S(e)MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 14, стр. 1524–1532 (2017) УДК 512.554.7
DOI 10.17377/semi.2017.12.131 MSC 17C20

ROTA—BAXTER OPERATORS OF WEIGHT ZERO ON SIMPLE JORDAN ALGEBRA OF CLIFFORD TYPE

V.YU. GUBAREV

Abstract. It is proved that every Rota—Baxter operator of weight zero on the Jordan algebra of a nondegenerate bilinear symmetric form is nilpotent of index less or equal three. We found exact value of nilpotency index of Rota—Baxter operators of weight zero on simple Jordan algebra of Clifford type over the fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$. For $\mathbb{Z}_p$, we essentially use the results from number theory concerned quadratic residues and Chevalley—Warning theorem.

Keywords: Rota—Baxter operator, Jordan algebra of Clifford type, quadratic residue, Chevalley—Warning theorem.

1. Introduction

In 1933, a physicist P. Jordan introduced a notion of Jordan algebra [1] in quantum mechanics. A Jordan algebra J over a field F of characteristic $\neq 2$ is a commutative (nonassociative) algebra over F satisfying the Jordan identity $(x^2y)x = x^2(yx)$.

Every associative algebra under the new product $x \circ y = \frac{1}{2}(xy + yx)$ is a Jordan algebra. For example, a matrix algebra under $\circ$ turns into a simple Jordan algebra. Moreover, the set of matrices symmetric relative to an involution is a Jordan algebra (of Hermitian type) under the product $\circ$. The set of self-adjoint matrices of order three over the octonions under $\circ$ is a 27-dimensional simple Jordan algebra (of Albert type). Given a vector space V with nondegenerate bilinear symmetric form f, one can equip a space $F \oplus V$ with a structure of simple Jordan algebra provided $\dim V \geq 2$ (Clifford type).

Gubarev, V.Yu., Rota—Baxter operators of weight zero on simple Jordan algebra of Clifford type.

Received October, 6, 2017, published December, 29, 2017.

The first classification theorem for Jordan algebras was obtained by P. Jordan, J. von Neumann, and E. Wigner in 1934 [2]. In 1978–1983, E. Zelmanov was a leader of so called “Russian revolution” [3] in the theory of Jordan algebras. In particular, he proved that every prime (including simple) Jordan algebra without absolute zero divisors is either of Hermitian, Albert or Clifford type [4].

Jordan algebras are deeply connected with Lie algebras and finite simple groups via Kantor—Köcher—Tits construction, the Freudenthal—Tits magic square, etc. Let us refer a reader to the excellent monographs on Jordan algebras [3, 5, 6].

The notions of Rota—Baxter operator and Rota—Baxter algebra were introduced by G. Baxter in 1960 [7] as a generalization of integration by parts formula. Since 1960s, J.-C. Rota, P. Cartier, L. Guo, C. Bai and others studied combinatorial and algebraic properties of Rota—Baxter algebras (see details in the monograph on the subject written by L. Guo in 2012 [8]).

There is a deep connection (see [9–14]) of Rota—Baxter algebras with Yang—Baxter equation, quantum field theory and Loday algebras.

In 1935, C. Chevalley [15] and E. Warning [16] proved two very close theorems which are now gathered in one Chevalley—Warning theorem. This theorem states that over a finite field every system of polynomial equations with sufficiently large number of variables has a solution. From that time, a lot of connections and applications of Chevalley—Warning theorem were found, including combinatorial Nullstellensatz, Erdös—Ginzburg—Ziv theorem, etc. (see, e.g. [17]).

The aim of the current work is to investigate Rota—Baxter operators of weight zero on simple Jordan algebras of nondegenerate bilinear forms. Firstly, we prove that all such operators are nilpotent of index not greater than 3. Secondly, we are interested on the exact value of nilpotency index of Rota—Baxter operators of weight zero. It occurs that the nilpotency index of Rota—Baxter operators of weight zero on simple Jordan algebra of Clifford type depends on the dimension of algebra and on the ground field. For a finite field, we essentially use the Chevalley—Warning theorem and its corollaries.

Throughout of the paper, the characteristic of the ground field F does not equal 2.

2. Preliminaries

2.1. **RB-operator.** Given an algebra A and scalar $\lambda \in F$, where F is a ground field, a linear operator $R\colon A \to A$ is called a Rota—Baxter operator (RB-operator, for short) on A of weight λ if the following identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy) \tag{1}$$

holds for all $x, y \in A$.

Let us list some examples of RB-operators of weight zero (see, e.g., [8]):

Example 1. *Given an algebra A of continuous functions on $\mathbb{R}$, an integration operator $R(f)(x) = \int\limits_0^x f(t)\,dt$ is an RB-operator on A of weight zero.*

Example 2. *A linear map R on the polynomial algebra $F[x]$ defined as $R(x^n) = \frac{x^{n+1}}{n+1}$ is an RB-operator on $F[x]$ of weight zero.*

Example 3. *Given an invertible derivation d on an algebra A, d^{-1} is an RB-operator on A of weight zero.*

Lemma 1. *Let A be a unital algebra, P be an RB-operator on A of weight zero. Then*

a) $1 \notin \operatorname{Im} P$;

b) if A is a simple finite-dimensional algebra and $\dim A > 1$, *then* $\dim\ker P \geq 2$;

c) if $P(1) \in F$, *then* $P(1) = 0$, $P^2 = 0$, *and* $\operatorname{Im} P \subset \ker P$;

d) $P(1)P(1) = 2P^2(1)$.

Proof. a) Assume there exists x such that $P(x) = 1$. We have

$$1 = 1 \cdot 1 = P(x)P(x) = P(P(x)x + xP(x)) = 2P(x) = 2 \cdot 1,$$

a contradiction.

b) Let A be simple, $\dim A = n < \infty$. Suppose that $\dim \operatorname{Im} P = n - 1$. Given a nonzero $x \in \ker P$, we compute

$$0 = P(x)P(y) = P(xP(y)), \quad 0 = P(y)P(x) = P(P(y)x).$$

Therefore, $\ker P$ is closed under multiplication on $\operatorname{Im} P$. As $A = \operatorname{Span}\{1, \operatorname{Im} P\}$, we have a proper ideal $\ker P$ in A, this contradicts simplicity of A.

c) From a), $P(1) = 0$. Other statements follow from

$$0 = P(1)P(x) = P(P(1)x + 1 \cdot P(x)) = P(P(x)).$$

d) Follows directly from (1). □

Lemma 2. *Let an algebra A equal $B \oplus C$ (as vector spaces), where B is a subalgebra of A with an RB-operator $P\colon B \to B$ of weight zero, $P(B)C, CP(B) \subseteq \ker P \oplus C$. Then the linear operator R on A defined as $R(b + c) = P(b)$, $b \in B$, $c \in C$, is an RB-operator of weight zero.*

Proof. Straightforward. □

Remark that the condition $P(B)C, CP(B) \subseteq \ker P \oplus C$ from Lemma 2 holds in the particular case when C is a B-bimodule.

2.2. **Chevalley—Warning theorem.** For our purposes, let us write down the following version of Chevalley—Warning theorem:

Theorem 1 (Chevalley—Warning theorem, [15, 16])**.** *Let F be a finite field and $f_i \in F[x_1, \ldots, x_n]$, $i = 1, \ldots, r$, be homogeneous polynomials of degree d_i respectively. If $d_1 + \ldots + d_r < n$, then there is a nonzero solution in F to $f_1(x) = \ldots = f_r(x) = 0$.*

We also need the following result

Proposition 1 ([17])**.** *Let F be a field of characteristic different from 2 in which each quadratic form in three variables has a nontrivial solution. Then, for any $a, b, c \in F^*$, there exist $x, y \in F$ such that $ax^2 + by^2 = c$.*

For a finite field, Proposition 1 could be derived from [18] as a corollary of Chevalley—Warning theorem.

2.3. **Simple Jordan algebra of Clifford type.** Let V be a n-dimensional vector space, $n \geq 2$, and f be a nondegenerate bilinear symmetric form acting on V. We introduce simple $(n+1)$-dimensional Jordan algebra $J_{n+1}(f)$ as the space $F \cdot 1 \oplus V$ with the following product

$$(\alpha \cdot 1 + v)(\beta \cdot 1 + u) = (\alpha\beta + f(v, u)) \cdot 1 + (\alpha u + \beta v).$$

Let us choose such basis $e_1, e_2, \ldots, e_n$ of V that the matrix of the form f in this basis is diagonal with elements $d_1, d_2, \ldots, d_n$ on the diagonal. As f is nondegenerate, $d_i \neq 0$ for each i. Let us identify the form f with the elements d_i: $f = (d_1, \ldots, d_n)$. By (v, u) for $v = \sum\limits_{i=1}^{n} v_i e_i$, $u = \sum\limits_{i=1}^{n} u_i e_i$, we denote the sum $\sum\limits_{i=1}^{n} d_i v_i u_i$.

It is well-known [6] that $J_{n+1}(f)$ is a quadratic algebra, i.e., every element $x \in J_{n+1}(f)$ satisfies a quadratic equation $x^2 - t(x)x + n(x)1 = 0$ for some $t(x), n(x) \in F$. For $x = \alpha \cdot 1 + v$, $\alpha \in F$, $v \in V$, we have $t(x) = 2\alpha$, $n(x) = \alpha^2 - (v, v)$.

3. Bound theorem and 3-dimensional case

Theorem 2. *Let J be a (not necessary simple or finite-dimensional) Jordan algebra of a bilinear form. Then $R^3 = 0$ for every RB-operator of weight zero on J.*

Proof. Let R be an RB-operator on J of weight zero. Firstly, if $R(1) = 0$, then $R^2 = 0$ by Lemma 1, c).

Secondly, if $R(1) \neq 0$, let us prove that $t = t(R(1)) = 0$. Indeed, assume $t \neq 0$. Then $R(1) = \alpha \cdot 1 + a$, $\alpha = t/2$, $a \in V$. Lemma 1, d) implies $R(a) = \frac{1}{2}((a, a) \cdot 1 - \alpha^2 \cdot 1) \in F$. Therefore, $R(a) = 0$, and $(a, a) = \alpha^2$. Relation (1) leads to

$$0 = R(1)R(a) = R(R(1)a + R(a)) = \alpha^2 R(1),$$

so $\alpha = t = 0$.

Suppose that $t = 0$. By Lemma 1, d)

$$\begin{aligned} 0 = R^2(1)R(x) &= R(R^2(1)x + R(1)R(x)) \\ &= R^2(R(1)x + R(x)) = R^2(R(1)x) + R^3(x) \end{aligned} \tag{2}$$

for every $x \in V$. As $R(1) \in V$, we have that $R(1)x \in F$ for all $x \in V$. So $R^2(R(1)x) = 0 = R^3(x)$, $x \in V$. Together with $R^3(1) = R(R^2(1)) = 0$ we obtain that $R^3 = 0$ on the entire J.

Corollary 1. *Let J be a Jordan algebra of a bilinear form and R be an RB-operator on J of weight zero. Then a) $R(1) \in V$, b) $R^2(1) = R(1)R(1) = 0$.*

Proposition 2. *All RB-operators of weight zero on simple Jordan algebra $J_3(f)$, $f = (d_1, d_2)$, are the following*

$$R(1) = 0, \quad R(e_1) = k(\alpha \cdot 1 + \beta e_1 + \gamma e_2), \quad R(e_2) = l(\alpha \cdot 1 + \beta e_1 + \gamma e_2) \tag{3}$$

for such $\alpha, \beta, \gamma, k, l \in F$ that $\alpha^2 - d_1\beta^2 - d_2\gamma^2 = 0$ and $k\beta + l\gamma = 0$. Moreover, $R^2 = 0$.

Proof. Let R be a nonzero RB-operator on $J_3(f)$ of weight zero. By Lemma 1, b), $\dim \ker R = 2$ and $\dim \operatorname{Im} R = 1$. So, by Theorem 2 we have $R^2 = 0$ and $\operatorname{Im} R \subset \ker R$. Proving $R(1) = 0$, we have exactly (3). The condition $\alpha^2 - d_1\beta^2 - d_2\gamma^2 = 0$ follows from the fact that $\operatorname{Im} R$ is a subalgebra of $J_3(f)$ and $k\beta + l\gamma = 0$ is a consequence of the equation (1) considered for $R(1)R(e_1)$.

Assume that $R(1) \neq 0$. By Corollary 1, a), $R(1) = \alpha e_1 + \beta e_2$ for some nonzero $\alpha, \beta \in F$. Let $x = \chi \cdot 1 + \gamma e_1 + \delta e_2$ be such an element of $\ker R$ that the vectors $R(1)$ and $\gamma e_1 + \delta e_2$ are linear independent. From

$$0 = R(1)R(x) = R(R(1)x) = R(\chi R(1) + \eta \cdot 1) = \eta R(1),$$

where $\eta = d_1\gamma\alpha + d_2\delta\beta$, we get $\eta = 0$. Together with $d_1\alpha^2 + d_2\beta^2 = 0$, a condition of $\operatorname{Im} R$ being a subalgebra of $J_3(f)$, we have $\alpha\delta = \beta\gamma$, i.e., $R(1)$ and $\gamma e_1 + \delta e_2$ are linear dependent, a contradiction.

In light of Theorem 2, define nilpotency index $\mathrm{rb}(J_{n+1}(f))$ as the minimal natural number s such that every RB-operator on $J_{n+1}(f)$ of weight zero is nilpotent of index s.

Corollary 2. *Let F be an algebraically closed or finite field, then $rb(J_3) = 2$.*

Proof. The main question is to find a nonzero solution (α, β, γ) of the equation $\alpha^2 - d_1\beta^2 - d_2\gamma^2 = 0$. Over an algebraically closed field it is enough to consider $\gamma = 1$, $\beta = 0$, $\alpha = \pm\sqrt{d_2}$. For a finite field, we apply Chevalley—Warning theorem.

The condition $k\beta + l\gamma = 0$ could be easily satisfied. At least one of the numbers β, γ is nonzero. Suppose that $\beta \neq 0$, choose any $l \neq 0$ and put $k = -l\gamma/\beta$.

Remark 1. *The statement of Corollary 2 could be not true for other fields. For example, in the case $F = \mathbb{R}$ we have $rb(J_3(f)) = 2$ if and only if f is not negative definite form.*

4. General case

Lemma 3. *Let $J_{n+1}(f)$ be $(n+1)$-dimensional simple Jordan algebra of the form $f = (d_1, \ldots, d_n)$. Let $R(1) = k_1e_1 + \ldots + k_ne_n$, $R(e_i) = \alpha_0 \cdot 1 + \alpha_1e_1 + \ldots + \alpha_ne_n$, $R(e_j) = \beta_0 \cdot 1 + \beta_1e_1 + \ldots + \beta_ne_n$, $k_i, \alpha_i, \beta_i \in F$. Suppose, $R(1) \neq 0$. Then*

a) $R(1)R(e_i) = \alpha_0R(1)$ and $(R(1), R(e_i) - \alpha_0 \cdot 1) = 0$;

b) $R^2(e_i) = (\alpha_0 - d_ik_i)R(1)$;

c) $R(e_i)R(e_j) = \alpha_0R(e_j) + \beta_0R(e_i)$, $(R(e_i) - \alpha_0 \cdot 1, R(e_j) - \beta_0 \cdot 1) = \alpha_0\beta_0$, and $d_j\alpha_j + d_i\beta_i = 0$.

d) $R(e_i)R(e_i) = 2\alpha_0R(e_i)$, $(R(e_i) - \alpha_0 \cdot 1, R(e_i) - \alpha_0 \cdot 1) = \alpha_0^2$, and $\alpha_i = 0$.

Proof. a) By Lemma 1, a), $R(1)R(e_i) = \alpha_0R(1) + (R(1), R(e_i) - \alpha_0 \cdot 1) \cdot 1 = \alpha_0R(1)$ and $(R(1), R(e_i) - \alpha_0 \cdot 1) = 0$.

b) Follows from a) and (1) applied for $R(1)R(e_i)$.

c) Let $v = R(e_i) - \alpha_0 \cdot 1$, $u = R(e_j) - \beta_0 \cdot 1$. On the one hand,

$$\begin{aligned} R(e_i)R(e_j) &= (\alpha_0\beta_0 + (v, u)) \cdot 1 + \alpha_0u + \beta_0v \\ &= (-\alpha_0\beta_0 + (v, u)) \cdot 1 + \alpha_0R(e_j) + \beta_0R(e_i). \end{aligned} \tag{4}$$

On the another hand,

$$\begin{aligned} R(R(e_i)e_j + e_iR(e_j)) &= R(\alpha_0e_j + d_j\alpha_j \cdot 1) + R(\beta_0e_i + d_i\beta_i \cdot 1) \\ &= (d_j\alpha_j + d_i\beta_i)R(1) + \alpha_0R(e_j) + \beta_0R(e_i). \end{aligned} \tag{5}$$

Comparing (4) and (5), we have done.

d) Follows from c) for $i = j$.

Remark 2. *Suppose that R is an RB-operator on $J_{n+1}(f)$ such that (I) $R(1) \neq 0$ and $R^2 = 0$; (II) x^2+1 has roots $(\pm i)$ in F and $\sqrt{d_j} \in F$ for all $j = 1, \ldots, n$. Define $e_i' = e_i/\sqrt{d_i}$. Let $A = (a_{kl}) \in M_{n+1}(F)$ be the matrix of R in the basis $1, e_1', \ldots, e_n'$ and let M be its submatrix of size $n \times n$ formed by all rows and columns except the first ones. By Lemma 3, c) M is skew-symmetric; by Lemma 3, b), the first column* *of A coincides with the transpose of the first row of A. Let us define a matrix $B = (b_{kl}) \in M_{n+1}(F)$ as follows:*

$$b_{kl} = \begin{cases} a_{kl}, & k = l = 1 \text{ or } k, l > 1, \\ i a_{kl}, & l = 1, k > 1, \\ -i a_{kl}, & k = 1, l > 1. \end{cases} \tag{6}$$

The matrix B is skew-symmetric. By Lemma 3, one can show the fact that R is an RB-operator on $J_{n+1}(f)$ satisfying the conditions (I) and (II) is equivalent to the equality $B^T B = 0$ or $B^2 = 0$ for the skew-symmetric matrix B defined by (6). In particular, over an algebraically closed field F, we have the correspondence between the set of RB-operators on $J_{n+1}(f)$ satisfying (I) and the set of all skew-symmetric matrices from $M_{n+1}(F)$ whose squares are zero.

Example 4. *Let $J_{n+1}(f)$ be $(n+1)$-dimensional simple Jordan algebra of the form $f = (d_1, \ldots, d_n)$. Let $l_1, \ldots, l_m, k_{m+1}, \ldots, k_n \in F$ be such that $k_j \neq 0$ for some j and*

$$\sum_{i=1}^{m} d_i l_i^2 = 1, \qquad \sum_{j=m+1}^{n} d_j k_j^2 = 0. \tag{7}$$

Then the the following linear map on $J_{n+1}(f)$

$$R(e_i) = \begin{cases} k_{m+1} e_{m+1} + \ldots + k_n e_n, & i = 0, \\ d_i l_i (k_{m+1} e_{m+1} + \ldots + k_n e_n), & 1 \leq i \leq p, \\ -d_i k_i (1 + l_1 e_1 + \ldots + l_m e_m), & m+1 \leq i \leq n, \end{cases} \tag{8}$$

is an RB-operator of weight zero. Moreover, $R^2 \neq 0$ and $rb(J_{n+1}(f)) = 3$.

Example 5. *Let $J_4(f)$ be 4-dimensional simple Jordan algebra of the form $f = (d_1, d_2, d_3)$. Moreover, there exist a solution x_0 of the equation $x^2 + d_1 d_2 d_3 = 0$ and such $k_1, k_2, k_3 \in F^*$ that $d_1 k_1^2 + d_2 k_2^2 + d_3 k_3^2 = 0$. Then the the following linear map on $J_4(f)$*

$$\begin{gathered} R(1) = d_1 k_1^2 e_1' + d_2 k_2^2 e_2' + d_3 k_3^2 e_3', \quad R(e_1') = -1 + \frac{\lambda}{d_1 k_1^2}(e_2' - e_3'), \\ R(e_2') = -1 + \frac{\lambda}{d_2 k_2^2}(-e_1' + e_3'), \quad R(e_3') = -1 + \frac{\lambda}{d_3 k_3^2}(e_1' - e_2'), \end{gathered} \tag{9}$$

for $\lambda = k_1 k_2 k_3 x_0$, $e' = e_i/(k_i d_i)$, is an RB-operator of weight zero and $rb(J_4(f))$=3.

Theorem 3. *Let F be a field, $J_{n+1}(f)$ be a simple Jordan algebra over F of the form $f = (d_1, \ldots, d_n)$, $n \geq 3$.*

a) If $x^2 + 1$ has roots in F and there exist pairwise different $j_1, j_2, j_3 \in \{1, \ldots, n\}$ such that $\sqrt{d_{j_s}} \in F$, $s = 1, 2, 3$, then $rb(J_{n+1}(f)) = 3$.

b) For $F = \mathbb{R}$, we have

$$rb(J_{n+1}(f)) = \begin{cases} 1, & d_1, \ldots, d_n < 0, \\ 2, & d_1, \ldots, d_n > 0 \text{ or } d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n < 0,\ d_i > 0, \\ 3, & \text{otherwise}. \end{cases}$$

c) If F is a finite field, then $rb(J_{n+1}(f)) = 3$ for all $n \geq 5$.

d) If $F = \mathbb{Z}_p$ for a prime p of the form $4t+1$, then we have

$$rb(J_{n+1}(f)) = \begin{cases} 2, & n=3, \textit{ even number of quadratic residues among } d_1, d_2, d_3, \\ 3, & \textit{otherwise}. \end{cases}$$

e) If $F = \mathbb{Z}_p$ for a prime p of the form $4t+3$, then we have

$$rb(J_{n+1}(f)) = \begin{cases} 2, & n=3, \textit{ odd number of quadratic residues among } d_1, d_2, d_3, \\ 3, & \textit{otherwise}. \end{cases}$$

Proof. a) Let $n = 3$. By Example 4, the following linear map P on $J_4(f)$

$$\begin{gathered} P(1) = \frac{e_2}{\sqrt{d_2}} + \frac{ie_3}{\sqrt{d_3}}, \quad P(e_1) = \sqrt{d_1}\left(\frac{e_2}{\sqrt{d_2}} + \frac{ie_3}{\sqrt{d_3}}\right), \\ P(e_2) = -\sqrt{d_2}\left(1 + \frac{e_1}{\sqrt{d_1}}\right), \quad P(e_3) = -\sqrt{d_3}\left(1 + \frac{e_1}{\sqrt{d_1}}\right) \end{gathered} \tag{10}$$

is an RB-operator of weight zero and $P^2 \neq 0$.

Let $n > 3$ and suppose that $j_s = s$, $s = 1, 2, 3$. Define a linear map R on J by Lemma 2 for $B = \mathrm{Span}\{1, e_1, e_2, e_3\}$ with P and $C = \mathrm{Span}\{e_4, \ldots, e_n\}$. This is an RB-operator on $J_{n+1}(f)$ by Lemma 2.

b) For all negative d_i, suppose there exists a nonzero RB-operator R of weight zero on $J_{n+1}(f)$, i.e., we can find $(0 \neq) a = \alpha_0 \cdot 1 + \sum\limits_{i=1}^{n} \alpha_i e_i \in \mathrm{Im}\,(R)$. So, $\alpha^2 - \sum\limits_{i=1}^{n} d_i \alpha_i^2 = 0$, a contradiction.

Let us consider the case when all d_i are positive. By Corollary 1, b), $R(1) = 0$ for an RB-operator R of weight zero on $J_{n+1}(f)$ and then $\mathrm{rb}(J_{n+1}(f)) \leq 2$. By Proposition 2, we can construct a nonzero RB-operator P on $J_2(f')$ for $f' = (d_1, d_2)$. Further, we extend P on the entire algebra $J_{n+1}(f)$ as in a). So, $\mathrm{rb}(J_{n+1}(f)) = 2$.

Suppose that $d_2, \ldots, d_n < 0$ and $d_1 > 0$. By Remark 1, we have that $\mathrm{rb}(J_{n+1}(f)) \geq 2$. Let R be such an RB-operator on $J_{n+1}(f)$ that $R(1) \neq 0$ (otherwise we have done by Lemma 1, c)). Denote $R(e_i) = \sum\limits_{j=0}^{n} \alpha_{ij} e_j$, $i = 1, \ldots, n$, where $e_0 = 1$. By Lemma 3, d), $\alpha_{11} = 0$ and so $R(e_1) = 0$. From Lemma 3, c), we get $\alpha_{i1} = 0$ for all $i = 1, \ldots, n$. As above, $R(e_i) = 0$ for all $i = 1, \ldots, n$. By Corollary 1, b), $R^2 = 0$. Thus, $\mathrm{rb}(J_{n+1}(f)) = 2$ by Proposition 2.

Finally, let $d_1, d_2 > 0$, $d_3 < 0$, all other d_i are of arbitrary sign. We apply Example 4 for $m = 1$ to obtain $\mathrm{rb}(J_{n+1}(f)) = 3$.

c) Let $n = 5$ and $m = 2$. By Chevalley—Warning theorem, the second equation (7) has a nonzero solution (k_3, k_4, k_5). By Proposition 1, the first equation of (7) also has a solution. Applying Example 4, we are done.

We deal with $n > 5$ in analogous way as in a).

d) Let $n = 3$. Suppose d_1 is a quadratic residue modulo p and d_2, d_3 are simultaneously quadratic residues or nonresidues. By Example 4, for $m = 1$, we have $\mathrm{rb}(J_4(f)) = 3$.

Suppose there is an odd number of quadratic residues through d_1, d_2, d_3. To the contrary, let R be an RB-operator of zero weight on $J_4(f)$ such that $R^2 \neq 0$. So $R(1) = k_1 e_1 + k_2 e_2 + k_3 e_3 \neq 0$. Without loss of generality we assume that $k_3 \neq 0$. By Lemma 1, b) and assumption that $R^2 \neq 0$, we have $\dim \mathrm{Im}\, R = \dim \ker R = 2$.

Consider $R(e_1) = \alpha_0 \cdot 1 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$. By Lemma 3, d), $\alpha_1 = 0$. By Lemma 3, a), $d_2k_2\alpha_2 + d_3k_3\alpha_3 = 0$ and by Lemma 3, d), $\alpha_0^2 = d_2\alpha_2^2 + d_3\alpha_3^2$. From the last two equalities and Corollary 1, b), we conclude that

$$(11)\quad \alpha_0^2 = d_2\alpha_2^2\left(1 + \frac{d_3}{d_2}\left(\frac{d_2k_2}{d_3k_3}\right)^2\right) = d_2\alpha_2^2\left(1 + \frac{d_2k_2^2}{d_3k_3^2}\right) = \frac{d_2\alpha_2^2}{d_3k_3^2}(d_2k_2^2 + d_3k_3^2) = -\frac{d_2d_1k_1^2\alpha_2^2}{d_3k_3^2}.$$

If all k_i are nonzero, then by (11), $(d_3k_3\alpha_0, k_1\alpha_2)$ is a solution of the equation $x^2 + d_1d_2d_3y^2 = 0$, it is a contradiction with a choice of d_1, d_2, d_3.

Let $k_1 = 0$ and k_2, k_3 be nonzero. Suppose that $R(1)$ and $R(e_1)$ form a basis of $\operatorname{Im} R$. By (11), $\alpha_0 = 0$ and $\operatorname{Im} R = \operatorname{Span}\{e_2, e_3\}$. So, coordinates of $R(e_2)$ and $R(e_3)$ on $1, e_1$ are zero. Applying Lemma 3, c), we have $R(e_1) = 0$, a contradiction.

Suppose that $R(1)$ and $R(e_2) = \beta_0 \cdot 1 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3$ form a basis of $\operatorname{Im} R$. By Lemma 3, d), $\beta_2 = 0$. As above, we have $\beta_3 = 0$. So, the equations $x^2 - d_1y^2 = 0$ and $d_2z^2 + d_3t^2 = 0$ have nonzero solutions (β_0, β_1) and (k_2, k_3) respectively, it contradicts a choice of d_1, d_2, d_3.

Let $n \geq 4$, then by the properties of quadratic residues we have that at least one of the equations $d_ix^2 + d_jy^2 = 0$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, has a nonzero solution. The rest follows due to the arguments from b).

e) Let $n = 3$. If d_1, d_2 are quadratic residues modulo p and d_3 is not, we have done by Example 4 for $m = 1$. If d_1, d_2, d_3 are quadratic nonresidues modulo p, we have $\operatorname{rb}(J_4(f)) = 3$ by Example 5. Indeed, by Chevalley—Warning theorem, the second equation (7) has a nonzero solution (k_1, k_2, k_3). The equation $x^2 + d_1d_2d_3 = 0$ has a solution by the properties of quadratic residues. The proof in the case when $R(1)$, $R(e_3)$ form a basis of $\operatorname{Im} R$ is analogous.

Supposing that there is an even number of quadratic residues through d_1, d_2, d_3, we have $\operatorname{rb}(J_4(f)) = 2$ by analogous arguments as in d).

For $n \geq 4$, if not all d_i are quadratic residues modulo p, we have $\operatorname{rb}(J_{n+1}(f)) = 3$ by extending by Lemma 2 an RB-operator from $n = 3$. When all d_i are quadratic residues modulo p, we have done by Example 4 for $m = 1$.

Corollary 3. *Let F be an algebraically closed field and $J_{n+1}(f)$ be a simple Jordan algebra of a bilinear form f over F. We have* $\operatorname{rb}(J_{n+1}(f)) = \begin{cases} 2, & n = 2, \\ 3, & n \geq 3. \end{cases}$

Corollary 4. *Let F be an algebraically closed or finite field and $J_\infty(f)$ denote a simple infinite dimensional Jordan algebra over F of a diagonalized bilinear form f. Then $\operatorname{rb}(J_\infty(f)) = 3$.*

Example 6. *Let $F = \mathbb{Z}_7$, $J_4(f)$ be a 4-dimensional simple Jordan algebra of the form $f = (-1, -1, -1)$. The following RB-operator R on $J_4(f)$ is defined by Example 5:*

$$R(1) = e_1 + 2e_2 + 3e_3, \quad R(e_1) = 1 + 4e_2 + 2e_3,$$
$$R(e_2) = 2 + 3e_1 + 6e_3, \quad R(e_3) = 3 + 5e_1 + e_2.$$

For investigating $\operatorname{rb}(J_k(f))$ over the field $\mathbb{Q}$, the Hasse—Minkowski theorem could be used. It says that a quadratic form S has a rational nonzero solution if and only

S has nonzero solutions over the real numbers and over the p-adic numbers for every prime p.

Example 7. *Let $F = \mathbb{Q}$, $J_4(f)$ be a 4-dimensional simple Jordan algebra of the form $f = (1, -3, 1)$. Then $rb(J_4(f)) = 2$. Indeed, by Proposition 2, $rb(J_4(f)) \geq 2$. Suppose, there exists an RB-operator R on $J_4(f)$ such that $R^2 \neq 0$. Thus, $R(1) = k_1e_1 + k_2e_2 + k_3e_3 \neq 0$ and $k_1^2 - 3k_2^2 + k_3^2 = 0$. This equation has no nonzero solutions in $\mathbb{Q}$ because the Pell equation $x^2 - 3y^2 = -1$ has no integer solutions.*

References

[1] Jordan P., *Über Verallgemeinerungsmoglichkeiten des Formalismus der Quantenmechanik*, Nachr. Akad. Wiss. Gottingen. Math. Phys. Kl. I, **41** (1933), 209–217. Zbl 0007.08502

[2] Jordan P., von Neumann J., Wigner E., *On an Algebraic Generalization of the Quantum Mechanical Formalism*, Ann. Math., **35**:1 (1934), 29–64. MR1503141

[3] McCrimmon K., *A taste of Jordan algebras*, New-York: Springer-Verlag New York Inc., 2004. MR2014924

[4] Zelmanov E., *On prime Jordan algebras. II*, Siberian Math. J., **24**:1 (1983), 73–85. MR0688595

[5] Jacobson N. *Structure and representations of Jordan algebras*, AMS Colloquium Publications, **XXXIX**, Providence, 1968. MR0251099

[6] Zhevlakov K.A., Slin'ko A.M., Shestakov I.P., and Shirshov A.I., *Rings that are nearly associative*, New-York: Academic Press, 1982. MR0668355

[7] Baxter G., *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math., **10** (1960), 731–742. MR0119224

[8] Guo L., *An Introduction to Rota–Baxter Algebra*, Surveys of Modern Mathematics., **4**, International Press, Somerville; Higher education press, Beijing, 2012. MR3025028

[9] Belavin A.A., Drinfel'd V.G., *Solutions of the classical Yang-Baxter equation for simple Lie algebras*, Funct. Anal. Appl., **16**:3 (1982), 159–180. MR0674005

[10] Semenov-Tyan-Shanskii M.A., *What is a classical r-matrix?*, Funct. Anal. Appl., **17**:4 (1983), 259–272. MR0725413

[11] Aguiar M., *Pre-Poisson algebras*, Lett. Math. Phys., **54** (2000), 263–277. MR1846958

[12] Connes A., Kreimer D., *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys., **210**:1 (2000), 249–273. MR1748177

[13] Bai C., Bellier O., Guo L., Ni X., *Splitting of operations, Manin products, and Rota–Baxter operators*, Int. Math. Res. Notices, **3** (2013), 485–524. MR3021790

[14] Gubarev V., Kolesnikov P., *Embedding of dendriform algebras into Rota—Baxter algebras*, Cent. Eur. J. Math., **11**:2 (2013), 226–245. MR3000640

[15] Chevalley C., *Démonstration d'une hypothèse de M. Artin*, Abh. Math. Sem. Univ. Hamburg, **11** (1935), 73–75. MR3069644

[16] Warning E., *Bemerkung zur vorstehenden Arbeit von Herrn Chevalley*, Abh. Math. Sem. Univ. Hamburg, **11** (1935), 76–83. MR3069645

[17] Clark P.L., *The Chevalley-Warning Theorem (Featuring... the Erdös-Ginzburg-Ziv Theorem)*, http://math.uga.edu/ pete/4400ChevalleyWarning.pdf

[18] Heath-Brown D.R., *A Note on the Chevalley—Warning Theorems*, Russian Math. Surveys, **66**:2 (2011), 427–436. MR2847793

Vsevolod Yuryevich Gubarev
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
Novosibirsk State University,
Pirogova str., 2,
630090, Novosibirsk, Russia
E-mail address: `wsewolod89@gmail.com`