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ON ATTRACTORS OF ITERATED FUNCTION SYSTEMS IN UNIFORM SPACES

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Abstract. We prove Hutchinson's theorem for uniform spaces, that a finite system $\mathcal{S}$ of $\mathcal{B}$-contractions in complete, well-chained, Hausdorff uniform space defines unique compact attractor.

Keywords: self-similar set, uniform space, $\mathcal{B}$-contraction, asymptotical regularity.

1. Introduction

Classical definition of self-similar sets originally proposed by J. Hutchinson [5] (1981) was formulated for contraction maps in complete metric spaces. The attempts to extend this construction to topological spaces were made by many authors. A. Kameyama [6](1993) introduced and investigated self-similar symbolic spaces defined as quotient spaces of the abstract Cantor set I^∞, W. J. Charatonik and A. Dilks [1] (1994) studied four types of self-homeomorpfic spaces, L. Bartholdi, R. Grigorchuk and V. Nekrashevych [4] (2001) considered self-similar structures generated by finite systems of injective continuous maps. It was proved by Tetenov [9] (2010) in general case of Hausdorff topological space $\mathbf{X}$ that a semigroup G of self-maps of $\mathbf{X}$, satisfying (P)-condition defines unique compact attractor K. Similar ideas were used in papers [2, 7] (2012). The paper [3] (2015) contains reference to more recent development in the area.

A self-similar set in a complete metric space $\mathbf{X}$ is the unique non-empty compact set $K \subset \mathbf{X}$ satisfying equation $K = \bigcup\limits_{i=1}^{m} S_i(K)$, where $S_i : \mathbf{X} \to \mathbf{X}$ are contraction maps. Such set K may be considered as the fixed point of a contraction operator

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T in the hyperspace $C(\mathbf{X})$ defined by $T(A) = \bigcup_{i=1}^{m} S_i(A)$ (see [5]).

Instead of using topological conditions which serve as a replacement to completeness it is natural to get rid of metrics by means of passing to uniform spaces. As we found, in order to obtain meaningful results, we have to impose the requirement that $\mathbf{X}$ is complete, well-chained, Hausdorff uniform space and the maps S_i are $\mathcal{B}$-contractions studied by Taylor [8].

Carrying out this general approach for construction of self-similar sets in uniform spaces, we prove the following non-metric version of Hutchinson's theorem:

Theorem 1. *Let $(\mathbf{X}, \mathcal{U})$ be a complete, well-chained, Hausdorff uniform space and $S = \{f_1, ..., f_m\}$ be a system of $\mathcal{B}$-contractions in $\mathbf{X}$. There is a unique non-empty compact $K \subset \mathbf{X}$ such that $K = \bigcup S_i(K)$.*

The chain of statements presented in the following section, will give the proof of the Theorem.

2. The proof of Theorem 1

Here we use the argument proposed by Taylor [8] in his paper on fixed points of $\mathcal{B}$-contractions in uniform spaces and define attractors of finite systems of $\mathcal{B}$-contractions.

Let $(\mathbf{X}, \mathcal{U})$ be a uniform space, where $\mathcal{U}$ is a family of entourages defining the uniform structure on the set $\mathbf{X}$.

Definition 2. *Let $\mathcal{B}$ be a basis for the uniformity $\mathcal{U}$ of a uniform space $(\mathbf{X}, \mathcal{U})$. A mapping $f : \mathbf{X} \to \mathbf{X}$ is a $\mathcal{B}$ - contraction on $\mathbf{X}$, if for each $U \in \mathcal{B}$ there is a $V \in \mathcal{B}$ such that*

$$(x, y) \in U \circ V \text{ implies } (f(x), f(y)) \in U \tag{1}$$

This definition can be extended to finite system of mappings $\mathcal{S} = \{f_1, \ldots, f_m\}$:

Definition 3. *A system $\mathcal{S} = \{f_1, \ldots, f_m\}$, $f_i : \mathbf{X} \to \mathbf{X}$ is a system of $\mathcal{B}$ - contractions on $\mathbf{X}$, if for each $U \in \mathcal{B}$ there is a $V \in \mathcal{B}$ such that*

$$(x, y) \in U \circ V \text{ implies } (f_i(x), f_i(y)) \in U \text{ for each } f_i \in \mathcal{S} \tag{2}$$

From now on $\mathcal{S} = \{f_1, \ldots, f_m\}$ will denote a system of $\mathcal{B}$-contractions of $\mathbf{X}$.

Some notation. Let I be the set $\{1, \ldots, m\}$, I^p be the set of ordered p-tuples $i_1 \ldots i_p$, and $I^* = \sum_{p=1}^{\infty} I^p$. Let I^∞ be the set of all infinite sequences formed by symbols $i_k \in \{1, ..., m\}$ supplied with the topology of infinite product of discrete sets.

For $\mathbf{j} \in I^p$ and $\alpha \in I^\infty$, we write $\mathbf{j} \prec \alpha$ if $j_1, \ldots j_p$ are first p elements of the sequence α.

For any p-tuple $\mathbf{j} = j_1 \ldots j_p \in I^*$ we write $f_{\mathbf{j}} = f_{j_1 \ldots j_p} := f_{j_1} \circ \ldots \circ f_{j_p}$.

We also extend the definition of asymptotical regularity to systems of contractions:

Definition 4. *A system $\mathcal{S} = \{f_1, \ldots, f_m\}$ of $\mathcal{B}$-contractions is* asymptotically regular *on $\mathbf{X}$ iff for each $x \in \mathbf{X}$ and entourage $U \in \mathcal{U}$, there is an integer n_0 such that for any sequence $\alpha = \{i_1, \ldots, i_n, \ldots\} \in I^\infty$ and for any $n \geq n_0$,*

$$(f_{i_1 \ldots i_n}(x), f_{i_1 \ldots i_{n+1}}(x)) \in U$$

Definition 5. *A uniform space $(\mathbf{X}, \mathcal{U})$ is well-chained provided for each entourage $U \in \mathcal{U}$ and pair of points x and y in $\mathbf{X}$, there is a positive integer n such that $(x, y) \in U^n$.*

Proposition 6. *Let $(\mathbf{X}, \mathcal{U})$ be a complete uniform space. Assume $\mathcal{B}$ is a basis for $\mathcal{U}$ such that $\mathcal{S} = \{f_1, \ldots, f_m\}$ is an asymptotically regular system of $\mathcal{B}$-contractions on X. Then for each sequence $\alpha = \{i_1, \ldots, i_n, \ldots\} \in I^\infty$ and each $x \in \mathbf{X}$, the sequence $\{f_{i_1 \ldots i_n}(x); i_1 \ldots i_n \prec \alpha\}$ converges.*

Proof. Take $x \in \mathbf{X}$ and $\alpha \in I^\infty$; we will show that $\{f_{i_1 \ldots i_n}(x); i_1 \ldots i_n \prec \alpha\}$ is a Cauchy sequence. Let W be an arbitrary symmetric entourage (these of course are cofinal in $\mathcal{U}$) and pick $U \in \mathcal{B}$ such that $U \subset W$. Also pick $V \in \mathcal{B}$ satisfying Definition 3, with respect to U. Since $\mathcal{S}$ is asymptotically regular, by Definition 4, we may choose n_0 such that $(f_{i_2 \ldots i_n}(x), f_{i_2 \ldots i_{n+1}}(x)) \in V$ for $n > n_0$. Fix $n > n_0$. Since W is symmetric, it clearly suffices to show that

$$(f_{i_1 \ldots i_n}(x), f_{i_1 \ldots i_{n+k}}(x)) \in U \subset W \text{ for all } k = 0, 1, 2, \ldots \tag{3}$$

Since (3) holds trivially for $k = 0$ we may proceed by induction on k. Assume (3) is valid for some fixed k. Then we have $(f_{i_2 \ldots i_n}(x), f_{i_2 \ldots i_{n+1}}(x)) \in V$ and $(f_{i_2 \ldots i_{n+1}}(x),\, f_{i_2 \ldots i_{n+k+1}}(x)) \in U$ so that

$$(f_{i_2 \ldots i_n}(x), f_{i_2 \ldots i_{n+k+1}}(x)) \in U \circ V$$

implies

$$(f_{i_1 \ldots i_n}(x), f_{i_1 \ldots i_{n+k+1}}(x)) \in U$$

by Definition 3, thus supporting our inductive assumption. Therefore there is unique $a \in \mathbf{X}$, such that $f_{i_1 \ldots i_n}(x) \to a$. □

To finish the proof, we need to show that if the space $\mathbf{X}$ is well-chained, then $\lim\limits_{n \to \infty} f_{i_1 \ldots i_n}(x)$ does not depend on the initial point $x \in \mathbf{X}$.

Lemma 7. *Let $\mathbf{X}$ be a well-chained uniform space and $\mathcal{S} = \{f_1, \ldots, f_m\}$ be a system of $\mathcal{B}$-contractions on $\mathbf{X}$. Then for any $x, y \in \mathbf{X}$ and for any $U \in \mathcal{B}$ there is such n_0, that for any $n > n_0$ and any n-tuple $i_1 \ldots i_n$,*

$$(f_{i_1 \ldots i_n}(x), f_{i_1 \ldots i_n}(y)) \in U \tag{4}$$

Proof. Given $U \in \mathcal{U}$, take $V \in \mathcal{B}$ for which (2) holds.
Observe that if $(x, y) \in U \circ V^n$, then for any $i_1 \ldots i_n \in I^*$, $(f_{i_1 \ldots i_n}(x),\, f_{i_1 \ldots i_n}(y)) \in U$. Indeed, this holds trivially for $n = 1$. Assume this is true for some fixed k, and let $(x, y) \in U \circ V^{k+1}$. Since $U \circ V^{k+1} = U \circ V^k \circ V$, there is a $z \in \mathbf{X}$ such that $(x, z) \in V$ and $(z, y) \in U \circ V^k$. Clearly $(x, z) \in V$ together with Definition 2 insures

that $(f_{i_1\dots i_{k+1}}(x), f_{i_1\dots i_{k+1}}(z)) \in U$ by our inductive assumption. Thus $(f_{i_1\dots i_{k+1}}(x),$ $(f_{i_1\dots i_{k+1}}(y)) \in U \circ V$ implies $(f_{i_1\dots i_{k+1}}(x), (f_{i_1\dots i_{k+1}}(y)) \in U$ by Definition 2, establishing our induction. $\square$

Since $\mathbf{X}$ is well-chained, there is n_0 such that $V^{n_0} \ni (x, y)$. Then $(x, y) \in U \circ V^{n_0}$. Therefore, for $n > n_0$, the inclusion (4) is valid.

Proposition 8. *Let $(\mathbf{X}, \mathcal{U})$ be a complete well-chained uniform space. Assume $\mathcal{B}$ is a basis for $\mathcal{U}$ such that $\mathcal{S} = \{f_1, \dots, f_m\}$ is a system of $\mathcal{B}$-contractions on X. Then for each sequence $\alpha = \{i_1, \dots, i_n, \dots\} \in I^\infty$ there is $c_\alpha \in \mathbf{X}$ such that for any $x \in \mathbf{X}$, $\lim\limits_{n\to\infty} \{f_{i_1\dots i_n}(x), i_1 i_2 \dots i_n \prec \alpha\} = c_\alpha$.*

Proof. Given $U \in \mathcal{U}$, take $V \in \mathcal{B}$ for which (2) holds. Take $x \in \mathbf{X}$ and find n_0 such that $(x, f_i(x)) \in V^{n_0}$ for any $i \in I$. By Lemma 7, if $n > n_0$ then for any $i_1 \dots i_{n+1} \in I^*$, $(f_{i_1\dots i_n}(x), f_{i_1\dots i_{n+1}}(x)) \in U$, so the system $\mathcal{S}$ is asymptotically regular. By Proposition 6, given $\alpha \in I^\infty$, for any x there is $\lim\limits_{n\to\infty} \{f_{i_1\dots i_n}(x), i_1 i_2 \dots i_n \prec \alpha\}$. By Lemma 7, this limit is the same for all $x \in \mathbf{X}$. We denote it by c_α. $\square$

Corollary 9. *Under the assumptions of Proposition 8, for any compact $A \subset \mathbf{X}$ and for each sequence $\alpha = \{i_1, \dots, i_n, \dots\} \in I^\infty$, there is a topological limit $\lim\limits_{n\to\infty} \{f_{i_1\dots i_n}(A), i_1 i_2 \dots i_n \prec \alpha\} = \{c_\alpha\}$.*

Proposition 10. *The set $K = \{c_\alpha, \alpha \in I^\infty\}$ is compact and satisfies the equation*

$$K = \bigcup_{i\in I} f_i(K).$$

Proof. For a given entourage $W \in \mathcal{U}$, take $U \in \mathcal{B}$ satisfying $U \circ U \subset W$. Take some $x \in \mathbf{X}$ and the number n so that $(f_{i_1\dots i_n}(x), f_{i_1\dots i_{n+k}}(x)) \in U$ for all $k = 0, 1, 2, \dots$ and all $i_1 \dots i_n \dots \in I^\infty$. Then the set K lies in the closure of the set $\bigcup\limits_{i_1\dots i_n \in I^n} U(f_{i_1\dots i_n}(x))$. Therefore the set $\{f_{i_1\dots i_n}(x), i_1 \dots i_n \in I^n\}$ is a finite W-net in K. Thus the set K is totally bounded. Therefore the set $\bar{K}$ is compact.

Take point $x \in \bar{K}$. Since $\bar{K} = \bigcup\limits_{i\in I} f_i(\bar{K})$, $x \in f_i(\bar{K})$ for some $i \in I$. Proceeding by induction we get a nested sequence of compact sets

$$\bar{K} \supset f_{i_1}(\bar{K}) \supset \dots \supset f_{i_1\dots i_n}(\bar{K}) \supset \dots$$

whose intersection is $\{x\}$ by Corollary 9. Let $\alpha = i_1 \dots i_n \dots$. Then $x = c_\alpha$ which shows K is closed and therefore compact. $\square$

The following statements follow from the previous proof.

Corollary 11. 1. *The map $\pi : I^\infty \to K$ defined by $\pi(\alpha) = s_\alpha$ is continuous.*
2. *The set K is the closure of the set of all fixed point of maps $f_{i_1 i_2 .. i_n}$*

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