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LATTICES OF SUBCLASSES. III

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Abstract. We prove that for certain Q-universal quasivarieties $\mathbf{K}$, the lattice of $\mathbf{K}$-quasivarieties contains continuum many subquasivarieties with the undecidable quasi-equational theory and for which the finite membership problem is also undecidable. Moreover, we prove that certain Q-universal quasivarieties have continuum many subquasivarieties with no independent quasi-equational basis.

Keywords: Abelian group, differential groupoid, finite membership problem, graph, independent basis, quasi-identity, quasi-equational theory, quasivariety, Q-universal, undecidable theory.

1. Introduction

This paper is a continuation of [37]-[38] and contributes to investigation of complexity of quasivariety lattices.

G. Birkhoff [12] and A. I. Maltsev [28] have independently asked which lattices are isomorphic to lattices of quasivarieties. This intriguing question (which is still open) triggered research devoted to investigation of structure and properties of quasivariety lattices which still remains to be one of the mainstreams in the area of universal algebra. It turned out quite soon that quasivariety lattices have quite a complicated inner structure; and some of those have the "highest" complexity amongst all the quasivariety lattices.

One of measures of complexity, the notion of Q-universality, was introduced for quasivarieties by M. V. Sapir in [35]. On can extend this notion to an arbitrary

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classes $\mathbf{K}$ of algebraic structures, thus defining the lattice of $\mathbf{K}$-quasivarieties. Up to now, there are many Q-universal classes known. Among them are the class of all unars (algebras with one unary operation) [17], the class of all commutative rings [1], the class of all graphs [1, 22, 24], the class of all differential groupoids [25], the class of pointed Abelian groups [31], and many others, cf. [1, 15, 17, 26]. Q-universality can be established in almost all the known cases by finding a so-called AD-class, cf. Definition 1 and Theorem 1.

According to Theorem 1(ii), almost all the known Q-universal classes possess another common property which reflects another face of complexity of quasivariety lattices. Each of these classes has a subclass $\mathbf{K}$ for which the set of finite sublattices (up to isomorphism) of the lattice of $\mathbf{K}$-quasivarieties is not computable, cf. [26, 30, 31, 36, 37, 38]. We prove here that in a quasivariety containing a computable AD-class, there are continuum many subquasivarieties with the undecidable finite membership problem, cf. Corollary 1. We note that A. M. Nurakunov [30] proved that there are continuum many quasivarieties of unars with undecidable quasi-equational theory for which the finite membership problem is also undecidable. We establish an analogous result for graphs, differential groupoids, and for some other certain types of algebraic structures, cf. Section 3. In Section 4, we prove that those classes mentioned above contain also continuum many quasivarieties which have no covers in the corresponding quasivariety lattice and thus no independent quasi-equational basis. Examples of quasivarieties with no independent basis exist in the class of unars (V. K. Kartashov [19]), lattices (V. I. Tumanov [44]), distributive p-lattices (M. P. Tropin [42, 43]), directed graphs (S. V. Sizyĭ [40]); for other examples, we refer to the monograph of V. A. Gorbunov [17, Chapter 6], to A. V. Kravchenko [26], and to A. V. Kartashova [20].

For all concepts, which are not defined here, we refer to G. Birkhoff [12], A. I. Maltsev [28, 29], W. Dziobiak [15], and V. A. Gorbunov [17]. One can also find them in [37]-[38].

2. The finite membership problem

Proposition 1. *Let $\mathbf{C} = \{\mathcal{C}_n \mid n < \omega\}$ be a computable class of finite structures of a finite signature with the following properties:*

(E_0) *$\mathcal{C}_n$ is a nontrivial structure for any $n < \omega$;*

(E_1) *if $k < \omega$ and n, n_0, $\dots$, $n_k < \omega$, then $\mathcal{C}_n \in \mathbf{SP}(\mathcal{C}_{n_0}, \dots, \mathcal{C}_{n_k})$ if and only if $n \in \{n_0, \dots, n_k\}$.*

Then there are continuum many quasivarieties $\mathbf{K} \subseteq \mathbf{Q}(\mathbf{C})$ such that the finite membership problem for $\mathbf{K}$ is undecidable.

Proof. Let $N \subseteq \omega$ be an arbitrary set. We put

$$\mathbf{K}_N = \{\mathcal{C}_i \mid i \in N\} \quad \text{and} \quad \mathbf{R}_N = \mathbf{Q}(\mathbf{K}_N).$$

Let $(\mathbf{R}_N)_{fin}$ denote the class of finite members of $\mathbf{R}_N$.

Claim 1. $\mathbf{R}_N \cap \mathbf{C} = \mathbf{K}_N$.

Proof of Claim. Obviously, $\mathbf{K}_N \subseteq \mathbf{R}_N \cap \mathbf{C}$. Conversely, let $\mathcal{C}_n \in \mathbf{R}_N \cap \mathbf{C}$ for some integer $n < \omega$. Then $\mathcal{C}_n \in \mathbf{L}_s\mathbf{P}_s(\mathbf{K}_N)$. As $\mathcal{C}_n$ is a finite structure, it is l-projective, whence $\mathcal{C}_n \in \mathbf{SP}(\mathbf{K}_N)$. Again, as $\mathcal{C}_n$ is finite, there are integers n_0, $\dots$, $n_k \in N$ such that $\mathcal{C}_k \in \mathbf{SP}(\mathcal{C}_{n_0}, \dots, \mathcal{C}_{n_k})$. We obtain therefore by our assumption about the class $\mathbf{C}$ that $n \in \{n_0, \ldots, n_k\} \subseteq N$; that is, $\mathcal{C}_n \in \mathbf{K}_N$ and the desired equality follows. ☐

Directly from Claim 1, we get

Claim 2. *If* $\mathbf{R}_{N_0} = \mathbf{R}_{N_1}$*, then* $N_0 = N_1$*.*

Claim 3. *If the set* $(\mathbf{R}_N)_{fin}$ *is computable, then* N *is computable.*

Proof of Claim. If $(\mathbf{R}_N)_{fin}$ is a computable set, then $N' = \{n < \omega \mid \mathcal{C}_n \in \mathbf{R}_N\}$ is a computable set by our assumption. By Claim 1, $N = N'$, whence the set N is also computable. ☐

As there are continuum many sets $N \subseteq \omega$ which are not computable, we get the desired statement from Claims 1-3. ☐

Definition 1. [1, 36] If a class $\mathbf{A} = \{\mathcal{A}_X \mid X \in \mathcal{P}_{fin}(\omega)\}$ of structures possesses the following properties:

- (P_0) for any $X \in \mathcal{P}_{fin}(\omega)$, the structure $\mathcal{A}_X$ is l-projective in $\mathbf{Q}(\mathbf{A})$ and the trivial congruence is a dually compact element in the relative congruence lattice $\mathrm{Con}_{\mathbf{Q}(\mathbf{A})}\mathcal{A}_X$;
- (P_1) $\mathcal{A}_\varnothing$ is a trivial structure;
- (P_2) if $X = Y \cup Z$ in $\mathcal{P}_{fin}(\omega)$, then $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y, \mathcal{A}_Z)$;
- (P_3) if $\varnothing \neq X \in \mathcal{P}_{fin}(\omega)$ and $\mathcal{A}_X \in \mathbf{Q}(\mathcal{A}_Y)$, then $X = Y$;
- (P_4) if $\mathcal{A}_X \leq \mathcal{B}_0 \times \mathcal{B}_1$ for some structures $\mathcal{B}_0$, $\mathcal{B}_1 \in \mathbf{Q}(\mathbf{A})$, then there are Y_0, $Y_1 \in \mathcal{P}_{fin}(\omega)$ such that $\mathcal{A}_{Y_0} \in \mathbf{Q}(\mathcal{B}_0)$, $\mathcal{A}_{Y_1} \in \mathbf{Q}(\mathcal{B}_1)$, and $X = Y_0 \cup Y_1$,

then $\mathbf{A}$ is called an AD-*class*. An AD-class consisting of finite structures of a finite signature is called a *finite* AD-*class*.

Corollary 1. *For any computable finite* AD-*class* $\mathbf{A}$*, the quasivariety* $\mathbf{Q}(\mathbf{A})$ *contains continuum many quasivarieties for which the finite membership problem is undecidable.*

Proof. For any $n < \omega$, we put $\mathcal{C}_n = \mathcal{A}_{\{n\}}$. By [36, Lemma 3.2], for any $k < \omega$ and any n, n_0, $\ldots$, $n_k < \omega$, one has $\mathcal{C}_n \in \mathbf{SP}(\mathcal{C}_{n_0}, \ldots, \mathcal{C}_{n_k})$ if and only if $n \in \{n_0, \ldots, n_k\}$. The desired statement follows from Proposition 1. ☐

We would like to recall some known properties of quasivarieties that contain an AD-class. For the first claim in following statement, we refer to M. E. Adams and W. Dziobiak [1, Theorem 3.3] as well as to [36, Corollary 3.5], and for the second—to [36, Theorem 4.4].

Theorem 1. *Let a quasivariety* $\mathbf{K}$ *contain an* AD-*class. Then*

- (i) $\mathbf{K}$ *is Q-universal;*
- (ii) $\mathbf{K}$ *contains continuum many classes* $\mathbf{K}' \subseteq \mathbf{K}$ *such that the set of* [*isomorphism types*] *of finite sublattices of the quasivariety lattice* $\mathrm{Lq}(\mathbf{K}')$ *is not computably enumerable.*

3. Examples of quasivarieties for which the finite membership problem and the quasi-equational theory are undecidable

3.1. **Graphs.**

Definition 2. A *graph* [a *directed graph*] is an algebraic structure $\mathcal{G} = \langle G; E\rangle$, where G is a non-empty set and E is a symmetric binary relation [a binary relation, respectively]. A graph is *antireflexive* if it satisfies the following quasi-identity:

$$\forall\, xy\ E(x,x) \ \longrightarrow\ x = y.$$

Let $\mathbf{C}$ be the quasivariety of antireflexive directed graphs defined by the following quasi-identities:

$$\forall\, xyz\ E(x,z)\ \&\ E(y,z) \ \longrightarrow\ x = y;$$
$$\forall\, xyz\ E(z,x)\ \&\ E(z,y) \ \longrightarrow\ x = y.$$

In [22], A. V. Kravchenko showed that the quasivariety $\mathbf{C}$ is $\mathcal{Q}$-universal. Moreover, according to [6, 24, 36], there are continuum many classes $\mathbf{K}$ of graphs such that the set of all finite sublattices of the lattice $\mathrm{Lq}(\mathbf{K})$ is not computably enumerable.

For an integer $n > 1$, let $\mathcal{C}_n$ denote the directed graph $\langle\{0,\dots,n-1\}; E\rangle$ such that for any i, $j < n$, $(i,j) \in E$ whenever $j \equiv i+1 (\mathrm{mod}\ n)$. By $\mathcal{C}_1$, we denote the trivial graph; that is, $\mathcal{C}_1 = \big\langle\{0\}; \{(0,0)\}\big\rangle$. We call $\mathcal{C}_n$ the *cycle of length* n. It is obvious that $\mathcal{C}_n \in \mathbf{C}$ for any $n > 0$. For a finite set $F = \{n_1,\dots,n_k\}$ of positive integers, we use any of $[F]$, $[n \mid n \in F]$, $[n_1,\dots,n_k]$ to denote the least common multiple of F. We assume that $[\varnothing] = 1$.

Lemma 1. *Let $n > 0$, $k > 1$, $k_1 > 1$, $\dots$, $k_n > 1$ be integers such that the set $\{k_1,\dots,k_n\}$ is minimal with respect to the property that $\mathcal{C}_k \in \mathbf{SP}(\mathcal{C}_{k_1},\dots,\mathcal{C}_{k_n})$. Then $k = [k_1,\dots,k_n]$. Conversely, if $k = [k_1,\dots,k_n]$, then $\mathcal{C}_k \in \mathbf{SP}(\mathcal{C}_{k_1},\dots,\mathcal{C}_{k_n})$.*

Proof. If $\mathcal{C}_k \in \mathbf{SP}(\mathcal{C}_{k_1},\dots,\mathcal{C}_{k_n})$, then it is easy to see that $\mathcal{C}_k$ embeds into $\mathcal{C}_{k_1} \times \dots \times \mathcal{C}_{k_n}$; let ψ be the corresponding embedding.

Let $\psi(i) = \langle a_j^i \mid 1 \leqslant j \leqslant n\rangle$ for $i < k$ and let $\psi_j = \pi_j\psi$ for $j \in \{1,\dots,n\}$. We have for all $i < k$

$$\big(\langle a_j^i \mid 1 \leqslant j \leqslant n\rangle, \langle a_j^{i+1} \mid 1 \leqslant j \leqslant n\rangle\big) \in E,$$

whence $(a_j^i, a_j^{i+1}) \in E$ for all $i < k$ and for all $j \in \{1,\dots,n\}$. Therefore, k_j divides k for all $j \in \{1,\dots,n\}$. Moreover, as k is minimal with the property above, we conclude that $m = [k_1,\dots,k_j]$. The last statement is straightforward to prove. $\square$

Theorem 2. *There are continuum many quasivarieties $\mathbf{R}$ of directed graphs such that the finite membership problem for $\mathbf{R}$ and the quasi-equational theory of $\mathbf{R}$ are undecidable.*

Proof. Let $P = \{p_i \mid i < \omega\}$ be the set of primes such that $p_i \leqslant p_j$ for all $i \leqslant j < \omega$ and let $\mathbf{K} = \{\mathcal{C}_{p_i} \mid i < \omega\}$. It follows from Lemma 1 that the class $\mathbf{K}$ satisfies the conditions (E_0)-(E_1). Let $N \subseteq \omega$ be a set of positive integers. We put

$$\mathbf{K}_N = \{\mathcal{C}_{p_i} \mid i \in N\} \quad \text{and} \quad \mathbf{R}_N = \mathbf{Q}(\mathbf{K}_N).$$

For any positive integer $n < \omega$, let ψ_n denote the following quasi-identity:

$$\forall\, x_0 \dots x_{p_n-1} \quad \&_{i<p_n} E(x_i, x_{i+1}) \ \longrightarrow\ x_0 = x_{p_n-1},$$

where $i+1$ is calculated modulo p_n. Let $\mathrm{Th}_q(\mathbf{R}_N)$ be the quasi-equational theory of $\mathbf{R}_N$; then $\mathrm{Th}_q(\mathbf{R}_N) = \mathrm{Th}_q(\mathbf{K}_N)$.

Claim 1. *For any positive integer $n < \omega$, $\psi_n \in \mathrm{Th}_q(\mathbf{R}_N)$ if and only if $n \notin N$.*

Proof. Let $n < \omega$ be a positive integer. If $n \in N$, then $\mathcal{C}_{p_n} \in \mathbf{K}_N \subseteq \mathbf{R}_N$ and $\mathcal{C}_{p_n} \not\models \psi_n$, whence $\psi_n \notin \mathrm{Th}_q(\mathbf{R}_N)$. Suppose now that $n \notin N$. In this case, p_i does not divide p_n for any $i \in N$. This means that the premise of ψ_n holds on $\mathcal{C}_t \in \mathbf{K}_N$ if and only if $t = 1$. But then $\mathcal{C}_t \models \psi_n$. Therefore, $\mathbf{K}_N \models \psi_n$ in this case. $\square$

Claim 2. *If* $\mathrm{Th}_q(\mathbf{R}_N)$ *is decidable, then* N *is computable.*

Proof of Claim. If $\mathrm{Th}_q(\mathbf{R}_N)$ is decidable, then the set $\{n < \omega \mid \psi_n \in \mathrm{Th}_q(\mathbf{R}_N)\}$ is computable. By Claim 1, the complement of N is computable, whence N also is. $\square$

Since there are uncountably many sets N which are not computable, the statements of Theorem follow now from Claim 2 and Proposition 1. $\square$

3.2. **Differential groupoids.**

Definition 3. [33] A differential groupoid is an algebra $\langle A; \cdot\rangle$ with a binary operation satisfying the identities:

$$x \cdot x = x \tag{1}$$

$$(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t) \tag{2}$$

$$x \cdot (x \cdot y) = x. \tag{3}$$

Differential groupoids were studied by many authors, also under the name of *LIR-groupoids* and with different basis for identities. For more information, see A. B. Romanowska, B. Roszkowska [33], A. V. Kravchenko [25], or the monograph of A. B. Romanowska and J. D. H. Smith [34].

Each proper non-trivial subvariety of the variety $\mathbf{D}$ of differential groupoids is relatively based by a unique identity of the form

$$(\dots((x\underbrace{y)y)\dots)y}_{i\ \text{times}} = xy^i = xy^{i+j}$$

for some $i \in \mathbb{N}$ and a positive integer j, see [33]. Denote such a variety by $\mathbf{D}_{i,i+j}$. In [25], A. V. Kravchenko showed that the variety $\mathbf{D}$ is $\mathcal{Q}$-universal. Moreover, according to [36], there are continuum many classes $\mathbf{K}$ of differential groupoids such that the set of all finite sublattices of the lattice $\mathrm{Lq}(\mathbf{K})$ is not computable.

Recall that a groupoid $\mathcal{G}$ is a *left zero band* if it satisfies the identity $x \cdot y = x$, i.e. if $\mathcal{G} \in \mathbf{D}_{0,1}$. In particular, any set G can be turned into a left zero band, when assuming $x \cdot y = x$ for all $x, y \in G$.

Definition 4. [33] Let I be a non-empty set and for each $i \in I$, let a non-empty set G_i be given. For each pair $(i, j) \in I \times I$, let $h_{ij} \colon G_i \to G_i$ be a mapping satisfying

(i) h_{ii} is an identity mapping on G_i;
(ii) $h_{ij} \circ h_{ik} = h_{ik} \circ h_{ij}$.

Define a groupoid structure on the disjoint union G of G_i for $i \in I$, by

(iii) $a_i \cdot a_j = h_{ij}(a_i)$, where $a_i \in G_i$, $a_j \in G_j$.

The groupoid $\mathcal{G}$ is said to be *the sum of left zero bands* $\mathcal{G}_i$ *over the left zero band* I by the mapping h_{ij} or briefly **Lz**-**Lz**-*sum* of $\mathcal{G}_i$, $i \in I$.

In particular, for any $i \in I$, $\mathcal{G}_i$ is a subgroupoid of $\mathcal{G}$ and is a left zero band. Moreover, the mapping $f \colon G \to I$, $a_i \mapsto i$, is a homomorphism. It was shown in [33] that $\mathcal{G}$ is a differential groupoid if and only if it is an **Lz**-**Lz**-sum of left zero bands $\mathcal{G}_i$ (over a left zero band I by some mappings h_{ij}). Let $\mathcal{G}_0$ denote the trivial groupoid whose universe is $\{\infty\}$.

For every positive integer n, let $\mathcal{D}_n$ denote the **Lz**-**Lz**-sum of $G_1 = \{0, 1, \dots, n-1\}$ and $G_2 = \{\infty\}$, where $h_{12}(k) \equiv k + 1 (\text{mod } n)$ for $k < n$ and h_{21} is the identity map. For example, $\mathcal{D}_3 = \langle \{0, 1, 2, \infty\}; \cdot \rangle$, where

$$0 \cdot 0 = 0 \cdot 1 = 0 \cdot 2 = 0;$$
$$1 \cdot 0 = 1 \cdot 1 = 1 \cdot 2 = 1;$$
$$2 \cdot 0 = 2 \cdot 1 = 2 \cdot 2 = 2;$$
$$0 \cdot \infty = 1, \quad 1 \cdot \infty = 2, \quad 2 \cdot \infty = 0;$$
$$\infty \cdot 0 = \infty \cdot 1 = \infty \cdot 1 = \infty \cdot \infty = \infty.$$

Obviously, $\mathcal{D}_n \in \mathbf{D}_{0,n}$ for each $n > 0$ and we call it a *cycle* of height n.

Lemma 2. *Let $n > 0$ be an integer, let $\mathcal{D} \in \mathbf{D}$, and let $ab^n = a$, $ba = b$ for some elements a, $b \in D$. Then $ab^i \cdot ab^j = ab^i$ and $b \cdot ab^i = b$ for any i, $j < \omega$. In particular, the substructure of $\mathcal{D}$ generated by elements a, b is isomorphic to $\mathcal{D}_k$ for some $k \leqslant n$.*

Proof. We assume first that $j \leqslant i$ and argue by induction on i. If $i = 0$ then $j = 0$, and we have by our assumption $ab^0 \cdot ab^0 = a \cdot a = a$ and $b \cdot ab^0 = b \cdot a = b$. Applying Definition 3 and the induction hypothesis, we get for $i > 0$ and $j > 0$

$$ab^i \cdot ab^0 = (ab^{i-1} \cdot b) \cdot (a \cdot a) = (ab^{i-1} \cdot a) \cdot (b \cdot a) = ab^{i-1} \cdot b = ab^i;$$
$$ab^i \cdot ab^j = (ab^{i-1} \cdot b) \cdot (ab^{j-1} \cdot b) = (ab^{i-1} \cdot ab^{j-1}) \cdot (b \cdot b) = ab^{i-1} \cdot b = ab^i;$$
$$b \cdot ab^i = (b \cdot b) \cdot (ab^{i-1} \cdot b) = (b \cdot ab^{i-1}) \cdot (b \cdot b) = b \cdot b = b.$$

If $j > i$, then there is a positive integer p such that $j \leq i + pn$. According to the above, we have $ab^i \cdot ab^j = ab^{i+pn} \cdot ab^j = ab^{i+pn} = ab^i$. Therefore, the map $a \mapsto 0$, $b \mapsto \infty$ extends to an isomorphism. □

A proof of the next statement can be found in [25, 38], cf. the proof of Lemma 1.

Lemma 3. [38, Lemma 4.1.3] *Let $n > 0$, $k > 1$, $k_1 > 1$, $\dots$, $k_n > 1$ be integers such that the set $\{k_1, \dots, k_n\}$ is minimal with respect to the property that $\mathcal{D}_k \in \mathbf{SP}(\mathcal{D}_{k_1}, \dots, \mathcal{D}_{k_n})$. Then $k = [k_1, \dots, k_n]$. Conversely, if $k = [k_1, \dots, k_n]$, then $\mathcal{D}_k \in \mathbf{SP}(\mathcal{D}_{k_1}, \dots, \mathcal{D}_{k_n})$.*

The following theorem is a groupoid analogue of Theorem 2.

Theorem 3. *There are uncountably many quasivarieties $\mathbf{K}$ of differential groupoids such that the finite membership problem for $\mathbf{K}$ and the quasi-equational theory of $\mathbf{K}$ are undecidable.*

Proof. For any positive integer $n < \omega$, let p_n denote the nth prime and let ψ_n denote the following quasi-identity:

$$\forall x \forall y \quad xy^{p_n} = x \;\&\; yx = y \;\longrightarrow\; xy^{p_n - 1} = x.$$

Let also

$$\Psi = \{\psi_n \mid 1 < n < \omega\}, \quad \mathbf{K} = \{\mathcal{D}_{p_i} \mid i < \omega\};$$
$$\mathbf{K}_N = \{\mathcal{D}_{p_i} \mid i \in N\}, \quad \mathbf{R}_N = \mathbf{Q}(\mathbf{K}_N) \quad \text{for an arbitrary set } N \subseteq \omega.$$

The argument repeats the one in the proof of Theorem 2 using Ψ and with a reference to Lemma 3. □

3.3. **Pointed Abelian groups.** A *pointed Abelian group* is a structure

$$\mathcal{A} = \langle A; +, -, 0, c\rangle$$

such that the structure $\langle A; +, -, 0\rangle$ is an Abelian group and c is a constant symbol. We put for any $a \in A$

$$0a = 0 \quad \text{and} \quad (i+1)a = ia + a \text{ for any } i \geqslant 0.$$

In [31], the second author proved that the variety of all pointed Abelian groups is Q-universal and that there are continuum many quasivarieties $\mathbf{K}$ of pointed Abelian groups such that the set of all finite sublattices of the lattice $\mathrm{Lq}(\mathbf{K})$ is not computable. For any positive integer $n > 1$, let $\mathcal{A}_n = \langle A_n; +, -, 0, c\rangle$, where $A_n = \{0, \dots, n-1\}$, $\langle A_n; +, -, 0\rangle$ is a cyclic group of order n, and $c = 1$.

The next lemma is an analogue of Lemmas 1 and 3. Its proof repeats the proof of Lemma 1.

Lemma 4. *Let $n > 0$, $k > 1$, $k_1 > 1$, $\dots$, $k_n > 1$ be integers such that the set $\{k_1, \dots, k_n\}$ is minimal with respect to the property that $\mathcal{A}_k \in \mathbf{SP}(\mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n})$. Then $k = [k_1, \dots, k_n]$. Conversely, if $k = [k_1, \dots, k_n]$, then $\mathcal{A}_k \in \mathbf{SP}(\mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n})$.*

The following theorem is a groupoid analogue of Theorem 2.

Theorem 4. *There are continuum many quasivarieties $\mathbf{K}$ of pointed Abelian groups such that the finite membership problem for $\mathbf{K}$ and the quasi-equational theory of $\mathbf{K}$ are undecidable.*

Proof. For any positive integer $n < \omega$, let ψ_n denote the following quasi-identity:

$$p_n c = 0 \;\longrightarrow\; (p_n - 1)c = 0.$$

Let also

$$\Psi = \{\psi_n \mid 1 < n < \omega\}, \quad \mathbf{K} = \{\mathcal{A}_{p_i} \mid i < \omega\};$$
$$\mathbf{K}_N = \{\mathcal{A}_{p_i} \mid i \in N\}, \quad \mathbf{R}_N = \mathbf{Q}(\mathbf{K}_N) \quad \text{for an arbitrary set } N \subseteq \omega.$$

The argument repeats the one in the proof of Theorem 2 using Ψ and with a reference to Lemma 4. $\square$

From Theorems 2, 3, and 4, on easily gets

Corollary 2. *There exist continuum many quasivarieties of graphs [differential groupoids, pointed Abelian groups, respectively] which have no computable basis of quasi-identities.*

Remark 1. Using Proposition 1 for the class of unars (algebras with one unary operation), one obtains the result of [30, Corollary 21] which states that there are continuum many quasivarieties of unars with no computable basis of quasi-identities.

From Corollary 1, we get the following statement.

Corollary 3. *The following quasivarieties contain continuum many subquasivarieties for which the finite membership problem is undecidable:*

(i) *the quasivariety $\mathbf{C}$ of undirected graphs;*
(ii) *the variety of commutative rings with unit;*
(iii) *the variety of unars;*
(iv) *the variety $\mathbf{D}$ of differential groupoids;*

(v) *the variety of MV-algebras;*
(vi) *the variety of pointed Abelian groups.*

4. Independent bases of quasi-identities

Definition 5. A quasivariety $\mathbf{K}$ has an *independent basis of quasi-identities*, if there is a set Φ of quasi-identities such that $\mathbf{K} = \mathrm{Mod}(\Phi)$ and $\mathbf{K} \neq \mathrm{Mod}(\Phi - \varphi)$ for any $\varphi \in \Phi$.

It is straightforward that a finitely based quasivariety has an independent basis of quasi-identities.

Proposition 2. [17, Proposition 6.3.1] *Let $\mathbf{K} \subseteq \mathbf{K}'$ be quasivarieties. If $\mathbf{K}$ has an independent quasi-equational basis Σ relative to $\mathbf{K}'$, then $\mathbf{K}$ has at least $|\Sigma|$ upper covers in the lattice $\mathrm{Lq}(\mathbf{K}')$.*

Theorem 5. *There are continuum many quasivarieties of differential groupoids with no independent quasi-equational basis.*

Proof. We use ideas of V. K. Kartashov from the proof of [19, Theorem 4]. For a set $F \in \mathcal{P}_{fin}(\omega)$, we put

$$[F] = \begin{cases} \prod_{i \in F} p_i, & \text{if } F \neq \varnothing; \\ 1, & \text{otherwise.} \end{cases}$$

We fix an infinite set $I \subset \omega$ and for each finite nonempty set $F \subseteq \omega$, consider the following quasi-identity which we denote by φ_F^I:

$$\forall x \forall y \quad xy^{[F]} = x \ \& \ yx = y \ \longrightarrow \ xy^{[F \cap I]} = x.$$

We put also

$$\Phi_I = \left\{ \varphi_F^I \mid \varnothing \neq F \in \mathcal{P}_{fin}(\omega) \right\} \quad \text{and} \quad \mathbf{K}_I = \mathrm{Mod}(\Phi_I).$$

Claim 1. *For any $F \in \mathcal{P}_{fin}(\omega)$, $\mathcal{D}_{[F]} \in \mathbf{K}_I$ if and only if $F \subseteq I$.*

Proof of Claim. Suppose first that $\mathcal{D}_{[F]} \in \mathbf{K}_I$. This means that $\mathcal{D}_{[F]} \models \varphi_F^I$. As $0\infty^{[F]} = 0$ in $\mathcal{D}_{[F]}$, we conclude that $0\infty^{[F \cap I]} = 0$. This is possible only when $[F]$ divides $[F \cap I]$; that is, when $F \subseteq I$.

Suppose now that $F \subseteq I$; we prove that $\mathcal{D}_{[F]} \in \mathrm{Mod}(\varphi_G^I)$ for any nonempty set $G \in \mathcal{P}_{fin}(\omega)$. Indeed, let $a, b \in D_{[F]}$ be such that $ab^{[G]} = a$ and $ba = b$. According to Lemma 2, $b(ab^i) = b$ for any $i < \omega$. This means that $[F]$ divides $[G]$, whence $F \subseteq G$ and $F \subseteq G \cap I$. Therefore, $[F]$ divides $[G \cap I]$ which implies that $ab^{[G \cap I]} = a$; that is, φ_G^I holds in $\mathcal{D}_{[F]}$. $\square$

Claim 2. *$\mathbf{K}_I$ has no upper cover in the lattice $\mathrm{Lq}(\mathbf{D})$.*

Proof of Claim. Let $\mathbf{K} \in \mathrm{Lq}(\mathbf{D})$ be such that $\mathbf{K}_I \subset \mathbf{K}$. Then there is a finite nonempty set $F \subseteq \omega$ and a structure $\mathcal{A} \in \mathbf{K}$ such that $\mathcal{A} \not\models \varphi_F^I$. This means that there are $a, b \in A$ such that $ab^{[F]} = a$ and $ba = b$. According to Lemma 2, $b(ab^i) = b$ for any $i < \omega$. Let $k < \omega$ be minimal with respect to the property that $ab^k = a$. Then $k > 0$ and k divides $[F]$. Therefore, $k = [G]$ for some nonempty $G \subseteq F$, and the substructure of $\mathcal{A}$ generated by elements a and b is isomorphic to $\mathcal{D}_{[G]}$. In particular, $\mathcal{D}_{[G]} \in \mathbf{K} \backslash \mathbf{K}_I$. Taking into account Claim 1, we conclude that $G \nsubseteq I$; let $j \in G \backslash I$.

Since I is infinite, there is $i \in I \backslash F$. According to Claim 1, $\mathcal{D}_{p_i} \in \mathbf{K}_I$. We put $H = G \cup \{i\}$. By Lemma 3, $\mathcal{D}_{[H]} \leq_s \mathcal{D}_{[G]} \times \mathcal{D}_{p_i}$, whence $\mathcal{D}_{[H]} \in \mathbf{K} \backslash \mathbf{K}_I$ according to Claim 1. We have therefore that $\mathbf{K}_I \subset \mathbf{Q}(\mathbf{K}_I, \mathcal{D}_{[H]}) \subseteq \mathbf{K}$. Consider the following quasi-identity which we denote by ψ_G:

$$\forall x \forall y \quad xy^{[G]} = x \;\&\; yx = y \;\longrightarrow\; xy^{[G\backslash\{j\}]} = x.$$

It is straightforward that $\mathcal{D}_{[G]} \not\models \psi_G$. As $\mathcal{D}_{[G]} \in \mathbf{K}$, we conclude that $\psi_G \notin \mathrm{Th}_q(\mathbf{K})$.

We prove that $\mathbf{Q}(\mathbf{K}_I, \mathcal{D}_{[H]}) \models \psi_G$. Indeed, let $\mathcal{D} \in \mathbf{K}_I$, let a, $b \in D$, and let $ab^{[G]} = a$, $ba = b$. As $\mathcal{D} \models \varphi^I_G$, we conclude that $ab^{[G\cap I]} = a$. Since $G \cap I \subseteq G\backslash\{j\}$, we get that $ab^{[G\backslash\{j\}]} = a$, whence $\mathcal{D} \models \psi_G$. Moreover, as the equality $ab^{[G]} = a$, where a, $b \in D_{[H]}$, implies that $a = \infty$, we conclude that $\mathcal{D}_{[H]} \models \psi_G$. Therefore, $\mathbf{K}_I \subset \mathbf{Q}(\mathbf{K}_I, \mathcal{D}_{[H]}) \subset \mathbf{K}$, whence $\mathbf{K}$ does not cover $\mathbf{K}_I$. □

According to Proposition 2, the quasivariety $\mathbf{K}_I$ has no independent basis of quasi-identities for any infinite set $I \subset \omega$.

Claim 3. *If I, $J \subset \omega$ are infinite sets and $I \not\subseteq J$, then $\mathbf{K}_I \not\subseteq \mathbf{K}_J$.*

Proof of Claim. Let $i \in I\backslash J$. Then according to Claim 1, $\mathcal{D}_{p_i} \in \mathbf{K}_I \backslash \mathbf{K}_J$. □

Claim 3 completes the proof. □

Theorem 6. [40] *There are continuum many quasivarieties of directed graphs in* $\mathbf{C}$ *with no independent quasi-equational basis.*

Proof. The proof goes along the lines of the proof of Theorem 5 with the quasi-identity φ^I_F defined as:

$$\forall x_0 \dots x_{[F]} \quad \&_{i<[F]} E(x_i, x_{i+1}) \;\longrightarrow\; x_0 = x_{[F\cap I]},$$

□

Remark 2. The result of Theorem 6 was proved by different methods by S. V. Sizyĭ in [40, Theorem 4]. A much stronger result than Theorem 6 was proved by A. V. Kravchenko and A. V. Yakovlev in [27]. Namely, they proved that for each quasivariety $\mathbf{K}$ of [undirected] graphs containing a non-bipartite graph, there is a quasivariety $\mathbf{K}' \subseteq \mathbf{K}$ which contains continuum many subquasivarieties with no independent quasi-equational basis relative to $\mathbf{K}'$.

Theorem 7. *There are continuum many quasivarieties of pointed Abelian groups with no independent quasi-equational basis.*

Proof. We define the quasi-identity φ^I_F as:

$$[F]c = 0 \;\longrightarrow\; [F \cap I]c = 0.$$

Then the proof repeats the proof of Theorem 5. □

We thank Aleksandr Kravchenko for drawing our attention to the paper of M. Demlová and V. Koubek [13]. The authors of [13] find, in particular, a finite AD-class in the quasivariety generated by a certain semigroup which was considered by M. V. Sapir [35] and was the first example of a Q-universal quasivariety. This result of M. Demlová and V. Koubek in combination with [36, Theorem 4.4] solves [36, Problem 2] in the positive.

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