector spaces: $I^{(1)} = I^2, I^{(i+1)} = (I^{(i)})^2$. An ideal I is called solvable if $I^{(n)} = 0$ for some natural number n.

Theorem 6. *Let J be a unital Jordan algebra and (D, D_1, D_2) be a ternary derivation with invertible values. If* $\ker D$ *does not contain non-zero ideals of the algebra J, then J is a simple algebra. If* $\ker D$ *contains non-zero ideals of J, then the quasi-regular radical $\mathcal{J}(J)$ of the algebra J lays inside* $\ker D$ *and $\mathcal{J}(J)$ is the largest proper ideal. Particularly, the quotient algebra $J/\mathcal{J}(J)$ is simple. If $D(1) = 0$, i.e., D is a derivation, then $\mathcal{J}(J)^2 = 0$. If $D(1) \neq 0$ and the characteristic of the base field $\neq 3, 5$, then $\mathcal{J}(J)$ is a solvable ideal.*

Proof. Let $\ker D$ does not contain non-zero ideals of the algebra J. Then by Lemma 6 J is a simple algebra.

Now, let $\ker D$ contains a non-zero ideal of the algebra J. Then by Lemma 6 every ideal different from J is contained in $\ker D$. Therefore, $\mathcal{J}(J) \subseteq \ker D$. If $D(1) = 0$ then by Lemma 6 $\mathcal{J}(J)^2 = 0$. Therefore, we can assume that $D(1) \neq 0$.

Let us prove that $\mathcal{J}(J)$ is the largest solvable ideal. Let $I \neq J$ be an ideal of the algebra J and $x \in I$. Then by Lemma 1 and Lemma 6 we obtain $xD(1)x = 0$ and $xD(1)x^2 = 0$. Due to the linearized Jordan identity we have

$$x^3 D(1) = -2[(D(1)x)x]x + 3xD(1)x^2 = 0.$$

Let $I_3(I)$ be a vector space spanned by the elements x^3, where $x \in I$. Then $I_3(I)$ is an ideal of I (see[14]). By virtue of [20] $I^{(n)} \subseteq I_3(I)$ for some natural number n. Therefore, $I^{[n]} \subseteq I^{(n)} \subseteq I_3(I)$. Hence, $D(1)I^{[n]} = 0$. Since $D(1) \neq 0$, then there is such $b \in J$ that $bD(1) = 1$. Then for $x, y \in I^{[n]}$ by virtue of (7) we obtain

$$xy = (xy)(bD(1)) = -(xy, b, D(1)) = (xD(1), b, y) + (D(1)y, b, x) = 0.$$

Hence, $(I^{[n]})^2 = 0$. Since $I^{(2n)} \subseteq I^{[n]}$, then $(I^{(2n)})^2 = 0$. Therefore, I is a solvable ideal.

Thus, $\mathcal{J}(J)$ is the largest solvable ideal. □

Let us give an example of a Jordan algebra J such that $\mathcal{J}(J) \neq 0$ and there is a derivation d of J with invertible values.

Let $F = \mathbb{R}$ be a field of real numbers and $\mathbb{H}$ be the quaternion division ring with standard basis $1, i, j, k$. Consider the Jordan algebra $\mathbb{H}^{(+)}$. Then $\mathbb{H}^{(+)}$ is a Jordan algebra of nondegenerate symmetric bilinear form f defined on a vector space $V = Fi + Fj + Fk$. Let I be a vector space over the field F. Extend the form f on $U = V \oplus I$ by

$$f(V, I) = f(I, V) = f(I, I) = 0.$$

Then $J = F \cdot 1 + U$ is a Jordan algebra with respect to the multiplication

$$(\alpha \cdot 1 + x)(\beta \cdot 1 + y) = (\alpha\beta + f(x, y)) \cdot 1 + \alpha y + \beta y.$$

Clearly, $J = \mathbb{H}^{(+)} + I$, $I^2 = 0$ and I is the largest ideal of the algebra J. Define a mapping $d : J \mapsto J$ by $d(x) = (i, x, j)$, where (i, x, j) is the associator in the algebra J. Then d is a derivation of the algebra J with invertible values.

Now, let F be a field of characteristic other than 2, J be a Jordan algebra, and d be a derivation of the algebra J with invertible values. Then using the results presented in [21, 22, 23, 2, 3] we can describe the structure of the quotient algebra $J/\mathcal{J}(J)$ (see also [24]). Namely, $J/\mathcal{J}(J)$ is isomorphic to one of the following algebras:

1. $A^{(+)}$, where A is either an associative division algebra or an algebra of all 2×2 matrices over an associative division algebra;

2. $H(A,\star)$, the Jordan algebra of symmetric elements of A relative to an involution $\star$, where A is either an associative division algebra or an algebra of all 2×2 matrices over an associative division algebra, or a simple 16-dimensional algebra over an extension of a filed F, in this case, $\star$ is the symplectic involution;

3. $B(V,f)$, the Jordan algebra of a nondegenerate symmetric bilinear form over an extension of the field F;

4. 27-dimensional exceptional Albert division algebra.

Consider the case 2. Let $J = H(A,\star)$ be a simple Jordan algebra of Hermitian type. Assume that the algebra A is generated by the set J. Show that the locally nilpotent radical $L(A)$ of the algebra A is equal to zero. Indeed, $J \cap L(A) = 0$. Therefore, for every $a \in L(A)$ we obtain $a + a^\star \in J \cap L(A) = 0$. Hence, $a^\star = -a$ for every $a \in L(A)$. Let $a \in L(A)$ and $h \in J$. Then $[a,h] \in J \cap L(A) = 0$. Therefore, $a \in Z(A)$. Let $s \in A$ and $s^\star = -s$. Then

$$2as = as + sa \in L(A) \cap J = 0.$$

Since $[x,y]^\star = -[x,y]$ for all $x,y \in J$, then $a[x,y] = 0$ for all $x,y \in J$. Since for $x,y,z \in J$ their Jordan associator is represented as $(x,y,z)^+ = \frac{1}{4}[y,[x,z]]$, then $a(x,y,z)^+ = 0$. Since the associator ideal of the algebra J coincides with J, then $a = 0$. Hence, $L(A) = 0$. Let I be an ideal of the algebra A and $I^\star \subseteq I$. If $J \cap I = 0$ then $r^2 = 0$ for every $r \in I$. Therefore, $I^4 = 0$. Hence we obtain that the algebra A is $\star$-simple. By virtue of [23] either A is a simple 16-dimensional algebra over an extension of the field F and $\star$ is a symplectic involution, or d extends to the derivation of the algebra A. Then by virtue of [3] we obtain two more cases, specifically, either A is an associative division algebra or an algebra of all 2×2 matrices over a division algebra.

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