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CLAW-FREE STRICTLY DEZA GRAPHS

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Abstract. A Deza graph with parameters (v, k, b, a) is a k-regular graph, which has exactly v vertices and any two distinct vertices have either a or b common neighbors. A strictly Deza graph is a Deza graph of diameter 2 that is not strongly regular. A claw-free graph is a graph in which no induced subgraph is a complete bipartite graph $K_{1,3}$. We proved if graph G is a claw-free strictly Deza graph which contains a 3-coclique then G is either an $4 \times n$-lattice, where $n > 2$, $n \neq 4$, or the 2-extension of the 3×3-lattice, or two strictly Deza graphs with the parameters $(9, 4, 2, 1)$, or two strictly Deza graphs with the parameters $(12, 6, 3, 2)$, or a Deza line graph with the parameters $(20, 6, 2, 1)$.

Keywords: strictly Deza graphs, claw-free graphs.

1. Introduction

In this paper we consider only undirected graphs in which both multiple edges and loops are disallowed. *A claw* is a short name for the complete bipartite graph $K_{1,3}$. *A claw-free graph* is a graph in which no induced subgraph is isomorphic to the claw. Claw-free graphs were studied as a generalization of line graphs. *The line graph* $L(G)$ of a graph G has the set of edges of a G as the set of vertices and two distinct vertices are adjacent whenever the correspondent edges contain a common vertex in G. Beineke (see [1]) gives a characterization of line graphs in terms of forbidden induced subgraphs and the claw is one of nine of those subgraphs. Later additional motivation are discovered about claw-free graphs. Sumner (1974, [2]) and, independently, Las Vergnas (1975, [3]) proved that every claw-free connected

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graph with an even number of vertices has a perfect matching. Another fact is the discovery of polynomial time algorithms for finding maximum independent sets in claw-free graphs [4], [5]. There is a characterization of claw-free perfect graphs [6]. Also there are hundreds of mathematical research papers and several surveys about claw-free graphs (see, for example, a survey [8]).

Recently Chudnovsky and Seymour published the series of papers in which they developed a structure theory of claw-free graphs (see, for example, [9]). They have shown that every connected claw-free graph can be obtained from one of the basic claw-free graphs by simple expansion operations.

Aside from line graphs there are another claw-free graphs. For example interval graphs, the graph of the icosahedron, the Schläfli graph (the strongly regular graph with parameters $(27, 16, 10, 8)$), complements of triangle-free graphs and more. It is clear the complement of triangle-free graph doesn‘t have a 3-coclique. Trivially such graph is a claw-free graph. Often in investigations of claw-free graphs authors suppose that such graphs contain 3-coclique.

Edge-regular claw-free graphs were studied by Brouwer and Numata in [10]. And coedge-regular claw-free graphs were studied by Kabanov and Makhnev in [14]. In paper [16] Makhnev described claw-free graphs with regular μ-graphs. And in paper [12] Kabanov described claw-free graphs in which μ-graphs have radius more than 1. All these results are very useful in our study, and the main theorems of the mentioned papers will be presented in the next section.

Let γ be a vertex in a graph. Denote by $[\gamma]$ the set of all neighbors of γ. The subgraph on the set $[\gamma] \cup \{\gamma\}$ is called the closed neighborhood of γ and is denoted by $\gamma^{\perp}$.

Let v, k, b, and a be integers such that $0 \leq a \leq b \leq k < v$. A graph G is called a (v, k, b, a) *Deza graph* if:

1) G contains exactly v vertices;

2) for any vertices $u, w \in G$: $|[u] \cap [w]| = \begin{cases} a \text{ or } b \text{ if } u \neq w; \\ k \text{ if } u = w. \end{cases}$

A *strictly Deza graph* is a Deza graph of diameter 2 that is not strongly regular.

A *m-extension* of a graph G is the graph obtained from G by replacing each vertex by m-clique, and two vertices from different m-cliques are adjacent if and only if they are adjacent in G.

The main result of this paper is a description of all claw-free strictly Deza graphs, which contains a 3-coclique.

Main theorem. *Let G be a claw-free strictly Deza graph which contains a 3-coclique. Then G is one of the following graphs:*

(1) *an $4 \times n$-lattice, where $n > 2$, $n \neq 4$;*

(2) *the 2-extension of the 3×3-lattice;*

(3) *two strictly Deza graphs with the parameters $(9, 4, 2, 1)$ (fig. 1.a, 1.b);*

(4) *two strictly Deza graphs with the parameters $(12, 6, 3, 2)$ (fig. 1.c, 1.d);*

(5) *one strictly Deza line graph with the parameters $(20, 6, 2, 1)$ (fig. 2).*

2. Preliminary results

In paper [7] authors introduced two additional parameters for Deza graphs: $\alpha = |\{w \in G : |[u] \cap [w]| = a\}|$, $\beta = |\{w \in G : |[u] \cap [w]| = b\}|$ for any vertex $u \in G$.

There are formulas for α and β in proposition 1.1 in [7]. If $a \neq b$ formulas take the form

$$\alpha = \frac{b(v-1)-k(k-1)}{b-a};\ \beta = \frac{a(v-1)-k(k-1)}{a-b}.$$

Let us call two vertices of Deza graph by a pair of "type a" if these vertices have a common neighbors. Similarly, we define a pair of "type b". Note that each vertex is contained in exactly α pairs of "type a" and in exactly β pairs of "type b".

In [13] it was described the class of strictly Deza line graphs. The main theorem of this paper is given below.

Theorem 1. *(Kabanov V.V., Mityanina A.V.) A graph G is strictly Deza line graph if and only if it is:*

(1) *the $4 \times n$ lattice graph, where $n > 1$ and $n \neq 4$ or;*

(2) *strictly Deza graph with the parameters either* $(9, 4, 2, 1)$ *(fig. 1.a), or* $(12, 6, 3, 2)$ *(fig. 1.c), or* $(20, 6, 2, 1)$ *(fig. 2).*

The induced subgraph on $[a] \cap [b]$ is called a μ-graph if the distance between a and b is equal to 2.

In the three theorems below, the claw-free graphs with some restrictions are described.

Theorem 2. *(Kabanov V.V., [12]) Let Γ be a connected claw-free graph that contains a 3-coclique. If every μ-graph of Γ has radius greater than 1, then Γ is the clique extension of one of the following graphs:*

(1) *the $m \times n$-lattice, where $m \geq 3$, $n \geq 3$;*

(2) *the triangular graph $T(n)$ with $n \geq 6$ (the strongly regular graph with parameters $(\frac{n(n-1)}{2}, 2(n-2), n-2, 4)$);*

(3) *the Schläfli graph (the strongly regular graph with parameters $(27, 16, 10, 8)$).*

Theorem 3. *(Kabanov V.V., Mahknev A.A., [14]) Let Γ be a connected coedge regular claw-free graph that contains a 3-coclique. Then Γ is an α-extension ($\alpha \geq 1$) of one of the following graphs:*

(1) *an $m \times n$-lattice ($m \geq 3$, $n \geq 3$) or a triangular graph $T(m)$ ($m \geq 6$);*

(2) *the Schläfli graph.*

Theorem 4. *(Mahknev A.A., [16]) Let Γ be a connected reduced claw-free graph that contains a 3-coclique. If all μ-graphs of Γ are regular of valency $\alpha > 0$, then Γ is one of the following graphs:*

(1) *a triangular graph $T(m)$ ($m \geq 6$);*

(2) *the Schläfli graph;*

(3) *the icosahedron graph.*

3. General remarks

Suppose that G satisfies the hypothesis of the main theorem, i.e., it is a claw-free strictly Deza graph with parameters (v, k, b, a) and G contains a 3-coclique.

If two non-adjacent vertices such that G is the union of the closed neighborhoods of these vertices exist in G then G is a graph of the statement of the main theorem in [11] and has the parameters either $(9, 4, 2, 1)$ or $(12, 6, 3, 2)$ (see fig.1). According to the list of all strictly Deza graphs with the number of vertices not greater than 13 (see [7]), there are two non-isomorphic Deza graphs accurate for each of the parameters set above. If we restore the structure of the resulting graph it will be easy to see that all of these graphs are the union of the closed neighborhoods of two non-adjacent vertices.

On the next figure all vertices lying on the same line $i-i$ are pairwise adjacent.

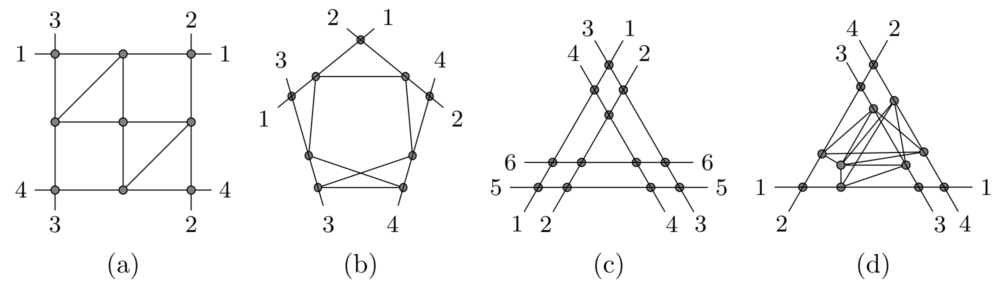

FIGURE 1

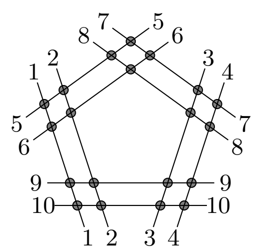

FIGURE 2

Further we will consider only the graphs where any two non-adjacent vertices belong to a 3-coclique.

In our terms a disconnected graph has a diameter of more than two.

Let γ and δ be non-adjacent vertices of G. Since G has the diameter 2, there exists vertex ε such that $\varepsilon \in ([\gamma] \cap [\delta])$. Up to the end of the paper let:

$$x = |[\gamma] \cap [\delta] \cap [\varepsilon]|, \qquad y = |[\gamma] \cap [\varepsilon]|, \qquad z = |[\delta] \cap [\varepsilon]|.$$

Note that the neighborhood of ε is entirely inside $\gamma^{\perp} \cup \delta^{\perp}$, otherwise there is a claw in G. But then we can express k for ε (as for the rest vertices of G) in terms of variables x, y, z, namely, $k = y + z - x + 2$ where $y, z \in \{a, b\}$.

In two lemmas below we consider graphs satisfying the hypothesis of the main theorem with certain restrictions on μ-graphs.

Lemma 1. *Let G be a graph satisfying the hypothesis of the main theorem. If every μ-graph has radius greater than 1 in G, then G is the one of the following graphs:*

(1) *an $4 \times n$-lattice, where $n > 2$, $n \neq 4$;*

(2) *the 2-extension of the 3×3-lattice.*

Proof. Suppose that G satisfies the hypothesis of the main theorem. According to Theorem 2, G is a clique extension of one of the graphs from the conclusion of Theorem 2. Consider a clique extension of graphs referred in theorem 2.

1. Let G be a clique ϕ-extension of $m \times n$-lattice with $n \geq 3$, $m \geq 3$ (conclusion 1 of Theorem 2). If $\phi = 1$ then G is an $m \times n$-lattice, so it is a line graph. According to the main theorem from [13] G is an $4 \times n$-lattice with $n \geq 3$, $n \neq 4$.

Further, we consider the case $\phi > 1$. Let us determine the number of common neighbors for distinct pairs of vertices in G:

- for any two vertices from the same ϕ-clique, the number of vertices adjacent to them is $\lambda_1 = (m-1)\phi + (n-1)\phi + (\phi - 2) = (m+n-1)\phi - 2$;
- for any two adjacent vertices from different ϕ-cliques, the number of vertices adjacent to them is either $\lambda_2 = (m-2)\phi + (\phi-1)2 = m\phi - 2$ or $\lambda_3 = n\phi - 2$;
- for any two non-adjacent vertices, the number of vertices adjacent to them is $\lambda_4 = 2\phi$.

Note that λ_1, λ_2, λ_3, λ_4 must belong to the set $\{a, b\}$. If $m \neq n$, then $\lambda_2 \neq \lambda_3$, and therefore λ_1 must be equal to λ_2 or λ_3. But in this case we get either $m = 1$ or $n = 1$ which contradicts the condition $n \geq 3$, $m \geq 3$.

If $m = n$, then $\lambda_2 = \lambda_3$. We get three parameters λ_1, λ_3, λ_4, each of them belongs to the set $\{a, b\}$. This means that at least two parameters are equal. According to the previous arguments, λ_1 is not equal to λ_3. Then suppose that $\lambda_1 = \lambda_4$. Replacing λ_1 and λ_4 by formulas obtained previously and expressing n through them, we obtain $n = \frac{3}{2} + \frac{1}{\phi}$. As $\phi > 1$ and n is integer, we have ϕ is 2. Therefore $n = 2$, which contradicts the condition $n \geq 3$; hence, the assumption of the equality λ_1 and λ_4 is not correct.

Suppose that $\lambda_3 = \lambda_4$. Replacing λ_3 and λ_4 by formulas obtained previously, we obtain $(n-2)\phi = 2$. Taking onto account that n and ϕ are integers and $n \geq 3$ and $\phi \geq 2$, we obtain the following values for n and ϕ: $n = 3$, $\phi = 2$. Therefore G is the clique 2-extension of the 3×3-lattice.

2. Let G be a clique ϕ-extension of a triangular graph $T(n)$ (conclusion 2 of Theorem 2) where graph $T(n)$ has parameters $(\frac{n(n-1)}{2}, 2(n-2), n-2, 4)$, $n \geq 6$. If $\phi = 1$ then G is a strongly regular graph and we get the contradiction with the hypothesis of the main theorem. So we will consider only the case $\phi > 1$. Let us determine the number of common neighbors for the distinct pairs of vertices in G:

- for any two vertices from the same ϕ-clique, the number of vertices adjacent to them is $\lambda_1 = 2(n-2)\phi + (\phi - 2) = (2n-3)\phi - 2$;
- for any two adjacent vertices from different ϕ-cliques, the number of the vertices adjacent to them is $\lambda_2 = (n-2)\phi + (\phi - 1)2 = n\phi - 2$;
- for any two non-adjacent vertices, the number of vertices adjacent to them is $\lambda_3 = 4\phi$.

According to a Deza graph definition, λ_1, λ_2, λ_3 must belong to the set $\{a, b\}$. This means that at least two parameters are equal.

Suppose that $\lambda_1 = \lambda_2$. Then after replacing λ_1, λ_2, λ_3 by formulas obtained previously, we obtain $n = 3$ which contradicts $n \geq 6$.

Suppose that $\lambda_1 = \lambda_3$. Then we obtain the following formula for the parameter n: $n = \frac{7}{2} + \frac{1}{\phi}$. As $\phi > 1$ and n is integer, we have ϕ is 2. Therefore $n = 4$, which contradicts the condition $n \geq 6$.

And now suppose that $\lambda_2 = \lambda_3$. Then we obtain the following formula for the parameter ϕ: $\phi = \frac{2}{n-4}$. As $\phi > 1$ and n,ϕ are integers, we have $n - 4 = 1$. Therefore $n = 5$, which contradicts the condition $n \geq 6$.

3. Let G be a clique ϕ-extension of Schläfli graph (conclusion 3 of Theorem 2) where Schläfli graph has parameters $(27, 16, 10, 8)$. If $\phi = 1$ then G is a strongly regular graph and we get contradiction with the hypothesis of the main theorem. So we will consider only the case $\phi > 1$. Let us determine the number of common neighbors for the distinct pairs of vertices in G:

- for any two vertices from the same ϕ-clique, the number of vertices adjacent to them is $\lambda_1 = 16\phi + (\phi - 2) = 17\phi - 2$;
- for any two adjacent vertices from different ϕ-cliques, the number of vertices adjacent to them is $\lambda_2 = 10\phi + (\phi - 1)2 = 12\phi - 2$;
- for any two non-adjacent vertices, the number of vertices adjacent to them is $\lambda_3 = 8\phi$.

For G we get λ_1, λ_2, λ_3, which belong to the set $\{a, b\}$. So, at least two parameters are equal. Taking onto account that $\phi > 1$ we get a strict inequality for λ_1, λ_2, λ_3, namely, $\lambda_1 > \lambda_2 > \lambda_3$. This contradicts the formulas obtained earlier.

□

In what follows we assume that there is μ-graph in G such that has radius 1. Without loss of generality, let us assume that subgraph $[\gamma] \cap [\delta]$ has radius 1 and vertex $\varepsilon \in ([\gamma] \cap [\delta])$ is joined to all vertices from μ-graph.

Lemma 2. *Let G be a graph satisfying the hypothesis of the main theorem. If each μ-graph has the same size in G, then G is the one of the graphs from the conclusion of Lemma* 1.

Proof. If each μ-graph has the same size in G, then G is a coedge regular graph. According to the corollary of Theorem 3, G is either a complement of a triangle-free regular graph or a graph from the consequence of Theorem 3. If G is a complement of a triangle-free regular graph then there are no 3-coclique in G. We get the contradiction with the hypothesis of our theorem. If G is ϕ-extension of a totally disconnected graph with m number of vertices, where $m \neq 2$, then in case of $m = 1$ G is a clique with ϕ number of vertices or otherwise $m > 2$ and G is a disconnected graph. In any case we get the contradiction with the definition of Deza graph. The remaining cases of the conclusion of Theorem 3 were already considered in Lemma 1.

□

In what follows we assume that there are μ-graphs of size a and size b in G.

Lemma 3. *The parameters of G satisfy the restrictions: $a > 0$, $\alpha > 0$, $\beta > 0$.*

Proof. Let parameter $a = 0$ in G. As there are μ-graphs of size a and of size b in G, we have there are two non-adjacent vertices such that their μ-graphs contain $a = 0$ vertices. But then the distance between vertices exceeds 2, which contradicts the definition of strictly Deza graphs.

Since G has μ-graphs of size a and size b, there are pairs of vertices of "type a" and of "type b" in the graph. But then the parameters α, β satisfy the restrictions: $\alpha \geq 1$, $\beta \geq 1$. □

Consider γ, δ, ε of G such that $\gamma \nsim \delta$, $\varepsilon \in ([\gamma] \cap [\delta])$ and ε is adjacent with all the vertices of this μ-graph. Let us recall previously entered designations $x = |[\gamma] \cap [\delta] \cap [\varepsilon]|$, $y = |[\gamma] \cap [\varepsilon]|$, $z = |[\delta] \cap [\varepsilon]|$ and the resulting equality for the parameter

k of G: $k = y + z - x + 2$, where $y, z \in \{a, b\}$. Variable x can take one of two values $\{a-1, b-1\}$.

There are consideration all possible combinations of values x, y, z, and k in the next paragraphs.

1. Let $x = a - 1$
   - if $y = z = a$, then $k = a + 3$;
   - if $y = a$, $z = b$, then $k = b + 3$;
   - if $y = z = b$, then $k = 2b - a + 3$.
2. Let $x = b - 1$
   - if $y = z = a$, then $k = 2a - b + 3$;
   - if $y = a$, $z = b$, then $k = a + 3$;
   - if $y = z = b$, then $k = b + 3$.

Further we will consider graphs with all above presented combinations of the parameters x, y, z, and k.

4. Graphs with parameter $k \in \{2a - b + 3, a + 3, b + 3\}$

Suppose that G satisfies the hypothesis of the main theorem, contains μ-graphs of size a and size b, and there is at least one μ-graph radius 1 in G.

Lemma 4. *Parameter k is not equal to $2a - b + 3$ in G.*

Proof. Suppose that G has parameter $k = 2a - b + 3$, and the variables x, y, z take the values $y = z = a$, $x = b - 1$. Note that $([\gamma] \cap [\delta] \cap [\varepsilon]) \subseteq ([\gamma] \cap [\varepsilon])$ and $([\gamma] \cap [\delta] \cap [\varepsilon]) \subseteq ([\delta] \cap [\varepsilon])$, which mean that the inequalities $x \le y$ and $x \le z$ are correct in G. Substituting the values of x, y, z we get $b - 1 \le a$. By the hypothesis of the theorem $b > a$, so we obtain $b = a + 1$ from which it follows that $([\gamma] \cap [\varepsilon]) = ([\gamma] \cap [\delta] \cap [\varepsilon]) = ([\delta] \cap [\varepsilon])$ and $k = a + 2$. But then $|[\gamma] \backslash \varepsilon^{\perp}| = |[\delta] \backslash \varepsilon^{\perp}| = k - b = 1$.

By the hypothesis of the main theorem, for γ and δ there is η in G such that $\eta \nsim \gamma$, $\eta \nsim \delta$. γ and η must have at least a common neighbors which can only lie in $[\gamma] \setminus [\delta]$ (otherwise we get a claw). Note that $|[\gamma] \setminus [\delta]| = 1$, this leads to $a \le 1$. According to Lemma 3 $a > 0$, this leads to $a = 1$. Substituting the value of a, we obtain the set of parameters $(v, 3, 2, 1)$ for a strictly Deza G.

According to the formula given in [7], let us calculate the value of parameter β: $\beta = \frac{a(v-1) - k(k-1)}{a-b} = 7 - v$ for G. By Lemma 3 parameter β satisfies the restriction $\beta > 0$, so we have the inequality $v < 7$. According to [7] there is no strictly Deza graph with the parameters $(v, 3, 2, 1)$, where $v < 7$. Therefore, the assumption is incorrect. $\square$

Lemma 5. *Parameter k is not equal to $a + 3$ in G.*

Proof. Suppose that G has the parameter $k = a + 3$. There are μ-graphs of size a and size b in G, and hence, there are vertices u, w in G such that $u \nsim w$ and $|[u] \cap [w]| = b$. Then $|[u] \setminus [w]| = k - b = (a + 3) - b$. Given that $b \ge a + 1$ we obtain $|[u] \setminus [w]| \le 2$.

By the hypothesis of the main theorem, for u, w there is vertex s in G such that $s \nsim u$, $s \nsim w$. u and s must have at least a common neighbors, which can only lie in $[u] \setminus [w]$ (otherwise we get a claw in G), and hence there can be no more than 2. Therefore, $a \le 2$. According to Lemma 3 $a > 0$, which gives $a \in \{1, 2\}$.

1. Let $a = 2$, then $k = 5, b \geq 3$. For u there is inequality $|[u]| \geq |[u] \cap [w]| + |[u] \cap [s]|$, from which it follows that $k \geq b + |[u] \cap [s]|$. Since $|[u] \cap [s]| \geq a$ and $a = 2$, we get $k \geq b + 2$. For parameter b we get the restriction $b \leq 3$, but taking onto account that $b \geq 3$ we obtain $b = 3$. Thus G has the parameters $(v, 5, 3, 2)$.

Let us calculate the value of parameter β for G: $\beta = \frac{2(v-1)-20}{2-3} = 22 - 2v$. By Lemma 3 $\beta > 0$ which gives the restriction $v < 11$ by the number of the graph G vertices. According to the paper [7] there is no strictly Deza graphs with the parameters $(v, 5, 3, 2)$, where $v < 11$.

2. Let $a = 1$, then $k = 4$, $b \geq 2$. Let us calculate the value of the parameter β: $\beta = \frac{v-13}{1-b}$. Since $\beta > 0$, we obtain the restriction for the number of the graph G vertices: $v < 13$. Note that $b \in \{2, 3\}$, and we get two sets of the parameters for a strictly Deza graph G: $(v, 4, 2, 1)$ and $(v, 4, 3, 1)$, where $v < 13$. According to the paper [7], there are no strictly Deza graphs with the parameters $(v, 4, 3, 1)$ for $v < 13$, there are two strictly Deza graphs with the parameters $(9, 4, 2, 1)$ and one Deza graph with the parameters $(8, 4, 2, 1)$. But Deza graph with the parameters $(8, 4, 2, 1)$ does not contain a 3-coclique, and both Deza graphs with the parameters $(9, 4, 2, 1)$ are a union of the closed neighborhoods of two non-adjacent vertices. We obtained the contradiction with the conditions; therefore, the assumption is incorrect. □

Lemma 6. *Parameter k is not equal to $b + 3$ in G.*

Proof. Suppose that G has the parameter $k = b + 3$. There are μ-graphs of size a and of size b in G, hence, there are vertices u,w in G such that $u \nsim w$ and $|[u] \cap [w]| = b$. Then $|[u] \setminus [w]| = |[w] \setminus [u]| = k - b = (b + 3) - b = 3$. By the hypothesis of the main theorem, for u, w there is vertex s in G such that $s \nsim u$, $s \nsim w$. u and s must have at least a common neighbors, which can only lie in $[u] \setminus [w]$, and the number of common neighbors does not exceed 3. Therefore $a \leq 3$. By Lemma 3 $a > 0$, which gives $a \in \{1, 2, 3\}$.

1. Let $a \in \{2, 3\}$ and there are no vertex s_i in G such that s_i lies outside $u^\perp \cup w^\perp$ and $s \neq s_i$. Then the whole neighborhood of s lies in $u^\perp \cup w^\perp$. According to the hypothesis of the main theorem, there are no claws in G, hence, $[s] \subseteq ([u] \cup [w]) \setminus ([w] \cap [u])$. This gives a restriction on the size of s neighborhood: $|[s]| \leq |[u] \setminus [w]| + |[w] \setminus [u]|$, so $k \leq 6$. By lemma condition $k = b + 3$, it leads to the inequality $b \leq 3$.

If $a = 3$ then we obtain a contradiction with $b > a$. If $a = 2$ then we obtain $b = 3, k = 6$. Then we can calculate the value of β for G: $\beta = \frac{2(v-1)-30}{2-3} = 32 - 2v$. In accordance with Lemma 3 we get the inequality $32 - 2v > 0$ from which it follows that $v < 16$. According to the paper [7] and [15], there are only two graphs with the parameters $(v, 6, 3, 2)$, where $v < 16$. Namely, $(12, 6, 3, 2)$-Deza graphs. But both of these are a union of the closed neighborhoods of two non-adjacent vertices, which contradicts the condition.

2. Let $a \in \{2, 3\}$ and there is s_i in G such that s_i lies outside $u^\perp \cup w^\perp$ and $s \neq s_i$. According to $a \geq 2$, u and s must have at least two common neighbors, which lie in $u^\perp \setminus w^\perp$ (otherwise we get a claw in G). Similar reasoning is correct for u and s_i. And since $|[u] \setminus [w]| = 3$, s and s_i have at least one common neighbor in $[u] \setminus [w]$. By the theorem hypothesis, G ia claw-free graph, hence, s and s_i must be adjacent.

Since s and s_i are arbitrary vertices outside $u^\perp \cup w^\perp$, the above reasoning is valid for any vertex outside $u^\perp \cup w^\perp$. Thus all vertices outside $u^\perp \cup w^\perp$ form a clique.

If $a = 3$, then we have the relations $[s] \cap [u] = [s_i] \cap [u] = [u] \setminus [w]$ (it contains 3 vertices) and $[s] \cap [w] = [s_i] \cap [w] = [w] \setminus [u]$ (it also contains 3 vertices). For s we get 6 neighbors in $u^\perp \cup w^\perp$, so the remaining $k - 6 = b - 3$ neighbors are outside $u^\perp \cup w^\perp$. Similarly for s_i. We determine common neighbors for s and s_i: 6 vertices in $u^\perp \cup w^\perp$ and $b-4$ vertices outside $u^\perp \cup w^\perp$. Thus, $|[s] \cap [s_i]| = 6 + (b-4) = b+2$, which contradicts the parameters of G.

If $a = 2$, then $b \geq 3$ and we get two possible cases. The first case is either $|[s] \cap [u]| = 3$ or $|[s] \cap [w]| = 3$ or both numbers are equal to 3. Then $b = 3$, $k = 6$ and G has the parameters $(v, 6, 3, 2)$. We calculate the value of the parameter β for G: $\beta = 32 - 2v$. According to the restrictions obtained in Lemma 3, we get the inequality $v \leq 15$. Similarly to item 1, we get that there are no graphs satisfying the hypothesis of the main theorem and the conditions of this lemma.

The second case is $|[s] \cap [u]| = |[s] \cap [w]| = 2$. Then s has 4 neighbors in $u^\perp \cup w^\perp$, and the remaining $b - 1$ neighbors are outside $u^\perp \cup w^\perp$. Hence, there are b vertices outside $u^\perp \cup w^\perp$. Since $b \geq 3$, s has at least two neighbors outside $u^\perp \cup w^\perp$. Denote these vertices by s_1 and s_2.

Note that s and s_1 have at least one common neighbor in $[u] \setminus [w]$, at least one in $[w] \setminus [u]$ and $b - 2$ common neighbors outside $u^\perp \cup w^\perp$. Thus, for s and s_1 we have already determined b common neighbors, and there cannot be more of those. Continuing reasoning, we obtain that every pair of vertices outside $u^\perp \cup w^\perp$ has exactly one common neighbor in $[u] \setminus [w]$ and exactly one common neighbor in $[w] \setminus [u]$. Since $|[u] \setminus [w]| = 3$, there cannot be more than three vertices outside $u^\perp \cup w^\perp$, which satisfies all conditions. But there are vertices s, s_1, s_2 already determined outside $u^\perp \cup w^\perp$, hence, there are exactly 3 vertices and $b = 3$. We get the parameters $(v, 6, 3, 2)$ for G, which have already been discussed in the preceding paragraphs of the lemma.

3. Let $a = 1$. Consider p and q in G such that $p \nsim q$ and $|[p] \cap [q]| = a = 1$ and vertex r in G which lies in $[p] \cap [q]$. We are to determine the neighborhood of r. There should be $k = b + 3$ vertices, two of which are p,q, and remaining $b + 1$ vertices are in $([p] \setminus [q]) \cup ([q] \setminus [p])$. Note that $|[p] \cap [r]|, |[q] \cap [r]| \in \{1, b\}$. Without loss of generality, assume that $|[p] \cap [r]| = 1$ and $|[q] \cap [r]| = b$.

Let us introduce the notations $\Delta_1 = [p] \setminus r^\perp$ and $\Delta_2 = [q] \setminus r^\perp$, where $|\Delta_1| = b+1$, $|\Delta_2| = 2$. Denote by θ the vertex in $[p] \cap [r]$, by $\chi_1, ..., \chi_b$ the vertices in $[q] \cap [r]$, by $\xi_1, ..., \xi_{b+1}$ the vertices in Δ_1 and by φ_1,φ_2 the vertices in Δ_2. Note that Δ_1 and Δ_2 are cliques, otherwise, we get a claw in G. Thus, any vertex from Δ_1 has b common neighbors with p. Hence, there are no vertices from Δ_1 adjacent to θ. So p is contained in at least $b + 1$ pairs of "type b". Let us determine the parameter $\beta = \frac{(b+3)(b+2)-(v-1)}{b-1}$ for G, where $\beta \geq b + 1$. Hence we obtain $v \leq 5b + 8$. Note that $|[p] \cup [q]| = 2 + (b+3) + (b+3) = 2b + 7$, and there cannot be more than $3b + 1$ vertices outside $[p] \cup [q]$.

For every vertex from Δ_1 we have already determined $b + 1$ neighbors in $p^\perp$ so for each of them there are two neighbors outside $p^\perp$. Furthermore, for each vertex from Δ_1 the corresponding pair of vertices is an edge, otherwise we get a claw in G. But these edges cannot have common vertices with each other, otherwise we obtain the contradiction with parameter b for the vertices from Δ_1.

We have to consider various cases of neighbor location outside $p^{\perp}$ for ξ_i, $i = 1, ..., b+1$. Provided that every ξ_i, $i = 1, ..., b+1$ must have at least one common neighbor with q.

Assume that none of vertices ξ_i, $i = 1, ..., b+1$ has neighbors that lie in $[q] \cap [r]$. Since ξ_i and q must have at least one common neighbor, then at least one of the vertices adjacent to ξ_i and lying in $p^{\perp}$ should be in the $[q] \setminus r^{\perp}$. And for the various ξ_i, ξ_j, $i \neq j$ a vertex should have its own, otherwise we get a contradiction with the parameter b. But for ξ_1,ξ_2,ξ_3 it is not possible to find 3 different vertices in $[q] \setminus r^{\perp}$, because $|[q] \setminus r^{\perp}| = 2$. Consequently, the assumption is incorrect.

Suppose that there is vertex ξ_i such that both of its neighbors from outside $p^{\perp}$ lie in $[q] \cap [r]$. Without loss of generality, assume that ξ_1 is a required vertex, and its neighbors in the $[q] \cap [r]$ are denoted by ω_1 and ω_2. But then $[\xi_1] \cap [r] = \{p, \omega_1, \omega_2\}$ and $|[\xi_1] \cap [r]| = 3$, hence, $b = 3$. There are only two common neighbors (ω_1,ω_2) for ξ_1 and q, which contradicts the parameters a and b of G. Therefore, the assumption is incorrect.

Suppose that only ω_1 lies in $[q] \cap [r]$ (without loss of generality, assume that $\omega_1 = \chi_1$), but then $|[\xi_1] \cap [r]| = 2$, which means that $b = 2$, $k = 5$. Note that r has 4 neighbors, with which it forms a pair of "type b" (2 vertices of $[r] \cap [q]$, ξ_1, q), therefore for parameter $\beta = \frac{(v-1)-20}{1-2}$ of G the inequality $\beta \geq 4$ is satisfied. We get the restriction of parameter v: $v \leq 17$. In G there are 11 vertices already determined, which enables the restriction $v \geq 11$. Thus, a strictly Deza graph G has the parameters $(v, 5, 2, 1)$, where $11 \leq v \leq 17$.

According to the paper [7] for $v \leq 13$ there is only the 4×3-lattice with suitable parameters $(12, 5, 2, 1)$. But all its μ-graphs have a radius greater than 1, which contradicts the conditions of lemma. According to the paper [15] for $14 \leq v \leq 16$ there is a strictly Deza graph with the parameters $(16, 5, 2, 1)$. But some vertex of the graph has the neighborhood $K_{1,2} \cup K_2$ ([15], p.112), which means that there is a claw in the graph. Consequently, a strictly $(16, 5, 2, 1)$-Deza graph does not satisfy the hypothesis of the main theorem.

If G has $v = 17$ vertices, then we can calculate α and β for the graph: $\alpha = 12$, $\beta = 4$. Note that for r all vertices with which it forms a pair of "type b" are already determined: 2 vertices of $[r] \cap [q]$, q and ξ_1. Then ξ_2 and ξ_3 cannot be adjacent to any vertex of $[r] \cap [q]$, otherwise we obtain a contradiction with the parameter $\beta = 4$. But q must have at least one common neighbor with ξ_2 and ξ_3, which should lie outside $[r] \cap [q]$, i.e. in Δ_2. For ξ_2 and ξ_3 two common neighbors are already determined, and they are adjacent to different vertices in Δ_2. Without loss of generality, we assume that $\xi_2 \sim \varphi_1$, $\xi_3 \sim \varphi_2$. We got that the remaining neighbor for ξ_2 is outside $p^{\perp} \cup q^{\perp}$. We denote it by ω_3. Similarly for ξ_3 we denote a neighbor outside $p^{\perp} \cup q^{\perp}$ by ω_4.

Consider ξ_2 and χ_2. In accordance with the parameters a, b vertex χ_2 must have either one or two neighbors in the neighborhood of ξ_2. Vertices p, ξ_1, ξ_3 cannot be these neighbors, otherwise we obtain a contradiction with the number of vertices with which vertex r forms a pairs of "type b". Vertex ξ_2 cannot be adjacent to φ_1, otherwise we obtain the contradiction with $b = 2$ for q and χ_2. Therefore, $\chi_2 \sim \omega_3$. Taking onto account similar reasoning for ξ_3 and χ_2, we obtain $\chi_2 \sim \omega_4$. Then vertex q forms pairs of "type b" with ω_3, ω_4, r, χ_1, χ_2, which contradicts $\beta = 4$. Hence, the assumption is incorrect.

Thus, we have considered all possible cases of neighbors location outside $p^{\perp}$ for ξ_i, $i = 1, ..., b+1$. $\square$

By Lemmas 4, 5 and 6, there is no graph G which has $k \in \{2a-b+3, a+3, b+3\}$ and which satisfies all conditions.

5. Graphs with parameter $k = 2b - a + 3$

Suppose that G satisfies the hypothesis of the main theorem, contains μ-graphs of size a and size b, and there is at least one μ-graph with a radius 1 in G. Suppose that parameter k for G takes the following value $k = 2b - a + 3$. Then $y = z = b$, $x = a - 1$ is satisfied for γ, δ, ε.

Consider $[\gamma] \cap [\delta]$. It is μ-graph with radius 1 and size a, which contains vertex ε adjacent to all the rest vertices in μ-graph. For γ and δ there exists vertex η such that $\eta \nsim \gamma, \eta \nsim \delta$. G does not have a claw therefore common neighbors of ε and η lie in $[\varepsilon] \setminus (\gamma^{\perp} \cap \delta^{\perp})$. With that $|[\varepsilon] \setminus (\gamma^{\perp} \cap \delta^{\perp})| = 2b-a+3-(a-1)-2 = 2b-2a+2$. In addition $|[\varepsilon] \cap [\eta]| \geq a$ and this produces inequality $2b - 2a + 2 \geq a$, which allows to obtain the following formula for the parameters a, b: $2b \geq 3a - 2$.

Consider arbitrary μ-graph of two non-adjacent vertices ρ_1, ρ_2 of G. Let us set ρ_3 as an arbitrary vertex of this μ-graph and set t as $t = |[\rho_1] \cap [\rho_2] \cap [\rho_3]|$. As long as G does not have a 3-claw, the neighborhood of ρ_3 lies in $\rho_1^{\perp} \cup \rho_2^{\perp}$ and the relation for ρ_3 is the following $k = |[\rho_1] \cap [\rho_3]| + |[\rho_2] \cap [\rho_3]| - t + 2$. For vertices in question and t the following variants are possible:

(1) if $|[\rho_1] \cap [\rho_3]| = |[\rho_2] \cap [\rho_3]| = a$, then $k = 2a - t + 2$ and $t = 2a + 2 - k = 2a + 2 - (2b - a + 3) = 3a - 2b - 1 = t_1$;

(2) if $|[\rho_1] \cap [\rho_3]| = b, |[\rho_2] \cap [\rho_3]| = a$, then $k = a+b-t+2$ and $t = 2a-b-1 = t_2$;

(3) if $|[\rho_1] \cap [\rho_3]| = |[\rho_2] \cap [\rho_3]| = b$, then $k = 2b - t + 2$ and $t = a - 1 = t_3$.

Note that for the values of t_1, t_2, t_3 there is inequality $t_1 < t_2 < t_3$. As long as μ-graph and the vertex in it were chosen arbitrary, for any vertex from any μ-graph in G parameter t belongs to the set $\{3a - 2b - 1, 2a - b - 1, a - 1\}$. With that the vertex with $t = t_3$ has to be found in the graph, namely vertex ε from the μ-graph of size a and radius 1.

Lemma 7. *If there are no vertices with parameter $t = t_1$ and $t = t_2$ in G, then G is a strictly Deza line graph with the parameters* $(20, 6, 2, 1)$.

Proof. Let G satisfy lemma hypothesis, then for all vertices in μ-graphs $t = t_3$ is satisfied. This means that all μ-graphs are regular to valence $t_3 = a - 1$.

If $a > 1$, then according to Theorem 4 G is either a triangular graph, Schläfli graph or icosahedron graph. In the first case G is a strongly regular graph, in the second case G has the diameter 3 which contradicts the hypothesis of the main theorem. Therefore $a = 1$. Then $k = 2b + 2$ and for all vertices in any μ-graph parameter $t_3 = 0$ is satisfied. So that none of the vertices in μ-graph is adjacent to any other, which means that any μ-graph is a coclique.

Consider arbitrary μ-graph of the size b. As long as G is a claw-free graph and μ-graph is a coclique, the size of this μ-graph less or equal 2. But then $b = 2$, $k = 6$ and G has the parameters $(v, 6, 2, 1)$.

Arbitrary μ-graph represents either one vertex, or a 2-coclique. But this means that an induced subgraph is not possible in G, where an induced graph represents two triangles with a common edge. Then, according to the theorem from the [17] G is a line graph, and according to Theorem 1 the mentioned conditions are satisfied only with $v = 20$. Therefore G is a strictly Deza line graph with the parameters $(20, 6, 2, 1)$.

$\square$

Lemma 8. *There is no vertex with parameter $t = t_1$ in G.*

Proof. Suppose that there is a vertex from μ-graph in G with the parameter $t = t_1$. As long as $t_1 \geq 0$, there is inequality $2b \leq 3a - 1$, from which it follows that $2b \in \{3a - 2, 3a - 1\}$.

1. Let $2b = 3a - 2$, then $b = \frac{3}{2}a - 1$, where a is an even number. Here $b > a$ and therefore $\frac{3}{2}a - 1 > a$ and consequently $a \geq 4$, $b \geq 5$. Note that $t_1 = 1$ and $k = 2a + 1$.

Consider $[\gamma] \cap [\delta]$. It is μ-graph with radius 1 and size a which contains vertex ε adjacent to all the rest vertices in μ-graph. We can calculate the sizes of subgraphs for γ, δ, ε: $|[\gamma] \cap [\varepsilon] \setminus \delta^\perp| = |[\delta] \cap [\varepsilon] \setminus \gamma^\perp| = \frac{a}{2}$ and $|[\gamma] \setminus \varepsilon^\perp| = |[\delta] \setminus \varepsilon^\perp| = \frac{a}{2} + 1$.

For γ and δ there is a non-adjacent vertex η of G. To exclude a claw, common neighbors for ε and η can lie only in $([\gamma] \cap [\varepsilon] \setminus \delta^\perp) \cup ([\delta] \cap [\varepsilon] \setminus \gamma^\perp)$. Alongside with this, the number of common neighbors of ε and η must be not less than a, but $|([\gamma] \cap [\varepsilon] \setminus \delta^\perp) \cup ([\delta] \cap [\varepsilon] \setminus \gamma^\perp)| = a$, resulting in equations $[\varepsilon] \cap [\eta] = ([\gamma] \cap [\varepsilon] \setminus \delta^\perp) \cup ([\delta] \cap [\varepsilon] \setminus \gamma^\perp)$ and $|[\varepsilon] \cap [\eta]| = a$.

For η we have determined $\frac{a}{2}$ common neighbors with γ, which lie in $[\gamma] \cap [\varepsilon] \setminus \delta^\perp$. But $|[\gamma] \cap [\eta]|$ is not less than a, so that the rest neighbors are to lie in $[\gamma] \setminus \varepsilon^\perp$ and the number of them cannot exceed $|[\gamma] \setminus \varepsilon^\perp| = \frac{a}{2} + 1$. From which it follows that the number of common neighbors of γ and η belong to the set $\{a, a + 1\}$. Similar reasoning is shown for common neighbors of δ and η. But the situation $|[\gamma] \cap [\eta]| = |[\delta] \cap [\eta]| = a + 1$ is not possible, otherwise there is a contradiction for η with $k = 2a + 1$.

Suppose $|[\gamma] \cap [\eta]| = |[\delta] \cap [\eta]| = a$. Then there are $2a$ vertex neighborhood determined for η, therefore there is vertex $\eta_1 \in [\eta]$ such that η_1 lies outside $\gamma^\perp \cup \delta^\perp$. η_1 is also to have not less than a common neighbors with γ and δ, which will lie in $([\gamma] \setminus [\delta]) \cup ([\delta] \setminus [\gamma])$. As long as $|[\gamma] \setminus [\delta]| = |[\delta] \setminus [\gamma]| = a + 1$, η and η_1 have $a - 1$ common neighbor in $[\gamma] \setminus [\delta]$ and in $[\delta] \setminus [\gamma]$ respectively. Consequently, $\eta \cap \eta_1$ already has $2a - 2$ vertices. With regard for $|\eta \cap \eta_1| \geq a$ the following formulas are possible:

- if $a \geq 2a - 2$, then there is inequality $2 \geq a$;
- if $b \geq 2a - 2 > a$, then with regard for $b = \frac{a}{2} - 1$ there is inequality $\frac{a}{2} + 1 \geq 2a - 2$, from which it follows $2 \geq a$.

In both cases it contradicts the restriction for the parameter a: $a \geq 4$. Consequently, the assumption $|[\gamma] \cap [\eta]| = |[\delta] \cap [\eta]| = a$ is not correct. Then one of the numbers $\{|[\gamma] \cap [\eta]|, |[\delta] \cap [\eta]|\}$ takes the value a, and the other $a + 1$. Without loss of generality, assume that $|[\gamma] \cap [\eta]| = a, |[\delta] \cap [\eta]| = a + 1$. Then $b = a + 1$ where we obtain $\frac{3}{2}a - 1 = a + 1$, from which it follows that $a = 4$, and also $b = 5$, $k = 9$.

Consider vertex ω in $[\gamma] \setminus ([\eta]^\perp \cup \varepsilon^\perp)$. Common neighbors of vertices ω and δ cannot lie in $[\delta] \cap [\eta]$, otherwise there is a 3-claw. Therefore $[\omega] \cap [\delta] \subseteq [\delta] \setminus ([\eta]^\perp \cup \{\varepsilon\})$. But $|[\delta] \setminus ([\eta]^\perp \cup \{\varepsilon\})| = a - 1$, which contradicts to $|[\omega] \cap [\delta]| \geq a$. Thus, G does not have parameter b equal to $\frac{3}{2}a - 1$.

2. Suppose that $2b = 3a - 1$, then $b = \frac{3a-1}{2}$, where a is an odd number. In this case $b > a$, hence $\frac{3}{2}a - 1 > a$, so that $a \geq 3$, $b \geq 4$. Note that $t_1 = 0$ and $k = 2a + 2$.

Consider non-adjacent vertices φ_1 and φ_2 of G such that their μ-graph contains vertex ψ with parameter $t = t_1$. Then $[\psi] \setminus \{\varphi_1, \varphi_2\} \subseteq ([\varphi_1] \setminus [\varphi_2]) \cup ([\varphi_2] \setminus [\varphi_1])$, at that $|[\psi] \setminus \{\varphi_1, \varphi_2\}| = 2a$ and $|[\psi] \cap [\varphi_1]| \geq a$, $|[\psi] \cap [\varphi_2]| \geq a$. We obtain the only option of distribution of ψ neighborhood such that $|[\psi] \cap [\varphi_1| = |[\psi] \cap [\varphi_2| = a$.

The μ-graph of φ_1 and φ_2 contains at least a vertices, then subgraphs $[\varphi_1] \setminus [\varphi_2]$ and $[\varphi_2] \setminus [\varphi_1]$ contain not more than $a + 2$ vertices, of which in each of subgraph there are a vertices adjacent to ψ. We obtain that $|[\varphi_1] \setminus ([\varphi_2] \cup \psi^{\perp})| \leq 2$ and $|[\varphi_2] \setminus ([\varphi_1] \cup \psi^{\perp})| \leq 2$. Note that $|[\varphi_1] \setminus ([\varphi_2] \cup \psi^{\perp})| = |[\varphi_2] \setminus ([\varphi_1] \cup \psi^{\perp})|$.

For φ_1 and φ_2 in G there is vertex χ such that $\chi \nsim \varphi_1, \chi \nsim \varphi_2$. Neighbors of χ cannot lie in $\varphi_1 \cap \varphi_2$, otherwise there is a claw. In this case χ and φ_1 should have at least a common neighbors, therefore the number of common neighbors of χ and ψ in the neighborhood of φ_1 is to be not less than $a - 2$. Similar reasoning is shown for χ and ψ in the neighborhood of φ_2, hence the number of common neighbors of χ and ψ in $[\varphi_1] \cup [\varphi_2]$ is not less than $2a - 4$. Note that $|[\chi] \cap [\psi]|$ is at least a.

Consider different possible formulas between a and $2a - 4$:

- if $a \geq 2a - 4$, then $a \leq 4$. As long as $a \geq 3$ and a is an odd number, we obtain the following parameters $a = 3$, $b = 4$, $k = 8$.
- if $b \geq 2a - 4 > a$, then the first inequality leads to the restriction $a \leq 7$, the second one to $a > 4$. With regard for a is an odd number, we obtain two sets of parameters: $(v, 12, 7, 5)$ and $(v, 16, 10, 7)$.

Further we will consider different values for the number $|[\varphi_1] \setminus ([\varphi_2] \cup \psi^{\perp})|$, which belongs to the set $\{0, 1, 2\}$.

2.1. Let $|[\varphi_1] \setminus ([\varphi_2] \cup \psi^{\perp})| = |[\varphi_2] \setminus ([\varphi_1] \cup \psi^{\perp})| = 0$. In this case $|[\varphi_1] \cap [\varphi_2]| = a+2$, hence, $b = a + 2$. Then there is $a + 2 = \frac{3a-1}{2}$, from which it follows that $a = 5$, thus G has the set of parameters $(v, 12, 7, 5)$.

Note that $[\varphi_1] \setminus [\varphi_2]$ and $[\varphi_2] \setminus [\varphi_1]$ have a vertices, and all of them adjacent to ψ. But there is a non-adjacent vertex χ, to them, which should have at least a common neighbors both with φ_1, and with φ_2. We have $[\chi] \cap [\varphi_1] = [\psi] \cap [\varphi_1] = [\varphi_1] \setminus [\varphi_2]$ and $[\chi] \cap [\varphi_2] = [\psi] \cap [\varphi_2] = [\varphi_2] \setminus [\varphi_2]$, hence, $|[\psi] \cap [\chi]| = 2a = 10$, which contradicts the values of a and b. Therefore, the assumption is incorrect.

2.2. Let $|[\varphi_1] \setminus ([\varphi_2] \cup \psi^{\perp})| = |[\varphi_2] \setminus ([\varphi_1] \cup \psi^{\perp})| = 1$. In this case $|[\varphi_1] \cap [\varphi_2]| = a+1$, hence, $b = a + 1$. Then there is $a + 1 = \frac{3a-1}{2}$, from which it follows that $a = 3$, thus G has the set of parameters $(v, 8, 4, 3)$.

For φ_1 it has been known that $|[\varphi_1] \cap [\psi]| = a$ and $|[\varphi_1] \cap [\chi]| \geq a$, then $|[\psi] \cap [\chi] \cap [\varphi_1]| \geq a - 1$. Consider similar reasoning for φ_2, we obtain $|[\psi] \cap [\chi] \cap [\varphi_2]| \geq a - 1$. So that, $|[\psi] \cap [\chi]| \geq 2a - 2$, where $2a - 2 = 4 = b$. Therefore, $|[\psi] \cap [\chi] \cap [\varphi_1]| = |[\psi] \cap [\chi] \cap [\varphi_2]| = a - 1 = 2$ and $|[\varphi_1] \cap [\chi]| = |[\varphi_2] \cap [\chi]| = a$.

There were determined $2a$ neighbors for χ in $[\varphi_1] \cup [\varphi_2]$, consequently, the rest two vertices lie outside $[\varphi_1] \cup [\varphi_2]$. Denote them by χ_1, χ_2 and carry out similar reasoning in order to find common neighbors with φ_1 and φ_2. As a result we will obtain an inequality $|[\chi_1] \cap [\chi_2] \cap ([\varphi_1] \cup [\varphi_2])| \geq 2a - 2$, where $2a - 2 = b$. But χ is a common neighbor for χ_1 and χ_2, hence, $|[\chi_1] \cap [\chi_2]| \geq b + 1$, which contradicts the parameters of G. Therefore, the assumption is incorrect.

2.3. Let $|[\varphi_1] \setminus ([\varphi_2] \cup \psi^{\perp})| = |[\varphi_2] \setminus ([\varphi_1] \cup \psi^{\perp})| = 2$. In this case $|[\varphi_1] \cap [\varphi_2]| = a$.

In $[\varphi_1] \setminus [\varphi_2]$ the vertex ψ is adjacent to a vertices, and non-adjacent to two ones. Denote them by δ_1 and δ_2 and note that they are adjacent to each other, otherwise there is a claw. In $[\varphi_1] \cap [\varphi_2]$ the vertex ψ is not adjacent to anything, but then

$[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$ form a clique of size $a-1$, otherwise there is a claw. Moreover, the absence of a claw in G leads to all vertices from $[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$, δ_1 and δ_2 also form a clique of size $a+1$. Similar reasoning is for the vertices from the set φ_2, which are non-adjacent to ψ, but adjacent between each other. Denote them by γ_1, γ_2, and $\{\gamma_1, \gamma_2\} \cup ([\varphi_1] \cap [\varphi_2] \setminus \{\psi\})$ also form a clique of size $a+1$.

Let us fix two arbitrary vertices from $[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$ and calculate the number of common neighbors for them: $a-3$ vertices from $[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$, δ_1,δ_2,γ_1,γ_2,φ_1,φ_2. We have obtained at least $a+3$ vertices, hence, $b \geq a+3$. Out of the parameters determined earlier only $(v, 16, 10, 7)$ satisfy this hypothesis.

Consider an arbitrary vertex ψ_i from $[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$, alongside with it, its closed neighborhood lies entirely in $\varphi_1^\perp \cup \varphi_2^\perp$. There are some neighbors for ψ_i: $a-2$ vertices from $[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$, δ_1,δ_2,γ_1,γ_2,φ_1,φ_2, so that $a+4 = 11$ vertices. Therefore, the rest 5 neighbors lie in $([\varphi_1] \cap [\psi]) \cup ([\varphi_2] \cap [\psi])$.

For ψ_i and φ_1 a common neighbors are determined (vertices from $[\varphi_1] \cap [\varphi_2] \setminus \{\psi\}$, δ_1 and δ_2), and for ψ_i and φ_2 a common neighbors are determined as well. Thus the following finding of the rest five neighbors of ψ_i in $([\varphi_1] \cap [\psi]) \cup ([\varphi_2] \cap [\psi])$ leads to at least one subgraph from $\{[\varphi_1] \cap [\psi_i], [\varphi_2] \cap [\psi_i]\}$ should contain b vertices. Without loss of generality, assume that $|[\psi_i] \cap [\varphi_1]| = b = 10$, so that 3 out of 5 neighbors of ψ_i lie in $[\varphi_1] \cap [\psi]$. But then the rest two neighbors lie in $[\varphi_2] \cap [\psi]$, which leads to $|[\varphi_2] \cap [\psi_i]| = |[\varphi_1] \cap [\varphi_2] \setminus \{\psi, \psi_i\}| + |\{\gamma_1, \gamma_2\}| + 2 = a + 2 = 9$. We have obtained the contradiction with the parameters a and b of G, consequently, the assumption is incorrect.

Considered all possible options, we have obtained that there is no graph G, which can contain the vertex with the parameter $t = t_1$.

□

Lemma 9. *There is no graph G, where one can find the vertex with the parameter $t = t_2$.*

Proof. Suppose that there is a vertex from μ-graph in G with the parameter $t = t_2$. As long as $t_2 \geq 0$, there is inequality $b \leq 2a - 1$. Regarding the formula obtained in the beginning of the paragraph $2b \geq 3a - 2$, we have the restrictions for the parameter b: $\frac{3}{2}a - 1 \leq b \leq 2a - 1$. Consider two non-adjacent vertices φ_1 and φ_2 and their μ-graph, where there is vertex ψ with the parameter $t = t_2$. Note that according to conditions of obtaining the parameter t_2 it has been known that $|[\varphi_1] \cap [\psi]| = b$, $|[\varphi_2] \cap [\psi]| = a$. Further let us consider two different cases of the size of μ-graph $[\varphi_1] \cap [\varphi_2]$.

1. Suppose that $[\varphi_1] \cap [\varphi_2] = b$, then let us calculate the sizes of some subgraphs of G:

- $|[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp| = b - (2a - b - 1) - 1 = 2b - 2a$;
- $|[\varphi_1] \setminus [\varphi_2]| = k - b = b - a + 3$;
- $|[\varphi_1] \cap [\psi] \setminus [\varphi_2]| = b - (2a - b - 1) = 2b - 2a + 1$;
- $|[\varphi_2] \cap [\psi] \setminus [\varphi_1]| = a - (2a - b - 1) = b - a + 1$;
- $|[\varphi_1] \setminus \psi^\perp| = k - (b + 1) = b - a + 2$;
- $|[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)| = b - a + 2 - (2b - 2a) = a - b + 2$.

As long as $|[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)| \geq 0$, we obtain inequality $a + 2 \geq b$, from which it follows that $b \in \{a + 1, a + 2\}$.

1.1. Let $b = a+2$, then $k = a+7 = b+5$ and $|[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)| = 0$, so that for φ_1 there are no neighbors non-adjacent to ψ outside $[\varphi_1] \cap [\varphi_2]$. If we plug the formula for b in the expressions given above, we obtain $|[\varphi_1] \setminus [\varphi_2]| = 5$, $|[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp| = 4$ and $|[\varphi_1] \cap [\varphi_2] \cap [\psi]| = a - 3$. Consequently, $a \geq 3$.

For φ_1 and φ_2 in G there is a vertex χ non-adjacent to them. Note that $[\varphi_1] \cap [\chi] \subseteq [\varphi_1] \setminus [\varphi_2]$, but then, according to the obtained equations we should have $|[\varphi_1] \cap [\chi]| \leq 5$. But the number of common neighbors of φ_1 and χ is to be at least a, hence, $a \leq 5$.

Vertices in $[\varphi_2] \setminus \psi^\perp$ are non-adjacent to ψ and number of these vertices is $k - (a + 1) = 6$, hence, they form a 6-clique. Consider two arbitrary vertices from $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$. Their common neighbors are φ_1,φ_2 and 4 vertices from the 6-clique. So that there are at least six of them. And as long as $3 \leq a \leq 5$, then two vertices under consideration have b common neighbors and $b \geq 6$. Regarding $b = a + 2$ we obtain the restriction for the parameter a: $a \geq 4$.

Suppose that $a = 5$, then $[\varphi_1] \setminus [\varphi_2] = [\varphi_1] \cap [\chi]$ and $[\varphi_2] \setminus [\varphi_1] = [\varphi_2] \cap [\chi]$. Obtained equations lead to $|[\psi] \cap [\chi]| = 8$, and this contradicts the parameters of G. Consequently, the assumption is not correct.

Suppose that $a = 4$, then $b = 6$, $k = 11$. As long as $|[\varphi_1] \cap [\chi]| \leq |[\varphi_1] \setminus [\varphi_2]|$, we have $|[\varphi_1] \cap [\chi]| = a = 4$. Similarly, the number of common neighbors for φ_2 and χ is 4. Therefore, χ has 3 neighbors outside $\varphi_1^\perp \cup \varphi_2^\perp$. Denote them by χ_1,χ_2,χ_3. According to similar reasoning χ_1,χ_2 have four neighbors in $[\varphi_1] \setminus [\varphi_2]$ and $[\varphi_2] \setminus [\varphi_1]$, along with this $|([\varphi_1] \setminus [\varphi_2]) \cap [\chi_1] \cap [\chi_2]| \geq 3$ and $|([\varphi_2] \setminus [\varphi_1]) \cap [\chi_1] \cap [\chi_2]| \geq 3$. We obtained that χ_1 and χ_2 have at least 7 common neighbors, with regard to the common neighbor χ, which contradicts the parameters of G. Therefore, the assumption is incorrect.

1.2. Let $b = a+1$, then $k = a+5$ and $|[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)| = 1$. If we plug the formula for b in the expressions given above, we obtain $|[\varphi_1] \setminus [\varphi_2]| = 4$, $|[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp| = 2$, $|[\psi] \cap [\varphi_2] \setminus [\varphi_1]| = |[\varphi_2] \setminus (\psi^\perp \cup [\varphi_1])| = 2$ and $|[\varphi_1] \cap [\varphi_2] \cap [\psi]| = a-2$. Consequently, $a \geq 2$.

Consider vertex χ which is non-adjacent to φ_1, φ_2. As long as $[\varphi_1] \cap [\chi] \subseteq [\varphi_1] \setminus [\varphi_2]$, $|[\varphi_1] \setminus [\varphi_2]| = 4$ and $|[\varphi_1] \cap [\chi]| \geq a$, we obtain the restriction for the parameter a: $a \leq 4$.

Vertices in $[\varphi_1] \setminus \psi^\perp$ are non-adjacent to ψ and form a 3-clique, similarly the vertices from $[\varphi_2] \setminus \psi^\perp$ form a 4-clique (otherwise we have a claw in the graph). Let us determine the number of common neighbors for two vertices from $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$: one vertex from $[\varphi_1] \setminus (\psi^\perp \cup [\varphi_2])$, two vertices from $[\varphi_2] \setminus (\psi^\perp \cup [\varphi_1])$, φ_1, φ_2, so that there are at least 5 of them. But the parameter a lies in the interval $2 \leq a \leq 4$, hence, the number of common neighbors of vertices from $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$ is b and $b \geq 5$. Recalling that $b = a + 1$, we have inequality $a + 1 \geq 5$, from which it follows that $a \geq 4$. But regarding the restrictions for the parameter a we obtain $a = 4$, thus G has the parameters $(v, 9, 5, 4)$.

For χ we have that $[\varphi_1] \cap [\chi] = [\varphi_1] \setminus [\varphi_2]$, from which it follows that $|[\varphi_1] \cap [\chi]| = 4 = a$. Similarly, we obtain $[\varphi_2] \cap [\chi] = 4$. So that, there are 8 neighbors determined for χ in $\varphi_1^\perp \cup \varphi_2^\perp$, hence, χ has one neighbor outside $\varphi_1^\perp \cup \varphi_2^\perp$. Besides this neighbor should have 4 neighbors in the neighborhood φ_1 and 4 neighbors in the neighborhood φ_2 according to the similar reasoning. From which it follows that χ and its neighbor outside $\varphi_1^\perp \cup \varphi_2^\perp$ have 8 common neighbors in the subgraph $\varphi_1^\perp \cup \varphi_2^\perp$, which contradicts the parameters of G.

2. Suppose that $[\varphi_1] \cap [\varphi_2] = a$, then let us calculate the sizes of some subgraphs of G:

- $|[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp| = b - a$;
- $|[\varphi_1] \setminus [\varphi_2]| = |[\varphi_2] \setminus [\varphi_1]| = 2b - 2a + 3$;
- $|[\varphi_1] \cap [\psi] \setminus [\varphi_2]| = 2b - 2a + 1$;
- $|[\varphi_2] \cap [\psi] \setminus [\varphi_1]| = b - a + 1$;
- $|[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)| = 2$;
- $|[\varphi_2] \setminus (\psi^\perp \cup \varphi_1^\perp)| = b - a + 2$;
- $|[\varphi_1] \cap [\varphi_2] \cap [\psi]| = 2a - b + 1$.

The size of the subgraph $|[\varphi_1] \cap [\varphi_2] \cap [\psi]|$ cannot be less than zero, therefore, $2a - b + 1 \geq 0$. With regard to this inequality and the obtained restriction for the parameter b, calculated in the beginning of the paragraph, we have $\frac{3}{2}a - 1 \leq b \leq 2a - 1$.

Consider subgraph $|[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp|$, vertices in which are non-adjacent to ψ, hence, forming a $(b-a)$-clique (otherwise we have a claw in the graph). There are two options for this subgraph: $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$ contains at least one edge or contains none.

2.1. Let subgraph $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$ contain at least one edge. Then $b - a \geq 2$, hence, $b \geq a+2$. Note that $[\varphi_1] \setminus \psi^\perp$ and $[\varphi_2] \setminus \psi^\perp$ form the cliques of size $b-a+2$ and $2b - 2a + 2$, respectively.

Consider two vertices from $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$ and denote them by ω_1,ω_2. Determine common neighbors of ω_1 and ω_2: two vertices from $[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)$, $b-a+2$ vertices from $[\varphi_2] \setminus (\psi^\perp \cup \varphi_1^\perp)$, $b - a - 2$ vertices from $[\varphi_1] \cap [\varphi_2] \setminus \psi^\perp$ and φ_1,φ_2. We have $2b - 2a + 4$ vertices in $\varphi_1^\perp \cup \varphi_2^\perp$, but there cannot be common neighbors of ω_1,ω_2 outside $\varphi_1^\perp \cup \varphi_2^\perp$, otherwise we have a claw in G. Note that $|[\omega_1] \cap [\omega_2]|$ cannot be more than b, hence, $2b-2a+4 \leq b$. From which it follows the inequality $b \leq 2a-4$, with regard to which we have the restriction for the parameter b: $a+2 \leq b \leq 2a-4$. From here we can express the restriction for the parameter a: $a \geq 6$.

For φ_1 and φ_2 in G there is vertex χ non-adjacent to them. Along with this, the number of common neighbors of χ with φ_1 and φ_2 takes either value a or b, and $[\chi] \supseteq ([\varphi_1] \cap [\chi] \setminus [\varphi_2]) \cup ([\varphi_2] \cap [\chi] \setminus [\varphi_1])$. Note that $|([\varphi_1] \cap [\chi] \setminus [\varphi_2]) \cup ([\varphi_2] \cap [\chi] \setminus [\varphi_1])| \in \{2a, a+b, 2b\}$.

Consider different options of relations between the size of neighborhoods of χ and size $|([\varphi_1] \cap [\chi] \setminus [\varphi_2]) \cup ([\varphi_2] \cap [\chi] \setminus [\varphi_1])|$. If $k \geq 2b$, then $2b - a + 3 \geq 2b$, from which it follows the inequality $a \leq 3$. It contradicts the restriction obtained earlier $a \geq 6$. If $a + b \leq k < 2b$, then $2b - a + 3 \geq a + b$, from which it follows the inequality $b \geq 2a - 3$. It contradicts the restriction obtained earlier $b \leq 2a - 4$. Consequently, $k < a + b$.

But then we have that $|[\varphi_1] \cap [\chi]| = |[\varphi_2] \cap [\chi]| = a$. Therefore, there are $2a$ neighbors of χ in $\varphi_1^\perp \cup \varphi_2^\perp$, and the rest $2b - 3a + 3$ neighbors lie outside $\varphi_1^\perp \cup \varphi_2^\perp$.

Consider subgraph $[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)$, which contains two vertices (denoted by τ_1, τ_2). Suppose that χ is non-adjacent to at least one of τ_1, τ_2. Without loss of generality, assume that $\chi \nsim \tau_1$. Common neighbors of χ and τ_1 can lie in the neighborhood of χ inside the subgraph $[\varphi_1] \setminus (\psi^\perp \cup \varphi_2^\perp)$ and outside it. But τ_1 cannot be adjacent to neighbors of χ, which are adjacent to ψ or to φ_2, otherwise we have a claw in the graph. But then the maximum number of neighbors of χ, with which τ_1 can be adjacent is $(2b - 3a + 3) + 1$. And among these vertices there should be at least a common neighbors for χ and τ_1. Then $2b-3a+4 \geq a$, creating inequality $b \geq 2a-2$, which contradicts the restriction obtained earlier $b \leq 2a-4$. Consequently, the assumption is incorrect and χ is adjacent to both vertices τ_1, τ_2. Note that χ has $a-2$ vertices in the neighborhood φ_1 beside τ_1, τ_2. But ψ is also adjacent to these vertices.

Suppose that $|[\psi]\cap[\chi]| = b$, then $|[\psi]\cap[\chi]\cap[\varphi_2]| = b-(a-2) = b-a+2$. With that $|[\psi]\cap[\varphi_2]\setminus[\varphi_1]| = b-a+1$, hence, the inclusion $([\psi]\cap[\chi]\cap[\varphi_2]) \subseteq ([\psi]\cap[\varphi_2]\setminus[\varphi_1])$ cannot be satisfied. Therefore, assumption is incorrect and $|[\psi]\cap[\chi]| = a$. Then $|[\psi]\cap[\chi]\cap[\varphi_2]| = 2$. χ and φ_2 have a common neighbors, out of which $a-2$ vertices lie in $[\chi]\cap[\varphi_2]\setminus[\psi]$. But there is an equality obtained earlier $|[\varphi_2]\setminus\psi^{\perp}| = b-a+2$, from which it follows that $b-a+2 \geq a-2$. Therefore, we have the inequality for the parameter b: $b \geq 2a-4$, which creates $b = 2a-4$ with regard to the restriction $b \leq 2a-4$ obtained earlier.

It was calculated earlier that ω_1 and ω_2 from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$ have $2b-2a+4$ of common neighbors. If we plug the formula for b, we obtain $|[\omega_1]\cap[\omega_2]| = b$. ω_1 and ω_2 were chosen arbitrary, and this means that for any pair of vertices from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$ there are b common neighbors in $\varphi_1^{\perp}\cap\varphi_2^{\perp}\setminus\psi^{\perp}$.

Consider the arbitrary vertex from $([\varphi_2]\cap[\psi])\setminus([\chi]\cup[\varphi_1])$ and denote it by τ. This vertex should have at least a common neighbors with φ_1, which should lie in $[\varphi_1]\setminus[\chi]$,otherwise there is a claw. Alongside with this τ can have only one neighbor in $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$, otherwise we have the contradiction with the parameter b for vertices from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$. Therefore, common neighbors of φ_1 and τ lie in $\{$one vertex from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}\}\cup([\varphi_1]\cap[\varphi_2]\cap\psi^{\perp})\cup[\varphi_1]\setminus([\varphi_2]\cup[\chi])$. Then $|[\varphi_1]\cap[\tau]| \leq a-5+3+1+1 = a$, which leads to $[\varphi_1]\cap[\tau] = \{$one vertex from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}\}\cup([\varphi_1]\cap[\varphi_2]\cap\psi^{\perp})\cup[\varphi_1]\setminus([\varphi_2]\cup[\chi])$. Similar reasoning is correct for all vertices from $([\varphi_2]\cap[\psi])\setminus([\chi]\cup[\varphi_1])$. We obtained that vertices from $([\varphi_2]\cap[\psi]\setminus[\varphi_1])\cup([\varphi_1]\cap[\varphi_2]\cap[\psi])\cup([\varphi_1]\setminus([\varphi_2]\cup[\chi]))\cup\{\varphi_2\}$ are common neighbors for ψ and τ. Then $|[\psi]\cap[\tau]| = (a-4)+3+(a-5)+1 = 2a-5$, which contradicts the parameters of G.

2.2. Let subgraph $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$ not contain edges. Then $b-a = 1$, from which it follows that $b = a+1$ and $k = a+5$. In the first part of paragraph 2 we obtained the following restrictions for the parameter b: $\frac{3}{2}a-1 \leq b \leq 2a-1$, which create two inequalities $a+1 \leq 2a-1$ and $\frac{3}{2}a-1 \leq a+1$ with regard to $b = a+1$. Having transformed these inequalities, we have the restriction for the parameter a: $2 \leq a \leq 4$.

2.2.1. Let $a = 2$, then $b = 3$, $k = 7$. The neighborhoods for φ_1, φ_2, ψ and the vertex from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$ are completely determined.

Consider vertices from $[\varphi_2]\setminus(\psi^{\perp}\cup[\varphi_1])$ and denote them by ν_1, ν_2, ν_3. Note that ν_1,ν_2,ν_3,φ_2 and the vertex from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$ form a 5-clique, otherwise we have a claw. So, there are b common neighbors already determined for each pair of vertices from this 5-clique. So that, in the neighborhood φ_2 and in the neighborhood of a vertex from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$ vertices ν_1,ν_2,ν_3 cannot have more neighbors. Hence, the rest neighbors of ν_1,ν_2,ν_3 lie in $[\varphi_1]\cap[\psi]$ and with that these neighbors are not common for ν_1, ν_2, ν_3, otherwise we have the contradiction with the parameter b. Note that φ_1 has only one determined neighbor with ν_1,ν_2,ν_3, namely, the vertex from $[\varphi_1]\cap[\varphi_2]\setminus\psi^{\perp}$. In accordance with the value of parameter $a = 2$, we are to determine at least one common neighbor for φ_1 and each of ν_1,ν_2,ν_3. But this can be achieved only upon the hypothesis that each of ν_1,ν_2,ν_3 is adjacent only with one vertex from $[\varphi_1]\cap[\psi]$, provided that each vertex is adjacent to its own one. So that, ν_1,ν_2,ν_3 have five neighbors each determined in $\varphi_1^\perp \cup \varphi_2^\perp$. Hence, the rest two neighbors of each vertex lie outside $\varphi_1^\perp \cup \varphi_2^\perp$. We obtained that there are at least 20 vertices in G.

At the same time, the vertex from $[\varphi_2] \setminus (\psi^\perp \cup [\varphi_1])$ is contained in at least four pairs of "type b" in G, therefore $\beta \geq 4$. We have the inequality $\frac{2(v-1)-42}{2-3} \geq 4$, from which it follows that $20 \geq v$. If we compare two restrictions of v, we obtain $v = 20$.

When we reconstruct the neighborhoods for the vertices adjacent to ν_1,ν_2,ν_3 and which lie outside $\varphi_1^\perp \cup \varphi_2^\perp$, we have that G is the 5×4-lattice, but this contradicts that there is a μ-graph with radius 1 in G.

2.2.2. Let $a = 3$, then $b = 4$, $k = 8$.

Consider μ-graph of non-adjacent vertices γ and δ, which has a radius 1 and size a and contains vertex ε. It is to be recalled that for γ and δ in the graph there is vertex η non-adjacent to them. η and ε should have at least three common neighbors, hence, η will be adjacent either with the edge in $[\delta] \cap [\varepsilon] \setminus [\gamma]$, or with the edge in $[\gamma] \cap [\varepsilon] \setminus [\delta]$. Let us denote the vertices in these edges by ξ_1,ξ_2 and ξ_3,ξ_4 respectively. Without loss of generality, assume that $\eta \sim \xi_1$,$\eta \sim \xi_2$. As long as δ and η cannot have more than four common neighbors, then there is at least one vertex in $[\delta] \setminus [\gamma]$ non-adjacent to η. Denote the vertices in $[\delta] \setminus \varepsilon^\perp$ by ω_1,ω_2,ω_3 and consider that $\eta \nsim \omega_1$.

Suppose that $|[\eta] \cap [\varepsilon]| = 4$. Note that ω_1 cannot be adjacent to ξ_1,ξ_2,ξ_3,ξ_4, otherwise we have a claw in the graph. But ε and ω_1 are to have at least three common neighbors and those can only be δ and vertices from $[\delta] \cap [\gamma] \setminus \{\varepsilon\}$. According to similar reasoning for the vertex from $[\gamma] \setminus \varepsilon^\perp$, non-adjacent to η. Denote it by ν_1, and the rest vertices in $[\gamma] \setminus \varepsilon^\perp$ by ν_2, ν_3. ε and ν_1 are to have at least three common neighbors, namely, γ and vertices from $[\delta] \cap [\gamma] \setminus \{\varepsilon\}$. But then vertices from $[\delta] \cap [\gamma] \setminus \{\varepsilon\}$ already have five neighbors determined (γ,δ,ε,ν_1,ω_1), which contradicts the parameters of G. Consequently, the assumption is incorrect and $|[\eta] \cap [\varepsilon]| = 3$. Therefore, η is not adjacent to any of ξ_1,ξ_2,ξ_3,ξ_4. Without loss of generality, assume that $\eta \nsim \xi_4$.

Suppose that $|[\eta] \cap [\gamma]| = |[\eta] \cap [\delta]| = b = 4$. Then the whole neighborhood of η lies in $\gamma^\perp \cup \delta^\perp$. With that, all vertices from $[\delta] \cap [\gamma] \setminus \{\xi_4, \omega_1\}$ cannot have neighbors outside $\gamma^\perp \cup \delta^\perp$, otherwise we have a claw in the graph. Consider common neighbors for ω_1,γ. ε and ν_1,ν_2,ν_3,ξ_3 cannot be common neighbors for them, otherwise we have a claw in the graph. Hence, only ξ_4 and two vertices from $[\delta] \cap [\gamma] \setminus \{\varepsilon\}$ can be common neighbors for ω_1,γ. The neighbors for ω_1 in $\gamma^\perp \cup \delta^\perp$ are completely determined: δ, ξ_4, ω_2, ω_3 and two vertices from $[\delta] \cap [\gamma] \setminus \{\varepsilon\}$. The rest vertices in the graph cannot be adjacent to ω_1, otherwise we have a claw in the graph. But then ω_1 and η have only two common neighbors (ω_2, ω_3), which contradicts the parameters of the graph. Consequently, the assumption is not correct.

Suppose that $|[\eta] \cap [\gamma]| = |[\eta] \cap [\delta]| = a = 3$. Without loss of generality, assume that $\eta \nsim \omega_2$ and $\eta \nsim \nu_1$. η in $\gamma^\perp \cup \delta^\perp$ has 6 neighbors determined, and as long as the size of the neighborhood of any vertex is 8, then the rest two neighbors of η lie outside $\gamma^\perp \cup \delta^\perp$. Denote these vertices by η_1,η_2.

ω_1,ω_2 should have at least three common neighbors with η. Provided that, neighbors can only lie in $([\delta] \cap [\eta] \setminus [\varepsilon]) \cup \{\eta_1, \eta_2\}$, otherwise we have a claw in the graph. With that $|([\delta] \cap [\eta] \setminus [\varepsilon]) \cup \{\eta_1, \eta_2\}| = 3$, hence, ω_1,ω_2 are adjacent to η_1,η_2. We have that the common neighbors for ω_1,ω_2 are ω_3,δ,η_1,η_2, which are b neighbors. This means that there are no other common neighbors.

ε and ω_1 should have at least two common neighbors beside δ, and they cannot lie in $[\varepsilon]\cap[\eta]$ (otherwise we have a claw in the graph). Therefore, these neighbors can be vertices from $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$ or ξ_4. Similar reasoning is applied to ε and ω_2. Among three vertices from $\{\xi_4\}\cup[\delta]\cap[\gamma]\setminus\{\varepsilon\}$ it is impossible to choose two neighbors for ω_1 and ω_2 so as they were different. Therefore, assumption is incorrect.

Suppose that, $|[\eta]\cap[\gamma]| = 4, |[\eta]\cap[\delta]| = 3$. Without loss of generality, assume that $\eta \nsim \omega_2$ and $\eta \sim \nu_1$. There are 7 vertices of the neighborhood determined for η in $\gamma^\perp \cup \delta^\perp$, then one vertex, which is adjacent to η, is outside $\gamma^\perp \cup \delta^\perp$. Let us determine common neighbors for η and ω_1: those can only be ω_3 and the neighbor of η outside $\gamma^\perp\cup\delta^\perp$ (otherwise we have a claw in the graph), this means that there are not more than two of them. This contradicts the parameters of G. Consequently, the assumption is not correct.

Suppose that $|[\eta]\cap[\gamma]| = 3$, $|[\eta]\cap[\delta]| = 4$. Without loss of generality, assume that $\eta \nsim \nu_1$, $\eta \nsim \omega_1$, and $\eta \sim \nu_2$, $\eta \sim \nu_3$, $\eta \sim \omega_2$, $\eta \sim \omega_3$. There are 7 vertices of the neighborhood determined for η in $\gamma^\perp \cup \delta^\perp$, hence, one vertex adjacent to η lies outside $\gamma^\perp \cup \delta^\perp$. Denote it by η_1.

Note that the common neighbors of η and ω_1 lie in $\{\eta_1\}\cup([\delta]\setminus(\{\omega_1\}\cup\varepsilon^\perp))$, but $|\{\eta_1\}\cup([\delta]\setminus(\{\omega_1\}\cup\varepsilon^\perp))| = 3$, hence, we have $[\eta]\cap[\omega_1] = \{\eta_1\}\cup([\delta]\setminus(\{\omega_1\}\cup\varepsilon^\perp))$. According to the similar reasoning for η and ν_1 we have that $\nu_1 \sim \eta_1$.

As long as we know that the previous cases for the vertices outside $\gamma^\perp \cup \delta^\perp$ are not correct, then η_1 has only one neighbor outside $\gamma^\perp \cup \delta^\perp$. Then we are to determine seven neighbors of η_1 in $\gamma^\perp \cup \delta^\perp$, where two are already known: ν_1 and ω_1. There are five neighbors to determine among ξ_4 and $[\eta]\cap([\delta]\cup[\gamma])$. But η_1 cannot have more than four neighbors in $[\eta]\cap([\delta]\cup[\gamma])$, hence, $\eta_1 \sim \xi_4$ and $|[\eta]\cap[\eta_1]\cap([\delta]\cup[\gamma])| = 4$. We have all vertices in G determined and $v = 17$.

Determine common neighbors for ν_1 and δ: ω_2,ω_3,ξ_1,ξ_2 cannot be common neighbors for them (otherwise we have a claw in the graph). The possible neighbors for ν_1 are ω_1 and two vertices from $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$. As long as $a = 3$, we have that ν_1 is adjacent to ω_1 and to two vertices from $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$. Note that the vertices from $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$ already have b common neighbors, namely, γ,δ,ε,ν_1.

Consider common neighbors for δ and ξ_4: excluding ω_2,ω_3,ξ_1,ξ_2, we have ω_1 and two vertices from $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$. There is one common neighbor for δ and ξ_4 (the vertex ε), so we are to determine at least two others. As long as all neighbors for the vertices from $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$ are already determined, then common neighbors for δ and ξ_4 are ω_1 and one vertex from the subgraph $[\delta]\cap[\gamma]\setminus\{\varepsilon\}$ (denote it by θ).

Consider common neighbors of η and ξ_4: those are ξ_3, η_1. We are to determine at least one other common neighbor, provided that it should lie outside $\delta^\perp$, otherwise we have a claw in the graph. Consequently, it can be either ν_2 or ν_3. Without loss of generality, assume that $\xi_4 \sim \nu_2$. But then η_1,γ,ω_1,θ,ν_2 are common neighbors for ξ_4, ν_1 (we obtained five neighbors), which contradicts the parameters of the graph.

2.2.3. Let $a = 4$, then $b = 5$, $k = 9$.

Consider μ-graph of non-adjacent vertices γ and δ, with radius 1, which as well contains ε. It is to be recalled that for γ and δ in G there is a vertex η non-adjacent to them. As long as $|\varepsilon^\perp \setminus (\gamma^\perp \cup \delta^\perp)| = 4$, and ε and η should have at least a common neighbors, also taking onto account that $[\eta]\cap[\varepsilon] \subseteq \varepsilon^\perp\setminus(\gamma^\perp\cup\delta^\perp)$, we have $[\eta]\cap[\varepsilon] = \varepsilon^\perp\setminus(\gamma^\perp\cup\delta^\perp)$ and $|[\varepsilon]\cap[\eta] = 4|$.

η should have at least four common neighbors with γ and δ, with that common neighbors cannot lie in $[\gamma]\cap[\delta]$. Therefore, there are 8 neighbors determined for η in $\gamma^\perp \cup \delta^\perp$. As long as parameter k for G takes the value $k = 9$ and $|[\gamma]\cup[\delta] \setminus ([\gamma]\cap[\delta])| = 10$, there is a vertex non-adjacent to η in $[\gamma]\cup[\delta]\setminus\varepsilon^\perp$. Without loss of generality, assume that this vertex lies in $[\delta]$ and denote it by ω. Let us determine common neighbors for ω and η: they lie in the neighborhood of η, but not in $[\eta]\cap[\gamma]$ and not in $[\eta]\cap[\varepsilon]$, otherwise we have a claw in the graph. Consequently, $[\eta]\cap[\omega] \subseteq (\delta^\perp \setminus (\{\omega\}\cup\varepsilon^\perp)) \cup ([\eta]\setminus(\gamma^\perp\cup\delta^\perp))$. Note that $\delta^\perp\setminus(\{\omega\}\cup\varepsilon^\perp)$ contains 2 vertices, while there can be no more than one vertex in $[\eta]\setminus(\gamma^\perp\cup\delta^\perp)$ which leads to the inequality $|[\omega]\cap[\eta]| \leq 3$. We obtained the contradiction with the parameters of G.

Having considered all possible hypotheses, we obtained that there is no graph G, where one can find vertices with parameter $t = t_2$. $\square$

We considered all possible cases for G. The main theorem is proved.

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