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RELATIVE e-SPECTRA AND RELATIVE CLOSURES FOR FAMILIES OF THEORIES

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ABSTRACT. We define the notions of relative e-spectra, with respect to E-operators, relative closures, and relative generating sets. We study properties connected with relative e-spectra and relative generating sets.

Keywords: E-operator, combination of theories, relative e-spectrum, disjoint families of theories, relative closure, relative generating set.

We continue to study structural properties of combinations of structures and their theories [2, 3, 4] generalizing the notions of e-spectra, closures and generating sets to relative ones. Properties of relative e-spectra and relative generating sets are investigated.

In Section 1 we recall preliminary notions and results on combinations of structures and their theories, e-spectra and closures. In Section 2 relative e-spectra are defined and their properties and values are described. In Section 3 we study families of theories with and without least generating sets. It is shown that the property of (non-)existence of least generating set is not preserved under extensions of families of theories. In Section 4 we present a topological characterization of the existence of relative least generating set and connect this property with values of e-spectra.

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1. Preliminaries

Throughout the paper we use the following terminology in [2, 3].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write

$$\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}},$$

where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write

$$\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$$

for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [3]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 1.1 [3]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure* Cl_E*) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Theorem 1.2 [3]. *For any sets $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$.*

Definition [3]. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$, where $\mathcal{O}_E(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}_E(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$. A subset $\mathcal{T}_0' \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}_0')$. The generating set $\mathcal{T}_0'$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}_0'$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}_0'$ is *least* if $\mathcal{T}_0'$ is contained in each generating set for $\mathcal{T}_0$.

Theorem 1.3 [3]. *If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

2. Relative e-spectra and their properties

Definition. For a structure $\mathcal{A}_E$ and a class K of structures, the number of *new* structures with respect to the structures $\mathcal{A}_i$ and to the class K, i. e., of the structures $\mathcal{B}$ forming E-classes of models of $\mathrm{Th}(\mathcal{A}_E)$ such that $\mathcal{B}$ are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$ in $\mathcal{A}_E$ as well as to the structures in K, is called the *relative e-spectrum* of $\mathcal{A}_E$ with respect to K and denoted by e_K-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e_K\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *relative e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ with respect to K and denoted by e_K-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

Similarly for a class $\mathcal{T}$ of theories and for a theory $T = \mathrm{Th}(\mathcal{A}_E)$ we denote by $e_{\mathcal{T}}$-$\mathrm{Sp}(T)$ the value e_K-$\mathrm{Sp}(T)$, where $K = K(\mathcal{T})$ is the class of all structures, each of which is a model of a theory in $\mathcal{T}$. The value $e_{\mathcal{T}}$-$\mathrm{Sp}(T)$ is called the *relative e-spectrum* of the theory T with respect to $\mathcal{T}$.

Remark 2.1. 1. The class $K(\mathcal{T})$, in the definition above, can be replaced by any subclass $K' \subseteq K(\mathcal{T})$ such that any structure in $K(\mathcal{T})$ is elementary equivalent to a structure in K'.

2. If $K_1 \subseteq K_2$ then e_{K_1}-$\mathrm{Sp}(T) \geq e_{K_2}$-$\mathrm{Sp}(T)$, and if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then $e_{\mathcal{T}_1}$-$\mathrm{Sp}(T) \geq e_{\mathcal{T}_2}$-$\mathrm{Sp}(T)$.

3. The value $e_{\mathcal{T}}$-$\mathrm{Sp}(T)$ is equal to the supremum $|\mathcal{T}_1 \setminus \mathcal{T}_0|$ for theories of E-classes of models of T such that $\mathcal{T}_1$ consists of all these theories and $\mathcal{T}_0 \subseteq \mathcal{T}_1$ with $\mathrm{Cl}_E(\mathcal{T}_0) = \mathcal{T}_1$.

Definition. Two theories T_1 and T_2 of a language Σ are *disjoint* modulo Σ_0, where $\Sigma_0 \subseteq \Sigma$, or Σ_0-*disjoint* if T_1 and T_2 are do not have common nonempty predicates for $\Sigma \setminus \Sigma_0$. If T_1 and T_2 are $\varnothing$-disjoint, these theories are called simply *disjoint*.

Families $\mathcal{T}_j$, $j \in J$, of theories in the language Σ are *disjoint* modulo Σ_0, or Σ_0-*disjoint* if T_{j_1} and T_{j_2} are Σ_0-disjoint for any $T_{j_1} \in \mathcal{T}_{j_1}$, $T_{j_2} \in \mathcal{T}_{j_2}$, $j_1 \neq j_2$. If T_{j_1} and T_{j_2} are disjoint for any $T_{j_1} \in \mathcal{T}_{j_1}$, $T_{j_2} \in \mathcal{T}_{j_2}$, $j_1 \neq j_2$, then the families $\mathcal{T}_j$, $j \in J$, are *disjoint* too.

The following properties are obvious.

1. Any families of theories in a language Σ are Σ-disjoint.

2. (Monotony) If $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma$ then disjoint families modulo Σ_0, in the language Σ, are disjoint modulo Σ_1.

3. (Monotony) If families $\mathcal{T}_{j_1}$ and $\mathcal{T}_{j_2}$ are Σ_0-disjoint then any subfamilies $\mathcal{T}'_{j_1} \subseteq \mathcal{T}_{j_1}$ and $\mathcal{T}'_{j_2} \subseteq \mathcal{T}_{j_2}$ are Σ_0-disjoint too.

Below we denote by K_Σ the class of all structures in languages containing Σ such that all predicates outside Σ are empty. Similarly we denote by $\mathcal{T}_\Sigma$ the class of all theories of structures in K_Σ.

Theorem 2.2. (Relative additivity for e-spectra) *If $\mathcal{T}_j$, $j \in J$, are Σ_0-disjoint families then for the E-combination $T = \mathrm{Comb}_E(T_i)_{i\in I}$ of $\{T_i \mid i \in I\} = \bigcup\limits_{j\in J} \mathcal{T}_j$ and for the E-combinations $T_j = \mathrm{Comb}_E(\mathcal{T}_j)$, $j \in J$,*

$$e_{\mathcal{T}_{\Sigma_0}}\text{-}\mathrm{Sp}(T) = \sum_{j\in J} (e_{\mathcal{T}_{\Sigma_0}}\text{-}\mathrm{Sp}(T_j)). \tag{1}$$

Proof. Denote by $\mathcal{T}$ the set of theories for E-classes of models of T. Since the families $\mathcal{T}_j$ are Σ_0-disjoint, applying Proposition 1.1 we have that a theory T^* belongs to $\mathrm{Cl}_E(\mathcal{T}^*)$, where $\mathcal{T}^* \subseteq \mathcal{T}$, if and only if some of the following conditions holds:

1) $T^* \in \mathcal{T}^*$;

2) for any formula $\varphi \in T^*$ without predicate symbols in $\Sigma \setminus \Sigma_0$, or with predicate symbols in $\Sigma \setminus \Sigma_0$ and saying that corresponding predicates are empty, there are infinitely many theories in $T \in \mathcal{T}^*$ containing φ;

3) for any formula $\varphi \in T^*$, saying that some predicates in $\Sigma \setminus \Sigma_0$ which used in φ are nonempty, there are infinitely many theories in $T \in \mathcal{T}^* \cap \mathcal{T}_j$, for some j, containing φ; moreover, the theories T belong to the unique $\mathcal{T}_j$.

Indeed, taking a formula φ in the language Σ we have finitely many symbols $R_1,\ldots,R_n$ in $\Sigma\setminus\Sigma_0$, used in φ. Considering formulas ψ_i saying that R_k are nonempty, $k=1,\ldots,n$, we get finitely many possibilities for $\chi^{\delta_1,\ldots,\delta_n}\rightleftharpoons\varphi\wedge\bigwedge\limits_{k=1}^{n}\psi_k^{\delta_k}$, $\delta_k\in\{0,1\}$. Since φ is equivalent to $\bigvee\limits_{\delta_1,\ldots,\delta_n}\chi^{\delta_1,\ldots,\delta_n}$ and only subdisjunctions with positive ψ_k related to the fixed $\mathcal{T}_j$ hold, we can divide the disjunction to disjoint parts related to $\mathcal{T}_j$. Since for φ there are finitely many related $\mathcal{T}_j$, we have finitely many cases for φ, each of which related to the fixed $\mathcal{T}_j$. These cases are described in Item 3. Item 2 deals with formulas in the language Σ_0 and with formulas for empty part in $\Sigma\setminus\Sigma_0$. In particular, by Proposition 1.1 these formulas define $\mathrm{Cl}_E(\mathcal{T}^*)\cap\mathcal{T}_{\Sigma_0}$.

Using Items 1–3 we have for $\mathcal{T}^*$ that a theory T^* belongs to $\mathcal{T}^*\setminus\mathcal{T}_{\Sigma_0}$ if and only if T^* belong to $(\mathcal{T}^*\cap\mathcal{T}_j)\setminus\mathcal{T}_{\Sigma_0}$ for unique $j\in J$. Thus theories witnessing the value $e_{\mathcal{T}_{\Sigma_0}}$-$\mathrm{Sp}(T)$ are divided into disjoint parts witnessing the values $e_{\mathcal{T}_{\Sigma_0}}$-$\mathrm{Sp}(T_j)$. Thus the equality (1) holds. □

Remark 2.3. Having positive ComLim [2] the equality (1) can fail if families $\mathcal{T}_j$ are not Σ_0-disjoint, even for finite sets J of indexes, producing

$$e_{\mathcal{T}_{\Sigma_0}}\text{-}\mathrm{Sp}(T')<\sum_{j\in J}(e_{\mathcal{T}_{\Sigma_0}}\text{-}\mathrm{Sp}(T_j)) \tag{2}$$

for appropriate T'.

Theorem 2.2 immediately implies

Corollary 2.4. *If $\mathcal{T}_j$, $j\in J$, are disjoint then for the E-combination*

$$T=\mathrm{Comb}_E(T_i)_{i\in I}$$

of $\{T_i\mid i\in I\}=\bigcup\limits_{j\in J}\mathcal{T}_j$ and for the E-combinations $T_j=\mathrm{Comb}_E(\mathcal{T}_j)$, $j\in J$,

$$e_{\mathcal{T}_\varnothing}\text{-}\mathrm{Sp}(T)=\sum_{j\in J}(e_{\mathcal{T}_\varnothing}\text{-}\mathrm{Sp}(T_j)). \tag{3}$$

Definition. The theory T in Theorem 2.2 is called the Σ_0-*disjoint E-union* of the theories T_j, $j\in J$, and the theory T in Corollary 2.4 is the *disjoint E-union* of the theories T_j, $j\in J$.

Remark 2.5. Additivity (1) and, in particular, (3) can be failed without indexes $\mathcal{T}_{\Sigma_0}$. Indeed, it is possible to find T_j with e-$\mathrm{Sp}(T_j)=0$ (for instance, with finite $\mathcal{T}_j$) while e-$\mathrm{Sp}(T)$ can be positive. Take, for example, disjoint singletons $\mathcal{T}_n=\{T_n\}$, $n\in\omega\setminus\{0\}$, such that T_n has n-element models. We have e-$\mathrm{Sp}(T_n)=0$ for each n while e-$\mathrm{Sp}(T)=1$, since the theory $T_\infty\in\mathcal{T}_\varnothing$ with infinite models belong to $\mathrm{Cl}_E(\{T_n\mid n\in\omega\setminus\{0\}\})$. Thus, for disjoint families $\mathcal{T}_j$, $j\in J$, the equality

$$e\text{-}\mathrm{Sp}(T)=\sum_{j\in J}(e\text{-}\mathrm{Sp}(T_j)) \tag{4}$$

can fail. Moreover, producing the effect above for definable subsets in models of T_j we get

$$e_{\mathcal{T}_{\Sigma_0}}\text{-}\mathrm{Sp}(T)>\sum_{j\in J}(e_{\mathcal{T}_{\Sigma_0}}\text{-}\mathrm{Sp}(T_j)).$$

At the same time, by Corollary 2.4 (respectively, by Theorem 2.2) the equality (4) holds for (Σ_0-)disjoint families $\mathcal{T}_j$, $j \in J$, if J is finite and each $\mathcal{T}_j$ does not generate theories in $\mathcal{T}_\varnothing$ (in $\mathcal{T}_{\Sigma_0}$).

Applying the equality (3) we take an E-combination T_0 with $e_{\mathcal{T}_\varnothing}$-Sp$(T_0) = \lambda$. Furthermore we consider disjoint copies T_j, $j \in J$, of T_0. Combining E-classes of all T_j we obtain a theory T such that if J is finite then $e_{\mathcal{T}_\varnothing}$-Sp$(T) = |J| \cdot \lambda$. We have the same formula if $|J| \geq \omega$ and $\lambda > 0$ since, in this case, the E-closure for theories of E-classes of models of T consists of theories of E-classes for theories T_j as well some theories in $\mathcal{T}_\varnothing$. If E-classes have a fixed finite or only infinite cardinalities, this theory has models whose cardinalities (finite or countable) are equal to the (either finite or countable) cardinality of models of T_j. Similarly, having theories T_λ of languages Σ with cardinalities $|\Sigma| = \lambda + 1$ and with e-Sp$(T_0) = \lambda > 0$ [2, Proposition 4.3] and taking E-combinations with their disjoint copies we get

Proposition 2.6. *For any positive cardinality λ there is a theory T such that E-classes of models of T form copies T_j, $j \in J$, of some E-combination T_0 with a language Σ in the cardinality $\lambda + 1$, with $e_{\mathcal{T}_\varnothing}$-Sp$(T_0) = \lambda$, and $e_{\mathcal{T}_\varnothing}$-Sp$(T) = |J| \cdot \lambda$.*

Remark 2.7. Since there are required theories T_0 which do not generate E-classes for $\mathcal{T}_\varnothing$, Proposition 2.6 can be reformulated without the index $\mathcal{T}_\varnothing$.

Remark 2.8. Extending the Σ_0-disjoint Σ_0-coordinated E-union T by definable bijections linking E-classes we can omit the additivity (1). Indeed, adding, for instance, bijections f_{jk} witnessing isomorphisms for models of disjoint copies T_j and T_j, have we $e_{\mathcal{T}_\varnothing}$-Sp$(T_j)$ instead of $e_{\mathcal{T}_\varnothing}$-Sp$(T_j) + e_{\mathcal{T}_\varnothing}$-Sp$(T_k)$. Thus, bijections f_{jk} allow to vary $e_{\mathcal{T}_\varnothing}$-Sp$(T)$ from λ to $|J| \cdot \lambda$ in terms of Proposition 2.6. Thus the equality (1) can fail again producing (2) for appropriate T'.

3. Families of theories with(out) least generating sets

Below we apply Theorem 1.3 characterizing the existence of e-least generating sets for Σ_0-disjoint families of theories.

The following natural questions arises:

Question 1. *When the existence of the least generating sets for the families $\mathcal{T}_j$, $j \in J$, is equivalent to the existence of the least generating set for the family $\bigcup\limits_{j\in J} \mathcal{T}_j$?*

Question 2. *Is it true that under conditions of Theorem 2.2 the existence of the least generating sets for the families $\mathcal{T}_j$, $j \in J$, is equivalent to the existence of the least generating set for the family $\bigcup\limits_{j\in J} \mathcal{T}_j$?*

Considering Question 2, we note below that the property of the (non)existence of the least generating sets is not preserving under expansions and extensions of families of theories.

Proposition 3.1. *Any E-closed family $\mathcal{T}_0$ of theories in a language Σ_0 can be transformed to an E-closed family $\mathcal{T}'_0$ in a language $\Sigma'_0 \supseteq \Sigma_0$ such that $\mathcal{T}'_0$ consists of expansions of theories in $\mathcal{T}_0$ and $\mathcal{T}'_0$ has the least generating set.*

Proof. Forming Σ'_0 it suffices to take new predicate symbols R_{T_0}, $T_0 \in \mathcal{T}_0$, such that $R_{T_0} \neq \varnothing$ for interpretations in the models of expansion T'_0 of T_0 and $R_{T_0} = \varnothing$ for interpretations in the models of expansion T'_1 of $T_1 \neq T_0$. Each formula $\exists \bar{x} R_{T_0}(\bar{x})$ isolates T'_0, and thus $\mathcal{T}'_0$ has the least generating set in view of Theorem 1.3. $\square$

Existence of families $\mathcal{T}_0$ without least generating sets implies

Corollary 3.2. *The property of non-existence of least generating sets is not preserved under expansions of theories.*

Remark 3.3. The expansion $\mathcal{T}'_0$ of $\mathcal{T}_0$ in the proof of Proposition 3.1 produces discrete topologies for sets of theories in $\mathcal{T}_0\cup\mathcal{T}'_0$. In fact, for this purpose it suffices to isolate finite sets in $\mathcal{T}_0$ since any two distinct elements $T_0, T_1\in\mathcal{T}_0$ are separated by formulas φ such that $\varphi\in T_i$ and $\neg\varphi\in T_{1-i}$, $i=0,1$.

Note also that these operators of discretization transform the given set $\mathcal{T}_0$ to a set $\mathcal{T}'_0$ with identical Cl_E.

Recall [4] that a theory T in a predicate language Σ is called *language uniform*, or a LU-*theory* if for each arity n any substitution on the set of non-empty n-ary predicates preserves T.

Clearly, if a set $\mathcal{T}_0$ has the discrete topology it can not be expanded to a set without the least generating set. At the same time, there are expansions that transform sets with the least generating sets to sets without the least generating sets. Indeed, take Example in [4, Remark 3] with countably many disjoint copies $\mathcal{F}_q$, $q\in\mathbb{Q}$, of linearly ordered sets isomorphic to $\langle\omega,\leq\rangle$ and ordering limits $J_q=\overline{\lim}\,F_q$ by the ordinary dense order on $\mathbb{Q}$ such that $\{J_q\mid q\in\mathbb{Q}\}$ is densely ordered. We have a dense interval $\{J_q\mid q\in\mathbb{Q}\}$ whereas the set $\cup\{F_q\mid q\in\mathbb{Q}\}$ forms the least generating set $\mathcal{T}_0$ of theories for $\mathrm{Cl}_E(\mathcal{T}_0)$. Now we expand the LU-theories for $\mathcal{F}_q$ and J_q by new predicate symbol R such that R is empty for all theories corresponding to $\mathcal{F}_q$ and $\forall\bar{x}R(\bar{x})$ is satisfied for all theories corresponding to J_q. The predicate R separates the set of theories for J_q with respect to Cl_E. At the same time the theories for J_q forms the dense interval producing the set without the least generating set in view of [4, Theorem 2]. Thus, we get the following

Proposition 3.4. *There is an E-closed family $\mathcal{T}_0$ of theories in a language Σ_0 and with the least generating set, which can be transformed to an E-closed family $\mathcal{T}'_0$ in a language $\Sigma'_0\supseteq\Sigma_0$ such that $\mathcal{T}'_0$ consists of expansions of theories in $\mathcal{T}_0$ and $\mathcal{T}'_0$ does not have the least generating set.*

Corollary 3.5. *The property of existence of least generating sets is not preserved under expansions of theories.*

Remark 3.6. Adding the predicate R which separates theories for J_q from theories for $\mathcal{F}_q$, we get a copy for each J_q containing empty R. This effect is based on the property that separating an accumulation point J_q for $\mathcal{F}_q$ we get new accumulation point preserving formulas in the initial language.

Introducing the predicate R together with the discretization for $\mathcal{F}_q$, E-closures do not generate new theories.

Proposition 3.7. *Any family $\mathcal{T}_0$ of theories in a language Σ, with infinitely many empty predicates for all theories in $\mathcal{T}_0$, can be extended to a family $\mathcal{T}'_0$ in the language Σ such that $\mathcal{T}'_0$ does not have the least generating set.*

Proof. Let $\Sigma_0\subseteq\Sigma$ consist of predicate symbols which are empty for all theories in $\mathcal{T}_0$. Now we consider a family $\mathcal{T}_1$ of LU-theories such that all these theories have empty predicates for $\Sigma\setminus\Sigma_0$, and, using Σ_0 as for [4, Theorem 2], $\mathcal{T}_1$ does not have the least generating set forming a dense interval. The family $\mathcal{T}'_0=\mathcal{T}_0\dot\cup\mathcal{T}_1$

extends $\mathcal{T}_0$ and does not have the least generating set since for any $\mathcal{T}_0'' \subseteq \mathcal{T}_0'$, $\mathrm{Cl}_E(\mathcal{T}_0'') = \mathrm{Cl}_E(\mathcal{T}_0'' \cap \mathcal{T}_0) \,\dot{\cup}\, \mathrm{Cl}_E(\mathcal{T}_0'' \cap \mathcal{T}_1)$. □

Corollary 3.8. *The property of existence of least generating sets is not preserved under extensions of sets of theories.*

In view of Theorem 1.3 any family consisting of all theories in a given infinite language both does not have the least generating set and does not have a proper extension in the given language. Thus there are families of theories without least generating sets and without extension having least generating sets. At the same time the following proposition holds.

Proposition 3.9. *There is an E-closed family $\mathcal{T}_0$ of theories in a language Σ and without the least generating set such that $\mathcal{T}_0$ can be extended to an E-closed family $\mathcal{T}_0'$ in the language Σ and with the least generating set.*

Proof. It suffices to take Example in [4, Remark 3] that we used for the proof of Proposition 3.4. The theories for $\{J_q \mid q \in \mathbb{Q}\}$ form a family without the least generating set whereas an extension of this family by the theories for $\mathcal{F}_q$ has the least generating set. □

Corollary 3.10. *The property of non-existence of least generating sets is not preserved under extensions of sets of theories.*

Remark 3.11. If an extension of an E-closed family $\mathcal{T}_0$ of theories transforms $\mathcal{T}_0$ with the least generating set to an E-closed family $\mathcal{T}_0'$ without the least generating set then, in view of Theorem 1.3, having the generating set in $\mathcal{T}_0$ consisting of isolated points we lose this property for $\mathcal{T}_0'$. If an extension of an E-closed family $\mathcal{T}_0$ of theories transforms $\mathcal{T}_0$ without the least generating set to an E-closed family $\mathcal{T}_0'$ with the least generating set then, again in view of Theorem 1.3, we add a set of isolated theories to $\mathcal{T}_0$ generating all theories in $\mathcal{T}_0'$.

Now we return to Questions 1 and 2.

Clearly, for any set $\mathcal{T}$ of theories, $\mathrm{Cl}_E(\mathcal{T} \cap \mathcal{T}_{\Sigma_0}) \subset \mathcal{T}_{\Sigma_0}$. Therefore $\mathrm{Cl}_E(\mathcal{T})$ and each its generating set are divided into parts: in $\mathcal{T}_{\Sigma_0}$ and disjoint with $\mathcal{T}_{\Sigma_0}$. Since $\mathcal{T}_j$, $j \in J$, are disjoint with respect to $\mathcal{T}_{\Sigma_0}$, each $\mathcal{T}_j$ has the least generating set if and only if both $\mathcal{T}_j \cap \mathcal{T}_{\Sigma_0}$ and $\mathcal{T}_j \setminus \mathcal{T}_{\Sigma_0}$ have the least generating sets. Since under conditions of Theorem 2.2 the sets $\mathcal{T}_j \setminus \mathcal{T}_{\Sigma_0}$ are disjoint, $j \in J$, we have the following proposition answering Question 1.

Proposition 3.12. *The set $\bigcup\limits_{j\in J} \mathcal{T}_j$ has the least generating set if and only if*

$$\left(\bigcup_{j\in J} \mathcal{T}_j\right) \cap \mathcal{T}_{\Sigma_0}$$

has the least generating set and each $\mathcal{T}_j \setminus \mathcal{T}_{\Sigma_0}$ has the least generating set.

Since $\left(\bigcup\limits_{j\in J} \mathcal{T}_j\right) \cap \mathcal{T}_{\Sigma_0}$ can be an arbitrary extension of each $\mathcal{T}_j \cap \mathcal{T}_{\Sigma_0}$, Propositions 3.7 and 3.12 imply the following corollary answering Question 2.

Corollary 3.13. *For any infinite language Σ_0 there are Σ_0-disjoint families $\mathcal{T}_j$, $j \in J$, with the least generating sets such that $\bigcup\limits_{j\in J} \mathcal{T}_j$ does not have the least generating set.*

4. Relative closures and relative least generating sets

Definition. Let $\mathcal{T}$ be a class of theories. For a set $\mathcal{T}_0 \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_{E,\mathcal{T}}(\mathcal{T}_0)$ the set $\mathrm{Cl}_E(\mathcal{T}_0) \setminus \mathcal{T}$. The set $\mathrm{Cl}_{E,\mathcal{T}}(\mathcal{T}_0)$ is called the *relative E-closure* of the set $\mathcal{T}_0$ with respect to $\mathcal{T}$, or *$\mathcal{T}$-relative E-closure*. If $\mathcal{T}_0 \setminus \mathcal{T} = \mathrm{Cl}_{E,\mathcal{T}}(\mathcal{T}_0)$ then $\mathcal{T}_0$ is said to be (relatively) *E-closed with respect to* $\mathcal{T}$, or *$\mathcal{T}$-relatively E-closed*.

Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$. A subset $\mathcal{T}'_0 \subseteq \mathcal{T}_0$ is said to be *generating* with respect to $\mathcal{T}$, or *$\mathcal{T}$-relatively generating*, if $\mathcal{T}_0 \setminus \mathcal{T} = \mathrm{Cl}_{E,\mathcal{T}}(\mathcal{T}'_0)$. The $\mathcal{T}$-relatively generating set $\mathcal{T}'_0$ (for $\mathcal{T}_0$) is *$\mathcal{T}$-minimal* if $\mathcal{T}'_0 \setminus \mathcal{T}$ does not contain proper subsets $\mathcal{T}''_0$ such that $\mathcal{T}_0 \setminus \mathcal{T} = \mathrm{Cl}_{E,\mathcal{T}}((\mathcal{T}'_0 \cap \mathcal{T}) \cup \mathcal{T}''_0)$. A $\mathcal{T}$-minimal $\mathcal{T}$-relatively generating set $\mathcal{T}'_0$ is *$\mathcal{T}$-least* if $\mathcal{T}'_0 \setminus \mathcal{T}$ is contained in $\mathcal{T}''_0 \setminus \mathcal{T}$ for each $\mathcal{T}$-relatively generating set $\mathcal{T}''_0$ for $\mathcal{T}_0$.

Remark 4.1. Note that for $\mathcal{T}$-least generating sets $\mathcal{T}'_0$, in general, we can say that $\mathcal{T}'_0$ are uniquely defined only with respect to $\mathcal{T}$. Moreover, since $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$ for any sets $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$ by Theorem 1.2, then for E-closed $\mathcal{T}$, $\mathrm{Cl}_E(\mathcal{T}'_0 \cup \mathcal{T}) = \mathrm{Cl}_E(\mathcal{T}'_0) \cup \mathcal{T}$ and $\mathcal{T}'_0$ is a $\mathcal{T}$-least generating set if and only if $\mathcal{T}'_0 \cup \mathcal{T}'$ is a $\mathcal{T}$-least generating set for some (any) $\mathcal{T}' \subseteq \mathcal{T}$, as well as if and only if $\mathcal{T}'_0 \setminus \mathcal{T}$ is a $\mathcal{T}$-least generating set.

The following theorem generalizes Theorem 1.3.

Theorem 4.2. *If $\mathcal{T}$ is a E-closed set and $\mathcal{T}'_0$ is a $\mathcal{T}$-relatively generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*
(1) *$\mathcal{T}'_0$ is the $\mathcal{T}$-least generating set for $\mathcal{T}_0$;*
(2) *$\mathcal{T}'_0$ is a $\mathcal{T}$-minimal generating set for $\mathcal{T}_0$;*
(3) *any theory in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some set $(\mathcal{T}'_0 \cup \mathcal{T})_\varphi$;*
(4) *any theory in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some set $(\mathcal{T}_0 \cup \mathcal{T})_\varphi$;*
(5) *any theory in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some set $(\mathcal{T}'_0)_\varphi$;*
(6) *any theory in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some set $(\mathcal{T}_0)_\varphi$.*

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Assume that $\mathcal{T}'_0$ is $\mathcal{T}$-minimal but not $\mathcal{T}$-least. Then there is a $\mathcal{T}$-relatively generating set $\mathcal{T}''_0$ such that $\mathcal{T}'_0 \setminus (\mathcal{T}''_0 \cup \mathcal{T}) \neq \varnothing$ and $\mathcal{T}''_0 \setminus (\mathcal{T}'_0 \cup \mathcal{T}) \neq \varnothing$. Take $T \in \mathcal{T}'_0 \setminus (\mathcal{T}''_0 \cup \mathcal{T})$.

We assert that $T \in \mathrm{Cl}_E(\mathcal{T}'_0 \setminus (\{T\} \cup \mathcal{T}))$, i.e., T is an accumulation point of $\mathcal{T}'_0 \setminus (\{T\} \cup \mathcal{T})$. Indeed, since $\mathcal{T}''_0 \setminus (\mathcal{T}'_0 \cup \mathcal{T}) \neq \varnothing$ and $\mathcal{T}''_0 \subset \mathrm{Cl}_E(\mathcal{T}'_0 \cup \mathcal{T}) = \mathrm{Cl}_E(\mathcal{T}'_0 \setminus \mathcal{T}) \cup \mathcal{T}$ (using that $\mathcal{T}$ is E-closed), then by [3, Proposition 1, (3)] (that every finite set $\mathcal{T} \subset \overline{\mathcal{T}}$ is E-closed), $\mathcal{T}'_0 \setminus \mathcal{T}$ is infinite and by Proposition 1.1 it suffices to prove that for any $\varphi \in T$, $((\mathcal{T}'_0 \setminus (\{T\} \cup \mathcal{T}))_\varphi$ is infinite. Assume on contrary that for some $\varphi \in T$, $((\mathcal{T}'_0 \setminus (\{T\} \cup \mathcal{T}))_\varphi$ is finite. Then $(\mathcal{T}'_0 \setminus \mathcal{T})_\varphi$ is finite and, moreover, as $\mathcal{T}'_0$ is $\mathcal{T}$-relatively generating for $\mathcal{T}_0$, by Proposition 1.1, $(\mathcal{T}_0 \setminus \mathcal{T})_\varphi$ is finite, too. So $(\mathcal{T}''_0 \setminus \mathcal{T})_\varphi$ is finite and, again by Proposition 1.1, T does not belong to $\mathrm{Cl}_E(\mathcal{T}''_0 \cup \mathcal{T})$ contradicting to $\mathrm{Cl}_E(\mathcal{T}''_0) = \mathcal{T}_0$.

Since $T \in \mathrm{Cl}_E(\mathcal{T}'_0 \setminus (\{T\} \cup \mathcal{T}))$ and $\mathcal{T}'_0$ is generating for $\mathcal{T}_0$, then $\mathcal{T}'_0 \setminus \{T\}$ is also generating for $\mathcal{T}_0$ contradicting the $\mathcal{T}$-minimality of $\mathcal{T}'_0$.

(2) $\Rightarrow$ (3). If $\mathcal{T}'_0 \setminus \mathcal{T}$ is finite then by Proposition 2.1 (3), $\mathcal{T}'_0 \setminus \mathcal{T} = \mathcal{T}_0 \setminus \mathcal{T}$. Since $\mathcal{T}_0 \setminus \mathcal{T}$ is finite and $\mathcal{T}$ is E-closed then for any $T \in \mathcal{T}_0 \setminus \mathcal{T}$ there is a formula $\varphi \in T$ negating all theories in $(\mathcal{T}_0 \setminus \{T\}) \cup \mathcal{T})$. Therefore, $(\mathcal{T}_0 \cup \mathcal{T})_\varphi = (\mathcal{T}'_0 \cup \mathcal{T})_\varphi$ is a singleton containing T and thus, $(\mathcal{T}'_0 \cup \mathcal{T})_\varphi$ isolates T.

Now let $\mathcal{T}'_0 \setminus \mathcal{T}$ be infinite. Assume that some $T \in \mathcal{T}'_0 \setminus \mathcal{T}$ is not isolated by the sets $(\mathcal{T}'_0 \cup \mathcal{T})_\varphi$. It implies that for any $\varphi \in T$, $((\mathcal{T}'_0 \setminus \{T\}) \cup \mathcal{T})_\varphi$ is infinite. Using Proposition 1.1 and the condition that $\mathcal{T}$ is E-closed we obtain $T \in \mathrm{Cl}_{E,\mathcal{T}}(\mathcal{T}'_0 \setminus \{T\})$ contradicting the $\mathcal{T}$-minimality of $\mathcal{T}'_0$.

(3) $\Rightarrow$ (2). Assume that any theory T in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some set $(\mathcal{T}'_0 \cup \mathcal{T})_\varphi$. By Proposition 1.1 it implies that $T \notin \mathrm{Cl}_E((\mathcal{T}'_0 \setminus \{T\}) \cup \mathcal{T})$. Thus, $\mathcal{T}'_0$ is a $\mathcal{T}$-minimal generating set for $\mathcal{T}_0$.

(3) $\Rightarrow$ (4) is obvious for finite $\mathcal{T}'_0 \setminus \mathcal{T}$. If $\mathcal{T}'_0 \setminus \mathcal{T}$ is infinite and any theory T in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some set $(\mathcal{T}'_0 \cup \mathcal{T})_\varphi$ then T is isolated by the set $(\mathcal{T}_0 \cup \mathcal{T})_\varphi$, since otherwise using Proposition 1.1 and the properties that $\mathcal{T}$ is E-closed and $\mathcal{T}'_0$ generates $\mathcal{T}_0$, there are infinitely many theories in $\mathcal{T}'_0$ containing φ that contradicts the equality $|(\mathcal{T}'_0 \cup \mathcal{T})_\varphi| = 1$.

(3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) are equivalent since $\mathcal{T}$ is E-closed. $\square$

Corollary 4.3. *If $\mathcal{T}_j$, $j \in J$, are Σ_0-disjoint families then $\bigcup\limits_{j\in J} \mathcal{T}_j$ has a $\mathcal{T}_{\Sigma_0}$-least generating set if and only if each $\mathcal{T}_j$ has a $\mathcal{T}_{\Sigma_0}$-least generating set. Moreover, if $\bigcup\limits_{j\in J} \mathcal{T}_j$ has a $\mathcal{T}_{\Sigma_0}$-least generating set $\mathcal{T}_0$ then $\mathcal{T}_0 \setminus \mathcal{T}_{\Sigma_0}$ can be represented as a disjoint union of $\mathcal{T}_{\Sigma_0}$-least generating sets for $\mathcal{T}_j$.*

Proof. Using Theorem 4.2 it suffices to note that $\mathcal{T}_{\Sigma_0}$ is E-closed and having $\mathcal{T}_0 \setminus \mathcal{T}_{\Sigma_0}$ it consists of isolated points each of which is related to exactly one set $\mathcal{T}_j$. $\square$

Clearly, any subset of $\mathcal{T}$-least generating set is again a $\mathcal{T}$-least generating set (for its E-closure). At the same time the property "to be a $\mathcal{T}$-least generating set" is preserved under finite extensions of generating sets $\mathcal{T}'_0$ disjoint with $\mathrm{Cl}_E(\mathcal{T}'_0)$:

Proposition 4.4. *If $\mathcal{T}$ is a E-closed set, $\mathcal{T}'_0$ is a $\mathcal{T}$-relatively generating set for a E-closed set $\mathcal{T}_0$, and $\mathcal{T}_f$ is a finite subset of $\overline{\mathcal{T}}$ disjoint with $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the $\mathcal{T}$-least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}'_0 \cup (\mathcal{T}_f \setminus \mathcal{T}_0)$ is the $\mathcal{T}$-least generating set for the E-closed set $\mathcal{T}_0 \cup \mathcal{T}_f$.*

Proof. (1) $\Rightarrow$ (2). If $\mathcal{T}'_0$ is a $\mathcal{T}$-least generating set for $\mathcal{T}_0$ then by Theorem 4.2 each theory T in $\mathcal{T}'_0 \setminus \mathcal{T}$ is isolated by some formula φ_T. Since $\mathcal{T}_f$ is finite then each theory T in $(\mathcal{T}'_0 \cup (\mathcal{T}_f \setminus \mathcal{T}_0)) \setminus \mathcal{T}$ is isolated by some formula ψ_T. Again by Theorem 4.2, $\mathcal{T}'_0 \cup \mathcal{T}_f$ is the $\mathcal{T}$-least generating set for $\mathcal{T}_0 \cup \mathcal{T}_f$ which is E-closed in view of Theorem 1.2.

(2) $\Rightarrow$ (1) is obvious. $\square$

Theorem 4.5. (Decomposition Theorem) *For any E-closed sets $\mathcal{T}$ and $\mathcal{T}'$ of a language Σ there is a $\mathcal{T}$-relatively generating set $\mathcal{T}'_0 \cup \mathcal{T}'_1$ for $\mathcal{T}'$, which is disjoint with $\mathcal{T}$ and satisfies the following conditions:*

(1) $|\mathcal{T}'_0 \cup \mathcal{T}'_1| \leq \max\{|\Sigma|, \omega\}$;

(2) *$\mathcal{T}'_0$ is the least generating set for its E-closure* $\mathrm{Cl}_E(\mathcal{T}'_0)$*;*

(3) $\mathrm{Cl}_E(\mathcal{T}'_0) \cap \mathcal{T}'_1 = \varnothing$;

(4) $\mathcal{T}_1'$ *is either empty or infinite and does not have infinite subsets satisfying* (2).

Proof. We denote by $\mathcal{T}_0'$ the set of isolated points in $\mathcal{T}' \setminus \mathcal{T}$ and by $\mathcal{T}_1'$ the subset of $\mathcal{T}' \setminus (\mathcal{T} \cup \mathrm{Cl}_E(\mathcal{T}_0'))$ with a cardinality $\leq \max\{|\Sigma|, \omega\}$ such that each sentence belonging to a theory in $\mathcal{T}' \setminus (\mathcal{T} \cup \mathrm{Cl}_E(\mathcal{T}_0'))$ belongs to a theory in $\mathcal{T}_1'$. Note that $|\mathcal{T}_0'|$ is bounded by the number of sentences in the language Σ, i. e., $|\mathcal{T}_0'| \leq \max\{|\Sigma|, \omega\}$, too. Thus the condition (1) holds and $\mathcal{T}_0' \cup \mathcal{T}_1'$ is a $\mathcal{T}$-relatively generating set for $\mathcal{T}'$ in view of Proposition 1.1.

By Theorem 4.2, $\mathcal{T}_0'$ is the least generating set for $\mathrm{Cl}_E(\mathcal{T}_0')$. Therefore the condition (2) holds. Now (3) and (4) are satisfied since $\mathcal{T}_1'$ is separated from $\mathrm{Cl}_E(\mathcal{T}_0')$ and does not have isolated points. □

Theorem 4.6. *If T is a E-combination of some theories T_i, $i \in I$, $\mathcal{T}$ is a E-closed set of theories, and $|e_{\mathcal{T}}\text{-}\mathrm{Sp}(T)| < 2^\omega$, then $\mathrm{Cl}_E(\mathcal{T} \cup \{T_i \mid i \in I\})$ has the $\mathcal{T}$-least generating set.*

Proof. By Theorem 4.2 we have to show that $\mathcal{T}' \rightleftharpoons \{T_i \mid i \in I\} \setminus \mathcal{T}$ has a generating set, modulo $\mathcal{T}$, of theories T_i being isolated points. Assume the contrary. Then we have sets $\mathcal{T}_0'$ and $\mathcal{T}_1'$ in terms of Theorem 4.5, where $|\mathcal{T}_0' \cup \mathcal{T}_1'| \leq \max\{|\Sigma|, \omega\}$ and $\mathcal{T}_1'$ is infinite. Thus T has a model $\mathcal{M}$ whose all E-classes satisfy theories in $\mathcal{T}_0' \cup \mathcal{T}_1'$.

Then we can construct a 2-tree [1] of sentences φ_δ, where δ are $\{0,1\}$-tuples, $\{\varphi_{\delta\hat{}0}, \varphi_{\delta\hat{}1}\}$ are inconsistent and $\varphi_\delta \equiv \varphi_{\delta\hat{}0}, \varphi_{\delta\hat{}1}$, such that all $(\mathcal{T}_1')_{\varphi_\delta}$ are infinite. Moreover, taking negations of formulas isolating theories in $\mathcal{T}_1'$ and applying Proposition 1.1 we can assume that for each $f \in 2^\omega$ the sequence of formulas $\varphi_{\langle f(0), \ldots, f(n)\rangle}$, $n \in \omega$, is contained in a theory belonging $\mathrm{Cl}_E(\mathcal{T}_1')$. Thus $|\mathrm{Cl}_E(\mathcal{T}_1')| \geq 2^\omega$ producing, by $\mathcal{M}$, $|e_{\mathcal{T}}\text{-}\mathrm{Sp}(T)| \geq 2^\omega$ that contradicts the assumption $|e_{\mathcal{T}}\text{-}\mathrm{Sp}(T)| < 2^\omega$. □

The following example shows that, in Theorem 4.6, the conditions $|e_{\mathcal{T}}\text{-}\mathrm{Sp}(T)| < 2^\omega$ and the existence of the $\mathcal{T}$-least generating set are not equivalent.

Example 4.7. Let Σ be a language with predicates P_i, Q_j, $i, j \in \omega$, of same arity (it suffices to take the arity 0). Now we consider a countable set of language uniform theories T_i [4] such that unique P_i is satisfied and Q_j are satisfied independently for the set $\mathcal{T} = \{T_i \mid i \in \omega\}$.

All theories T_i are isolated in $\mathrm{Cl}_E(\mathcal{T})$ by the formulas $\exists \bar{x} P_i(\bar{x})$. Hence, $\mathcal{T}$ is the least generating set for $\mathrm{Cl}_E(\mathcal{T})$. At the same time $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$ witnessed by theories with empty predicates P_i and independently satisfying Q_j. Thus $|e_{\mathcal{T}}\text{-}\mathrm{Sp}(T)| = 2^\omega$ for the theory T being the E-combination of T_i, $i \in \omega$.

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