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ON ROTA—BAXTER OPERATORS OF NON-ZERO WEIGHT ARISEN FROM THE SOLUTIONS OF THE CLASSICAL YANG—BAXTER EQUATION

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Abstract. Let L be a simple anti-commutative algebra. In this paper we prove that a non skew-symmetric solution of the classical Yang—Baxter equation on L with L-invariant symmetric part induces on L a Rota—Baxter operator of a non-zero weight.

Keywords: Rota—Baxter operator, anti-commutative algebra, Lie algebra, Malcev algebra, non-associative bialgebra, classical Yang—Baxter equation.

Given an algebra A over a field F and scalar $\lambda \in F$, a linear operator $R : A \to A$ is called a Rota—Baxter of the weight λ if for all $x, y \in A$ the following identity holds:

(1) $$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy).$$

As an example of a Rota—Baxter operation of weight zero one can consider the operation of integration on the algebra of continuous functions on $\mathbb{R}$: the equation (1) follows from the integration by parts formula.

An algebra with a Rota—Baxter operation is called a Rota—Baxter algebra. These algebras first appeared in the paper of Baxter [1]. The combinatorial properties of Rota—Baxter algebras and operations were studied in papers of F.V. Atkinson, P. Cartier, G.-C. Rota and the others (see [2]-[5]). For basic results and the main properties of Rota—Baxter algebras see [6].

There is a standard method for constructing Rota—Baxter operations of weight zero on a simple Lie algebra L from skew-symmetric solutions of the classical Yang—Baxter equations (CYBE): if $r = \sum a_i \otimes b_i$ is a skew-symmetric solution of CYBE,

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then one can define an operator R on L by $R(a) = \sum \langle a_i, a\rangle b_i$, where $\langle \cdot, \cdot\rangle$ is a Killing form on L. It turns out that R is a Rota—Baxter operator of weight 0 ([9, 10]).

In this paper we consider the case when L is a simple anti-commutative algebra and $r \in L \otimes L$ is a non skew-symmetric solution of CYBE with L-invariant symmetric part. It turns out that instead of the element r it is more convenient to consider structure that closely connected with the solutions of CYBE (skew-symmetric or not): the structure of bialgebra (a vector space with multiplication and comultiplication).

Lie bialgebras were introduced by Drinfeld [7] for studying the solutions of the classical Yang—Baxter equation on Lie algebras. Lie bialgebras are Lie algebras and Lie coalgebras at the same time, such that comultiplication is a 1-cocycle. Every skew-symmetric solution of the classical Yang—Baxter equation induces a structure of a Lie bialgebra on the corresponding Lie algebra L. It is known that in this case the Drinfeld double contains a non-zero radical.

If r is not a skew-symmetric solution of CYBE then the corresponding comultiplication gives a structure of a Lie bialgebra if and only if $r + \tau(r)$ is a L-invariant element of $L \otimes L$. Here τ is a switch morphism.

In section 2 we consider a structure of bialgebra on an arbitrary simple finite-dimensional algebra A over a field of characteristic zero such that the Drinfeld (classical) double is a direct sum of two simple ideals. We prove that the structure induces on A Rota—Baxter operators of a non-zero weight.

In section 3 we consider the case when A is an anti-commutative simple algebra. We prove that in this case Rota—Baxter operators from section 2 are induced by non skew-symmetric solutions of the classical Yang—Baxter equations with ad-invariant symmetric part. As a corollary we obtain that if L is a simple Lie algebra over an algebraically closed field and $r = \sum a_i \otimes b_i$ is a non skew-symmetric solutions of the classical Yang—Baxter equations such that $r + \tau(r)$ is L-invariant, then an operator R on L defined by $R(a) = \sum \langle a_i, a\rangle b_i$ is a Rota—Baxter operator of a non-zero weight.

In the last section we use the results from the section 3 and construct Rota—Baxter operators of non-zero weight on simple non-Lie Malcev algebra.

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1. Definitions and preliminary results

In the paper it is assumed that the characteristic of the ground field F is 0 and all spaces are supposed to be finite-dimensional.

Given vector spaces V and U over a field F, denote by $V \otimes U$ its tensor product over F. Define the linear mapping τ on V by $\tau(\sum\limits_i a_i \otimes b_i) = \sum\limits_i b_i \otimes a_i$. Denote by V^* the dual space of V.

Let L be an anti-commutative algebra with multiplication $[\cdot, \cdot]$. Recall that L acts on $L^{\otimes n}$ by

$$[x_1 \otimes x_2 \otimes \ldots \otimes x_n, y] = \sum_i x_1 \otimes \ldots \otimes [x_i, y] \otimes \ldots \otimes x_n$$

for all $x_i, y \in L$.

An element $r \in L^{\otimes n}$ is called L-invariant (or ad-invariant) if $[r, y] = 0$ for all $y \in L$.

Definition 1. *A pair* (A, Δ)*, where* A *is a vector space over* F *and* $\Delta : A \to A \otimes A$ *is a linear mapping, is called a coalgebra, while* Δ *is a comultiplication.*

Given $a \in A$, put $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.
Define a multiplication on A^* by

$$fg(a) = \sum f(a_{(1)})g(a_{(2)}),$$

where $f, g \in A^*$, $a \in A$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. The algebra obtained is *the dual algebra* of the coalgebra (A, Δ).

The following definition of a coalgebra related to some variety of algebras was given in [15].

Definition 2. *Let* $\mathcal{M}$ *be an arbitrary variety of algebras. The pair* (A, Δ) *is called a* $\mathcal{M}$*-coalgebra if* A^* *belongs to* $\mathcal{M}$*.*

The dual algebra A^* of (A, Δ) gives rise to the following bimodule actions on A:

$$f \rightharpoonup a = \sum a_{(1)} f(a_{(2)}) \text{ and } a \leftharpoonup f = \sum f(a_{(1)}) a_{(2)},$$

where $a \in A,\ f \in A^*$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

Let A be an arbitrary algebra with a comultiplication Δ, and let A^* be the dual algebra for (A, Δ). Then A induces the bimodule action on A^* by the formulas

$$f \leftharpoondown a(b) = f(ab) \text{ and } b \rightharpoondown f(a) = f(ab),$$

where $a, b \in A,\ f \in A^*$.

Consider the space $D(A) = A \oplus A^*$ and equip it with a multiplication by putting

$$(a + f)(b + g) = (ab + f \rightharpoonup b + a \leftharpoonup g) + (fg + f \leftharpoondown b + a \rightharpoondown g).$$

Then $D(A)$ is an ordinary algebra over F, A and A^* are some subalgebras in $D(A)$. It is called *the classical double* [8] *or Drinfeld double* [13].

Let Q be a bilinear form on $D(A)$ defined by

$$Q(a + f, b + g) = g(a) + f(b)$$

for all $a, b \in A$ and $f, g \in A^*$. It is easy to check that Q is a nondegenerate symmetric associative form, i.e. $Q(xy, z) = Q(x, yz)$.

In [7] the following definition was given:

Definition 3. *Let* L *be a Lie algebra with a comultiplication* Δ*. The pair* (L, Δ) *is called a Lie bialgebra if and only if* (L, Δ) *is a Lie coalgebra and* Δ *is a 1-cocycle, i.e., it satisfies*

$$\Delta([a, b]) = \sum([a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b]) + \sum([a, b_{(1)}] \otimes b_{(2)} + b_{(1)} \otimes [a, b_{(2)}])$$

for all $a, b \in L$*.*

In [7], it was proved that the pair (L, Δ) is a Lie bialgebra if and only if its Drinfeld double $D(L)$ is a Lie algebra.

There is an important type of Lie bialgebras called coboundary bialgebras. Namely, let L be a Lie algebra and $r = \sum\limits_i a_i \otimes b_i$ from $(id - \tau)(L \otimes L)$, that is, $\tau(r) = -r$. Define a comultiplication Δ_r on L by

$$\Delta_r(a) = [r, a] = \sum_i [a_i, a] \otimes b_i - a_i \otimes [a, b_i]$$

for all $a \in L$. It is easy to see that Δ_r is a 1-cocycle. In [9] it was proved that (L, Δ_r) is a Lie coalgebra if and only if the element

$$C_L(r) = [r_{12}, r_{13}] - [r_{23}, r_{12}] + [r_{13}, r_{23}]$$

is L-invariant. Here $[r_{12}, r_{13}] = \sum\limits_{ij}[a_i, a_j] \otimes b_i \otimes b_j$, $[r_{23}, r_{12}] = \sum\limits_{ij} a_i \otimes [a_j, b_i] \otimes b_j$, and $[r_{13}, r_{23}] = \sum\limits_{ij} a_i \otimes a_j \otimes [b_i, b_j]$. In particular, if

$$\sum_{ij}[a_i, a_j] \otimes b_i \otimes b_j - a_i \otimes [a_j, b_i] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j] = 0, \tag{2}$$

then the pair (L, Δ_r) is a Lie bialgebra. The equation (2) is called *the classical Yang—Baxter equation.*

The equation (2) can de considered for every variety of algebras. Let A be an arbitrary algebra and $r = \sum\limits_i a_i \otimes b_i \in A \otimes A$. Then the equation

$$C_A(r) = r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0 \tag{3}$$

is called the classical Yang—Baxter equation on A. Here the subscripts specify the way of embedding $A \otimes A$ into $A \otimes A \otimes A$, i.e. $r_{12} = \sum\limits_i a_i \otimes b_i \otimes 1$, $r_{13} = \sum_i a_i \otimes 1 \otimes b_i$, $r_{23} = \sum_i 1 \otimes a_i \otimes b_i$. Note that $C_A(r)$ is well defined even if A is non-unital. The equation for different varieties of algebras was considered in [12, 23, 13, 24, 14].

An element $r = \sum\limits_i a_i \otimes b_i \in A \otimes A$ induces a comultiplication Δ_r on A:

$$\Delta_r(a) = \sum a_i a \otimes b_i - a_i \otimes ab_i$$

for all $a \in A$.

In [11] the following definition of bialgebra in sense of Drinfeld for any variety of algebras was given:

Definition 4. *Let $\mathcal{M}$ be an arbitrary variety of algebras and let A be an algebra from $\mathcal{M}$ with a comultiplication Δ. The pair (A, Δ) is called* an $\mathcal{M}$-bialgebra (in the sense of Drinfeld) *if its Drinfeld double $D(A)$ belongs to $\mathcal{M}$.*

For Jordan, associative, alternative and Malcev bialgebras it is known that if r is a skew-symmetric solution of the classical Yang—Baxter equation (3) on A, then (A, Δ_r) is a bialgebra of corresponding variety.

In [13] it was proved that if A is an arbitrary simple algebra equipped with a comultiplication Δ and I is a proper ideal of $D(A)$, then $\dim(I) = \dim(A)$.

If L is a simple finite-dimensional Lie algebra over the field of complex numbers, then the Drinfild double $D(L)$ either contains a radical R such that $D(L) = L \oplus R$ (semidirect sum) and $R^2 = 0$ or $D(L)$ is a direct sum of two simple ideals that are isomorphic to L [8]. In the same paper bialgebra structures on L in both cases were described.

If (L, Δ) is a Lie bialgebra and the radical R of $D(L)$ is not zero then the comultiplication is induced by a skew-symmetric solution of the classical Yang—Baxter equation, i.e. $\Delta = \Delta_r$ for some $r = \sum a_i \otimes b_i \in (id - \tau)(L \otimes L)$ such that $C_L(r) = 0$ (see [8] for the case when F is the field of complex numbers $\mathbb{C}$ or [16] for the more general case when L is a Malcev algebra). Define an operator $R : L \to L$ by:

$$R(x) = \sum \langle a_i, x\rangle b_i, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ is the Killing form on L. It is well known that R is a Rota—Baxter operator of weight zero on L (see [9, 10]).

2. Bialgebras and Rota—Baxter operators of non-zero weights

In this section A is an arbitrary simple finite-dimensional algebra over a field of characteristic zero equipped with a comultiplication Δ such that (A,Δ) is a bialgebra such that $D(A)$ is a direct sum of simple ideals.

Since $\dim(I)=\dim(A)$ for every proper ideal I of $D(A)$, we have that $D(A)=A_1\oplus A_2$ where A_i are simple algebras and $\dim(A_i)=\dim(A)$ [13].

Proposition 1. *There exist two linear mapping $\varphi_i : A^*\to A$, $i=1,2$, such that for all $f\in A^*$: $f-\varphi_i(f)\in A_i$. Moreover, $A_i=\{f-\varphi_i(f)|\ f\in A^*\}$.*

Proof. Since A is simple, $A\cap A_i=(0)$ for $i=1,2$. It means that $D(A)=A\oplus A_1=A\oplus A_2$ (as vector spaces) and for every $f\in A^*$ there are unique elements $a_i\in A(i=1,2)$ such that $f+a_i\in A_i$. For $i=1,2$ define φ_i as

$$\varphi_i(f)=-a_i$$

if $f+a_i\in A_i$. Since $\dim(A_i)=\dim(A^*)$, we have that $A_i=\{f-\varphi_i(f)|\ f\in A^*\}$. □

Consider $a\in A$. There exist unique $l_i\in L_i(i=1,2)$ such that $a=l_1+l_2$. By proposition 1, $l_i=f_i-\varphi_i(f_i)(i=1,2)$ and $a=(f_1-\varphi_1(f_1))+(f_2-\varphi_2(f_2))$. Thus, $f_1=-f_2$ and we proved that for every $a\in A$ there is $f\in A^*$ such that:

$$a=-\varphi_1(f)+\varphi_2(f). \tag{5}$$

If for some $a\in A$ there exist two elements $f_1\in A^*$ and $f_2\in A^*$ such that $a=-\varphi_1(f_i)+\varphi_2(f_i)$ then we get $(f_1-\varphi_1(f_1))-(f_1-\varphi_2(f_1))=(f_2-\varphi_1(f_2))-(f_2-\varphi_2(f_2))$. Since $f-\varphi_i(f)\in A_i$, we obtain that $f_1=f_2$.

Define a map $\psi : A\to A^*$ as $\psi(a)=f$ if $a=-\varphi_1(f)+\varphi_2(f)$. It is easy to see that ψ is an isomorphism of vector spaces.

We will need the following properties of the maps φ_i and ψ:

Proposition 2. *1. For all $f,g\in A^*$ and $i=1,2$:*

$$\varphi_i(fg)=\varphi_i(f)\varphi_i(g). \tag{6}$$

2. For all $a,b\in A$:

$$\psi(ab)=\psi(a)\leftharpoondown b=a\rightharpoondown\psi(b). \tag{7}$$

3. For all $a,b\in A$:

$$\psi(ab)=\psi(a)\psi(b)-\varphi_1(\psi(a))\rightharpoondown\psi(b)-\psi(a)\leftharpoondown\varphi_1(\psi(b)), \tag{8}$$

$$\psi(ab)=-(\psi(a)\psi(b)-\varphi_2(\psi(a))\rightharpoondown\psi(b)-\psi(a)\leftharpoondown\varphi_2(\psi(b))). \tag{9}$$

Proof. Take $f,g\in A^*$ and fix $i=1,2$. From the definition of φ_i we have that $f-\varphi_i(f)=p\in A_i$ and $g-\varphi_i(g)=q\in A_i$. Then, since A_i is an ideal of $D(A)$, $fg=\varphi_i(f)\varphi_i(g)+s$ for some $s\in A_i$. In means that $fg-\varphi_i(f)\varphi_i(g)\in A_i$ and by the definition of φ_i we conclude that $\varphi_i(fg)=\varphi_i(f)\varphi_i(g)$.

Let us prove (7). Let $a,b\in A$. Then

$$ab=(\psi(a)-\varphi_1(\psi(a)))b-(\psi(a)-\varphi_2(\psi(a)))b$$

$$=(\psi(a)\leftharpoondown b+\psi(a)\rightharpoonup b-\varphi_1(\psi(a))b)-(\psi(a)\leftharpoondown b+\psi(a)\rightharpoonup b-\varphi_2(\psi(a))b). \tag{10}$$

Since A_1 is an ideal, the expression in the first brackets in (10) lies in A_1. Thus, $\psi(ab) = \psi(a) \leftharpoondown b$ by the definition of the map ψ. Similar arguments show that $\psi(ab) = a \rightharpoondown \psi(b)$ and (7) is proved.

In order to prove (8) and (9) consider elements $a, b \in A$. We have:

$$a = (\psi(a) - \varphi_1(\psi(a))) + (-\psi(a) + \varphi_2(\psi(a)))$$

$$b = (\psi(b) - \varphi_1(\psi(b))) + (-\psi(b) + \varphi_2(\psi(b)))$$

Multiplying a and b in $D(A)$ we get:

$$ab = (\psi(a)\psi(b) - \varphi_1(\psi(a)) \rightharpoondown \psi(b) - \psi(a) \leftharpoondown \varphi_1(\psi(b)) + x_1)$$

$$+((\psi(a)\psi(b) - \varphi_2(\psi(a)) \rightharpoondown \psi(b) - \psi(a) \leftharpoondown \varphi_2(\psi(b))) + x_2)$$

where $x_i \in A$ $(i = 1, 2)$. This proves (8) and (9). □

Theorem 1. *Let $R : A \to A$ be an operator defined as*

$$R(a) = \varphi_1(\psi(a)). \tag{11}$$

Then R is a Rota—Baxter operator of weight 1 on A.

Proof. Using (6)-(8) and (11) for all $a, b \in A$ we compute:

$$R(R(a)b + aR(b) + ab) - R(a)R(b)$$

$$= \varphi_1(\psi(R(a)b + aR(b) + ab) - \varphi_1(\psi(a))\varphi_1(\psi(b))$$

$$= \varphi_1(R(a) \rightharpoondown \psi(b) + \psi(a) \leftharpoondown R(b) + \psi(ab) - \psi(a)\psi(b)) = 0.$$

Thus, R is a Rota—Baxter operator of weight 1 on A. □

Using similar arguments one can proof the following:

Theorem 2. *Let Q be an operator $Q : A \to A$ defined as*

$$Q(a) = \varphi_2(\psi(a)). \tag{12}$$

Then Q is a Rota—Baxter operator of weight -1 on A.

Remark 1. Theorem 2 can be proved using the following well-known fact. Given a Rota—Baxter operator R of weight λ on an algebra A, we have that $R + \lambda id$ is a Rota—Baxter operator on A of weight $-\lambda$ [6]. By the definitions of maps φ_i and ψ we have that $-\varphi_1(\psi(a)) + \varphi_2(\psi(a)) = a$ for all $a \in A$. Thus, we obtain that Q is the Rota—Baxter operator on A of weight -1.

3. Rota—Baxter operators and classical Yang—Baxter equation for simple anti-commutative algebras

Let L be a simple finite-dimensional anti-commutative algebra over a field F of characteristic zero and $r = \sum a_i \otimes b_i \in L \otimes L$ is a solution of the classical Yang—Baxter equation (2) on L. Recall, that if Δ is a comultiplication on L then bialebra (L, Δ) is anti-commutative if and only if the Drinfeld double is anti-commutative.

Theorem 3. *Let (L, Δ) be a structure of an anti-commutative bialgebra on a simple anti-commutative algebra L such that the Drinfeld double is a direct sum of two simple ideals. If R is the Rota—Baxter operator defined as in* (11)*, then there is a non skew-symmetric solution of CYBE* (2) *$r = \sum a_i \otimes b_i$ such that $r + \tau(r)$ is*

L − invariant and a non-degenerate associative symmetric bilinear form ω such that for all $a \in L$:

$$R(a) = \sum_i \omega(a_i, a)b_i. \tag{13}$$

Proof. Let $D(L) = L_1 \oplus L_2$. Consider maps $\varphi_i : L^* \to L$ $(i = 1, 2)$ defined in proposition 1. Since L is a finite-dimensional algebra, there exists an element $r_1 = \sum\limits_i a_i \otimes b_i \in L \otimes L$ such that $\varphi_1(f) = \sum f(a_i)b_i$ for all $f \in L^*$.

Similarly, there exists an element $r_2 = \sum\limits_i c_i \otimes d_i \in L \otimes L$ such that $\varphi_2(f) = \sum f(c_i)d_i$.

Since L_1 and L_2 are simple and the form Q is associative, $Q(L_1, L_2) = 0$. Therefore for all $f, g \in L^*$ we have $Q(f - \varphi_1(f), g - \varphi_2(g)) = 0$. Hence,

$$\sum_i \langle f \otimes g, a_i \otimes b_i + d_i \otimes c_i \rangle = 0.$$

Consequently,

$$r_1 + \tau(r_2) = 0. \tag{14}$$

Also we have $(f - \varphi_1(f))(g - \varphi_2(g)) = 0$. Therefore,

$$fg - f \leftharpoondown \varphi_2(g) - \varphi_1(f) \rightharpoondown g = 0.$$

The last equality means that for all $a \in L$

$$fg(a) = \sum_i f(g(c_i)d_i a) + g(f(a_i)ab_i),$$

$$\langle f \otimes g, \Delta(a) \rangle = \langle f \otimes g, \sum_i d_i a \otimes c_i + a_i \otimes ab_i \rangle.$$

Using (14) we finally obtain

$$\langle f \otimes g, \Delta(a) \rangle = -\langle f \otimes g, \sum_i a_i a \otimes b_i - a_i \otimes ab_i \rangle.$$

Thus, $\Delta = -\Delta_{r_1}$.

Since φ_1 is a homomorphism, for all $f, g \in L^*$ we have

$$\varphi_1([f, g]) = \sum [f, g](a_i)b_i = -\sum (f([a_j, a_i])g(b_j)b_i + f(a_j)g([b_j, a_i])b_i)$$
$$= [\varphi_1(f), \varphi_1(g)] = \sum f(a_j)g(a_i)[b_j, b_i].$$

Therefore r_1 is a solution of (2). Similar arguments show that r_2 is also a solution of (2).

Anti-commutativity of $D(L)$ is equivalent to $\tau(\Delta_{r_1}(a)) = -\Delta_{r_1}(a)$. Therefore $[r_1 + \tau(r_1), a] = 0$ for all $a \in L$ and $r_1 + \tau(r_1)$ is L-invariant.

Define a form $\omega(\cdot, \cdot)$ on L by:

$$\omega(a, b) = Q(\psi(a), b)$$

for all $a, b \in L$. It is clear that ω is bilinear and non-degenerate. Let us prove that ω is associative and symmetric.

Let $a, b \in L$ and $f_1, f_2 \in L^*$ be such that $\psi(a) = f_1$ and $\psi(b) = f_2$. Since $Q(L_1, L_2) = 0$, we have:

$$0 = Q(f_1 - \varphi_2(f_1), f_2 - \varphi_1(f_2)) = -Q(f_1, \varphi_1(f_2)) - Q(\varphi_2(f_1), f_2).$$

Similarly one can prove that

$$Q(f_1, \varphi_2(f_2)) + Q(\varphi_1(f_1), f_2) = 0.$$

Summing up the last two equations and using (5) we obtain that $Q(f_1, b) - Q(a, f_2) = 0$. Thus, $\omega(a, b) = \omega(b, a)$ for all $a, b \in L$.

Let $a, b, c \in L$. Using (7) and associativity of the form Q we compute:

$$\omega([a, b], c) = Q(\psi([a, b]), c) = Q(\psi(a) \leftharpoondown b, c) = Q([\psi(a), b], c) =$$

$$= Q(\psi(a), [b, c]) = \omega(a, [b, c]).$$

Thus, ω is a bilinear non-degenerate symmetric associative form on L. And for all $a \in L$ we have

$$R(a) = \varphi_1(\psi(a)) = \sum Q(\psi(a), a_i)b_i = \sum \omega(a, a_i)b_i.$$

□

Corollary 1. *Let (L, Δ) be a structure of an anti-commutative bialgebra on simple Lie algebra L over an algebraically closed field F such that the Drinfeld double is a direct sum of two simple ideals. If R is the Rota—Baxter operator defined as in* (11)*, then there exists a non skew-symmetric solution of CYBE* (2) $r = \sum a_i \otimes b_i$ *such that $r + \tau(r)$ is L-invariant and for all $a \in L$:*

$$R(a) = \sum_i \langle a_i, a\rangle b_i.$$

Here $\langle \cdot, \cdot \rangle$ is the Killing form on L.

Proof. By theorem 3 there exists a non-skew symmetric solution of CYBE r_1 with L-invariant symmetric part and $R(a) = \sum \omega(a, a_i)b_i$ for some non-degenerate associative symmetric bilinear form ω on L. But since L is simple, there exists a non-zero $\lambda \in F$ such that $\omega(a, b) = \lambda\langle a, b\rangle$. Therefore

$$R(a) = \sum \omega(a, a_i)b_i = \lambda \sum \langle a, a_i\rangle b_i.$$

It is left to define r as $r = \frac{1}{\lambda} r_1$ to prove the theorem. □

Theorem 4. *Let L be a simple anti-commutative algebra and $r = \sum_i a_i \otimes b_i$ is a non skew-symmetric solution of CYBE* (2) *such that $r + \tau(r)$ is L-invariant. Then there exists a non-degenerate symmetric associative bilinear form ω on L such that an operator $R : L \to L$ defined as*

$$R(a) = \sum_i \omega(a_i, a)b_i$$

is a Rota—Baxter operator of non-zero weight.

Proof. Define a comultiplication Δ_r on L as

$$\Delta_r(a) = [r, a]$$

for all $a \in L$. Since $r + \tau(r)$ is L invariant, we have that $[r, a] = -[\tau(r), a]$ for all $a \in L$. Therefore

$$\tau(\Delta(a)) = -\Delta(a) \tag{15}$$

and $D(L)$ is anti-commutative.

We want to prove that Drinfeld double of the bialgebra is a direct sum of two simple ideals. For this consider a map $\varphi_1 : L^* \to L$ defined as

$$\varphi_1(f) = -\sum_i f(a_i)b_i. \tag{16}$$

Since r is a solution of (2), φ_1 is a homomorphism.

Consider a subspace $L_1 = \{f - \varphi_1(f) | \, f \in L^*\}$. Let us prove that L_1 is an ideal of $D(L)$. For every $a \in L$, using (15), we have

$$[f - \varphi_1(f), a] = f \leftharpoondown a + f \rightharpoonup a - [\varphi_1(f), a]$$

$$= f \leftharpoondown a - \sum [b_i, a] f(a_i) - \sum b_i f([a_i, a]) + \sum f(a_i)[b_i, a] = f \leftharpoondown a - \sum f([a_i, a])b_i.$$

On the other hand,

$$\varphi_1(f \leftharpoondown a) = -\sum (f \leftharpoondown a)(a_i)b_i = -\sum f([a, a_i])b_i = \sum f([a_i, a])b_i$$

and $[f - \varphi_1(f), a] \in L_1$.

Now take $g \in L^*$. We have

$$[f - \varphi_1(f), g] = [f, g] - \varphi_1(f) \rightharpoondown g - \varphi_1(f) \leftharpoonup g$$

Since φ_1 is a homomorphism, we have

$$\varphi_1([f, g] - \varphi_1(f) \rightharpoondown g) = [\varphi_1(f), \varphi_1(g)] - \varphi_1(\varphi_1(f) \rightharpoondown g)$$

$$= \sum (f(a_i)g(a_j)[b_i, b_j] - f(a_i)g([a_j, b_i])b_j) = \varphi_1(f) \leftharpoonup g$$

and L_1 is a proper ideal of $D(L)$. By definition $\dim(L_1) = \dim(L)$.

If $V \subset D(L)$ then by $V^\perp$ denote the complement of V with respect to Q, i.e. $V^\perp = \{l \in D(L) | \, Q(l, V) = 0\}$. If V is a proper subalgebra in $D(L)$ then $V^\perp$ is a proper ideal of $D(L)$.

Consider $L_1^\perp$. Since L_1 is an ideal of $D(L)$, then $L_1^\perp$ is also an ideal if $D(L)$. It means that $I = L_1 \cap L_1^\perp$ is a proper ideal of $D(L)$ and therefore $I = L_1$ or $I = 0$.

Since $r + \tau(r) \neq 0$, there exists $h = \sum f_j \otimes g_j \in L^* \otimes L^*$ such that $\sum_{i,j} f_j(a_i)g_j(b_i) + f_j(b_i)g_j(a_i) \neq 0$. Then $\sum Q(f_j + \varphi_1(f_j), g_j + \varphi_1(g_j)) \neq 0$ and $I = L_1 \cap L_1^\perp = 0$. Therefore $D(L) = L_1 \oplus L_1^\perp$ and $L_1^\perp$ is isomorphic to the quotient algebra $D(L)/L_1$.

But $D(L) = L \oplus L_1$ (as vector spaces), therefore L is also isomorphic to the quotient algebra $D(L)/L_1$. Thus, L and $L_1^\perp$ are isomorphic. Similar arguments show that L is isomorphic to L_1 and $D(L)$ is a sum of two simple ideals. And the statement of the theorem follows from the definition of maps φ_i and theorems 1 and 3.

□

Corollary 2. *Let L be a simple Lie algebra over an algebraically closed field F and $r = \sum_i a_i \otimes b_i$ is a non skew-symmetric solution of CYBE (2) such that $r + \tau(r)$ is L invariant. Then an operator $R : L \to L$ defined as*

$$R(a) = \sum_i \langle a_i, a \rangle b_i \tag{17}$$

is a Rota—Baxter operator of non-zero weight. Here $\langle \cdot, \cdot \rangle$ is the Killing form on L.

Remark 2. If r is a non skew-symmetric solution of the classical Yang—Baxter equation such that $\tau(r)+r$ is L-invariant, then so is an element $r_1 = \tau(r)$. Elements r and $-r_1$ induce the same bialgebra structure on L. And if Q is a Rota—Baxter operator with respect to r_1 then Q has weight λ and $R + Q = -\lambda\mathrm{id}$, where λ is a weight of operator R and id is the identity operator on L.

Example 1. Let $L = sl_2(\mathbb{C})$, x, h, y is the standard basis of L, i.e. $hx = 2x$, $hy = -2y$, $xy = h$.

Consider an element

$$r = \alpha(h \otimes x - x \otimes h) + \frac{1}{4} h \otimes h + x \otimes y,$$

where $\alpha \in F$. Then for every $\alpha \in F$ r is a solution of the classical Yang—Baxter equation and therefore induces on L a structure of a Lie bialgebra with semisimple Drinfeld double. By corollary 2 the operator R defined as (17) is a Rota—Baxter operator of non-zero weight.

We have:

$$R(x) = 0,\ R(h) = 2h + 8\alpha x,\quad R(y) = 4(y - \alpha h).$$

Direct computations show that R is a Rota—Baxter operator of weight -4.

In order to compute the second Rota—Baxter operator Q we need to consider

$$r_1 = \tau(r) = -\alpha(h \otimes x - x \otimes h) + \frac{1}{4} h \otimes h + y \otimes x.$$

Thus,

$$Q(x) = 4x,\ Q(h) = -8\alpha x + 2h,\ Q(y) = 4\alpha h$$

and Q is a Rota—Baxter operator of weight -4. Note that $Q + R = 4\mathrm{id}$.

The following example shows that in general a non skew-symmetric solutions of the CYBE not necessary induces a Rota—Baxter operator of non-zero weight.

Example 2. Let $L = sl_2(\mathbb{C})$ and let x, h, y be the standard basis of L. Consider an element $r = x \otimes x$. Obviously, r is a non skew-symmetric solution of (2). The corresponding operator R defined by (17) acts as follows:

$$R(x) = 0,\ R(h) = 0,\ R(y) = 4x$$

and is a Rota—Baxter operator of weight zero.

4. Rota—Baxter operators on the split simple Malcev algebra

In this section we consider simple non-Lie Malcev algebra.

Malcev algebras were introduced by A.I. Malcev [18] as tangent algebras for local analytic Moufang loops. The class of Malcev algebras generalizes the class of Lie algebras and has a well developed theory [19].

An important example of a non-Lie Malcev algebra is the vector space of zero trace elements of a Caley—Dickson algebra with the commutator bracket multiplication [20, 21]. In [22] some properties of Malcev bialgebras were studied. In particular, there were found conditions for a Malcev algebra with a comultiplication to be a Malcev bialgebra. In [16] it was found a connection between solutions of the classical Yang—Baxter equation on Malcev algebras with Malcev bialgebras.

Definition 5. *An anti-commutative algebra M is called a Malcev algebra if for all $x, y, z \in M$ the following equation holds:*

$$J(x, y, xz) = J(x, y, z)x, \tag{18}$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of elements x, y, z.

In [16] it was proved that non skew-symmetric solutions of the classical Yang—Baxter equation on simple Malcev algebra M induces on M a structure of Malcev bialgebra with semisimple Drinfeld double.

Let M be a simple Malcev algebra. Then M is a simple Lie algebra or the 7-dimentional Malcev algebra isomorphic to the commutator algebra of traceless elements of the split Caley—Dickson algebra [25].

Example 3. Let $\mathbb{M}$ be the simple Malcev algebra over the field of complex numbers $\mathbb{C}$. In this case $\mathbb{M}$ has a basis h, x, x', y, y', z, z' with the following table of multiplication:

$$hx = 2x,\ hy = 2y,\ hz = 2z,$$
$$hx' = -2x',\ hy' = -2y',\ hz' = -2z',$$
$$xx' = yy' = zz' = h,$$
$$xy = 2z',\ yz = 2x',\ zx = 2y',$$
$$x'y' = -2z,\ y'z' = -2x,\ z'x' = -2y.$$

The remaining products are zero. In [16] it was proved that up to automorphism any non skew-symmetric solution of the classical Yang—Baxter equation r has the following form:

$$r = r_0 + \frac{1}{4}h \otimes h + x \otimes x' + y' \otimes y + z \otimes z'$$

where

$$r_0 = \alpha(h \otimes x - x \otimes h) + \beta(h \otimes y' - y' \otimes h) + \gamma(h \otimes z - z \otimes h)$$
$$+\delta(x \otimes y' - y' \otimes x) - 2\beta(x \otimes z - z \otimes x) + \mu(y' \otimes z - z \otimes y').$$

Scalars $\alpha, \beta, \gamma, \delta, \mu$ are arbitrary. Then the element $\frac{r}{4}$ induces the following Rota—Baxter operator of weight -1 on $\mathbb{M}$:

$$R(h) = \frac{1}{2}h + 2\alpha x + 2\beta y' + 2\gamma z,\ R(x) = 0,\ R(x') = x' - \alpha h + \delta y' - 2\beta z,$$
$$R(y) = y - \beta h - \delta x + \mu z,\ R(y') = R(z) = 0,\ R(z') = z' - \gamma h + 2\beta x - \mu y'.$$

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