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THE GENERALIZED DETERMINANT AND ITS APPLICATION TO THE ENUMERATION OF PERMUTATIONS

D.B. EFIMOV

Abstract. In this work we present a method of enumeration of restricted permutations belonging to two arbitrary disjoint complementary classes. This method is based on combinatorial properties of the generalized determinant and the permanent. In particular, we apply this method to enumeration of restricted permutations with special character of inversions. We also show the connection between the generalized determinant and associative finite dimensional algebras with nilpotent generators.

Keywords: restricted permutations, determinant, permanent, algebras with nilpotent generators

1. Introduction

Let $A = (a_{ij})$ be a $(0,1)$-matrix of order n. Each such matrix defines a class $\mathcal{B}(A)$ of so called *restricted permutations*. Namely a permutation p belongs to $\mathcal{B}(A)$ if and only if the inequality $M_p \leq A$ holds for its incidence matrix M_p , i.e. each element of the matrix M_p is not more than the corresponding element of the matrix A. Denote by E_A and O_A, respectively, the number of even and odd permutations from the class $\mathcal{B}(A)$. It is easy to see that the number of elements in the class $\mathcal{B}(A)$ is equal to the permanent of the matrix A: $|\mathcal{B}(A)| = E_A + O_A = \operatorname{per} A$. It is easy to see also that $E_A - O_A = \det A$. This emplies following formulas for calculating the total number of even and odd permutations from the class $\mathcal{B}(A)$ (see [1], [2]):

$$E_A = \frac{\operatorname{per} A + \det A}{2}, \quad O_A = \frac{\operatorname{per} A - \det A}{2}. \tag{1}$$

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In [2] these formulas are extended with help of generalized permanents to cases of the enumeration of restricted permutations with a given residue of the decrement, full cycles and also permutations with restrictions on their cyclic structure. In this work we consider generalized functions of the determinant similar to the Schur functions or immanants (see [3], [4]). With help of this functions we extend (1) to the case of the enumeration of restricted permutations belonging to two arbitrary disjoint complementary classes. As an interesting special case we consider classes of restricted permutations with special character of inversions.

2. The generalized determinant

Let P be a subset of the symmetric group S_n and let $A = (a_{ij})$ be an $n \times n$ matrix. Consider the following matrix function, which we shall call *the generalized determinant*:

$$d_P(A) = \sum_{\sigma \in S_n \setminus P} \prod_{i=1}^{n} a_{i\sigma(i)} - \sum_{\delta \in P} \prod_{j=1}^{n} a_{j\delta(j)}. \tag{2}$$

Example 1.

If $P = \varnothing$ then $d_P(A) = \operatorname{per} A$.

If P is the set of odd permutations then $d_P(A) = \det A$.

If $n = 3$ and $P = \{(213), (312)\}$ then $d_P(A) = a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} + a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{21}a_{32}$.

Let now A be a $(0,1)$-matrix of order n and let $\mathcal{B}(A)$ be the corresponding class of restricted permutations. Denote by $E_{A,P}$ the number of permutations in the set $\mathcal{B}(A) \setminus P$ and denote by $O_{A,P}$ the number of permutations in the set $\mathcal{B}(A) \cap P$. Obvious that $E_{A,P} + O_{A,P} = |\mathcal{B}(A)| = \operatorname{per} A$. It is easy to see also that $E_{A,P} - O_{A,P} = d_P(A)$. Combine the last two formulas we get the generalization of (1):

$$E_{A,P} = \frac{\operatorname{per} A + d_P(A)}{2}, \quad O_{A,P} = \frac{\operatorname{per} A - d_P(A)}{2}. \tag{3}$$

Further we will focus on quite wide special case of these formulas (this case is connected with the enumeration of permutations with a special character of inversions). For this we will specify the structure of the subset P.

3. Permutations with a special character of inversions

Let $I_n = \{1, 2, \ldots, n\}$ be the set of the first n natural numbers. Select in $I_n \times I_n$ the set T of all pairs (k, l) such that $k < l$. Let $B \subset T$ be a subset in T. Consider in S_n permutations in which pairs of elements from B form an odd number of inversions. Let $S_{n,B}$ denote the set of all such permutations.

Example 2.

If $B = \varnothing$ then we assume, by definition, that $S_{n,B} = \varnothing$.

If $B = T$ then $S_{n,B}$ is the set of all odd permutations in S_n.

If $B = \{(k, l)\}$ then $S_{n,B}$ is the set of permutations $p \in S_n$, in which the element l stands left from the element k. So, if $n = 3$, $B = \{(1, 2)\}$ then $S_{n,B} = \{231, 213, 321\}$.

If $B = \{(k, l),\ (k, m)\}$ then $S_{n,B}$ is the set of permutations $p \in S_n$ in which the element k stands right from the element l, but left from m, or vice versa. So, if $n = 3$, $B = \{(1, 2), (1, 3)\}$ then $S_{n,B} = \{213, 312\}$.

Let us denote in this special case the number of restricted permutations from the class $\mathcal{B}(A)$ having the even and odd number of inversions on pairs of elements from the set B by $E_{A,B}$ и $O_{A,B}$, respectively, and rewrite the generalized determinant (2) in the following form:

$$d_B(A) = \sum_{\sigma\in S_n\setminus S_{n,B}} \prod_{i=1}^{n} a_{i\sigma(i)} - \sum_{\delta\in S_{n,B}} \prod_{j=1}^{n} a_{j\delta(j)}.$$

Then taking into account formulas (3), we get:

$$E_{A,B} = \frac{\operatorname{per} A + d_B(A)}{2}, \quad O_{A,B} = \frac{\operatorname{per} A - d_B(A)}{2}. \tag{4}$$

In further we will denote by $A(\alpha_1, \alpha_2, \dots, \alpha_k|\beta_1, \beta_2, \dots, \beta_l)$ a matrix obtained from a matrix A by removing rows with numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ and columns with numbers $\beta_1, \beta_2, \dots, \beta_l$. Let us give two properties of the function $d_B(A)$ simplifying in some cases its calculation. This properties are analogues of the expansion of the determinant along a row and a column.

Proposition 1. *Let $j \in I_n$ and the set B doesn't contain pairs of elements of the form (i, j) or (j, k), where i, k are arbitrary numbers from I_n. Then the following expansion of the function $d_B(A)$ along the j-th column holds:*

$$d_B(A) = \sum_{i=1}^{n} a_{ij}\, d_{\widetilde{B}}(A(i|j)), \tag{5}$$

where $\widetilde{B}$ is the set of pairs of elements, which is obtained from the set B as follows: each pair (a, b), $b < j$ of the set B transforms to the pair (a, b) of the set $\widetilde{B}$, each pair (s, t), $j < s$ of the set B transforms to the pair $(s-1, t-1)$ of the set $\widetilde{B}$, each pair (a, t), $a < j < t$ of the set B transforms to the pair $(a, t-1)$ of the set $\widetilde{B}$. In particular, if j is more than each element belonging to pairs of the set B, then we obtain the «stable» with respect of B formula:

$$d_B(A) = \sum_{i=1}^{n} a_{ij}\, d_B(A(i|j)).$$

Proof. Let π be a permutation of the numbers $1, 2, \dots, n$. Let us remove from it the number j and diminish on 1 all others numbers greater than j. As a result we obtain a new permutation $\tilde{\pi}$ of the numbers $1, 2, \dots, n-1$. For example, if $\pi = (14352)$ and $j = 3$ then $\tilde{\pi} = (1342)$. It is easy to see that the number of inversions of a permutation π with respect of pairs of elements from B is equal to the number of inversions of the permutation $\tilde{\pi}$ with respect of pairs of elements from $\widetilde{B}$. It follows that the sum of all addends in $d_B(A)$ containing the fixed element a_{ij} is equal to $a_{ij}\, d_{\widetilde{B}}(A(i|j))$. Since each addend in $d_B(A)$ contains exactly one element from the j-th column of the matrix A then we get the formula (5). $\square$

Example 3. Consider the following 4×4 matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Suppose $B = \{(1,2),(1,3)\}$. Then, expanding the generalized determinant of the matrix A along the last fourth column, we get:

$$d_B(A) = d_B\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} + d_B\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} + d_B\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} =$$
$$= 3 + 1 + 1 = 5.$$

Proposition 2. *Consider elements of the i-th row of the matrix A. Suppose that $a_{ij} = 0$ for all j belonging to a pair of elements of the set B. Then one can expand the generalized determinant of the matrix A along the i-th row by the formula:*

$$d_B(A) = \sum_j a_{ij}\, d_{B_j}(A(i|j)), \tag{6}$$

where summation is over all j not included in pairs of the set B, and B_j is the set of pairs of elements, which is obtained from the set B as follows: each pair (a,b), $b < j$ of the set B transforms to the pair (a,b) of the set B_j, each pair (s,t), $j < s$ of the set B transforms to the pair $(s-1,t-1)$ of the set B_j, each pair (a,t), $a < j < t$ of the set B transforms the pair $(a,t-1)$ of the set B_j.

Proof. The formula (6) follows from the proof of the previous proposition and from the fact that each summand in $d_B(A)$ includes exactly one nonzero element of i-th row. $\square$

Example 4. Consider the following 4×4 matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Suppose that $B = \{(1,3)\}$. Then, expanding the generalized determinant of the matrix A along the third row, we get:

$$d_B(A) = d_{B_2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + d_{B_4}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 0 + 0 = 0,$$

where $B_2 = (1,2)$, $B_4 = (1,3)$.

Let us give one more property of the function $d_B(A)$, allowing to simplify the calculation of generalized determinant in the case when a set B consists only from one pair of elements.

Proposition 3. *Let $B = \{(k,l)\}$. Then the function $d_B(A)$ can be calculated by the following formula:*

$$d_B(A) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_{ik}a_{jl} - a_{il}a_{jk})\mathrm{per}(A(i,j|k,l)). \tag{7}$$

Proof. Each addend in $d_B(A)$ contains as factors one element from k-th and one element from l-th column of the matrix A. The addends containing elements a_{ik} and a_{jl} as factors get into $d_B(A)$ with the plus sign if $i < j$ and with a minus sign if $i > j$. Herewith the sum of all addends in $d_B(A)$ containing elements a_{ik}, a_{jl} or

a_{il}, a_{jk}, $i < j$ as factors, is equal to $(a_{ik}a_{jl} - a_{il}a_{jk})\mathrm{per}(A(i,j|k,l))$. Going trough all ordered pairs of numbers of rows $i < j$, we get the formula (7). $\square$

Let us give some naive, but illustrative example demonstrating of using of given technique for the enumeration of restricted permutations with special character of inversions.

Task. Suppose that in a school at some day should be carried out the following subjects: M – mathematics, R – Russian, P – physics, G – geography, S – sport (physical education). The question: How many ways are there to draw up a schedule for the day if the mathematics and the Russian can not stand as last two lessons because of the complexity of these subjects, the sport should not be placed after the long break as the third lesson, the geography can not be placed as the first lesson and the physics as the second lesson because of the employment of teachers in other classes, and the physics must be stand in the schedule after the mathematics into force of the connectivity of studied themes?

Solution. Let us number the subjects as follows: mathematics – 1, physics – 2, Russian – 3, geography – 4, sport – 5. Each of the five subjects is assigned to one of the five lessons when scheduling. Thus each admissible variant of the schedule corresponds to a permutation p of the numbers $1, 2, \dots, 5$ satisfying the following two conditions:

(1) $p(1) \neq 4$, $p(2) \neq 2$, $p(3) \neq 5$, $p(4) \neq 1, 3$, $p(5) \neq 1, 3$;
(2) the number 1 stands in the permutation to the left of the number 2.

And the problem is reduced to the determination of the number of such permutations. To solve the task consider the class of restricted permutations satisfying the first condition. The matrix of restrictions of this class has the following form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

One can directly calculate that the total number of permutations of the given class is equal to $\mathrm{per}(A) = 24$. Assuming $B = \{(1,2)\}$ and using (7), we obtain:

$$d_B(A) = \sum_{i=1}^{4} \sum_{j=i+1}^{5} (a_{i1}a_{j2} - a_{i2}a_{j1})\mathrm{per}(A(i,j|1,2)) = -\mathrm{per}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
$$+\mathrm{per}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \mathrm{per}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
$$+\mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$
$$+\mathrm{per}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2 + 3 + 3 + 2 + 2 + 2 + 3 + 3 = 16.$$

Then using formulas (4), we get that the number of belonging to the given class restricted permutations in which the number 1 stands to the left of the number 2 is equal:

$$E_{A,B} = \frac{24+16}{2} = 20.$$

In terms of subjects these permutations are as follows: MRPGS, MRPSG, MRGPS, MRGSP, RMPGS, RMPSG, RMGPS, RMGSP, MGRPS, MGRSP, MSRPG, MSRGP, RGMPS, RGMSP, RSMPG, RSMGP, SMRPG, SMRGP, SRMPG, SRMGP. The four belonging to the given class restricted permutations, for which the last property doesn't hold, have the form: PMRSG, PRMSG, PMRGS, PRMSG.

4. The connection with associative algebras

We shall give first some definition. *The Pimenov algebra* with n generators is an associative algebra $P_n(\iota)$ generated over the field of real numbers by 1 and elements ι_k, $k = 1, \dots, n$ linked by the following generating relations: $\iota_k^2 = 0$, $\iota_k\iota_l = \iota_l\iota_k$, $k, l = 1, \dots, n$. By its structure, the algebra $P_n(\iota)$ reminds the Grassmann algebra with the only difference that its generators commute, but don't anticommute. Well known the relation between the permanent and Pimenov algebra and between the determinant and Grassmann algebra (see, for example, [3] and [5]). This relation one can generalized to the case of functions $d_B(A)$ and finite dimensional associative algebras with nilpotent generators of index 2.

Consider the finite dimensional associative algebras $GP_{n,B}$ generated over the field of real numbers by 1 and generators $\iota_1, \iota_2, \dots, \iota_n$ satisfying the commutation relations:

$$\begin{gathered} \iota_k^2 = 0,\ k = 1, 2, \dots, n; \\ \iota_k\iota_l = \iota_l\iota_k, \text{ if } (k,l) \in T \setminus B \text{ or } (l,k) \in T \setminus B; \\ \iota_s\iota_t = -\iota_t\iota_s,\ \text{ if } (s,t) \in B \text{ or } (t,s) \in B. \end{gathered}$$

Example 5. If $B = T$ then we get the Grassmann algebra. If $B = \varnothing$ then we get the Pimenov algebra.

Denote the r-th row vector of the matrix A by $\bar{a}_r$. Introduce formally the row vector $\bar{\iota} = (\iota_1, \iota_2, \dots, \iota_n)$. Consider the homogeneous elements of order 1 in the algebra $GP_{n,B}$:

$$a_r = \bar{a}_r \cdot \bar{\iota} = a_{r1}\iota_1 + a_{r2}\iota_2 + \cdots + a_{rn}\iota_n.$$

Then easy to see that

$$\prod_{r=1}^{n} a_r = d_B(A)\iota_1\iota_2 \dots \iota_n.$$

Example 6. Consider the following 3×3 matrix :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Suppose $B = \{(1,2),\ (1,3)\}$. Then

$$d_B(A) = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}.$$

The algebra $GP_{3,B}$ has the following generating relations:

$$\iota_1^2 = \iota_2^2 = \iota_3^2 = 0,\ \iota_1\iota_2 = -\iota_2\iota_1,\ \iota_1\iota_3 = -\iota_3\iota_1,\ \iota_2\iota_3 = \iota_3\iota_2.$$

We have the equality:

$$(a_{11}\iota_1 + a_{12}\iota_2 + a_{13}\iota_3)(a_{21}\iota_1 + a_{22}\iota_2 + a_{23}\iota_3)(a_{31}\iota_1 + a_{32}\iota_2 + a_{33}\iota_3) = d_B(A)\iota_1\iota_2\iota_3.$$

The given relation between functions $d_B(A)$ and finite dimensional associative algebras $GP_{n,B}$ can be use, for example, for upper bounds of functions $d_B(A)$ similar how it has been done for permanents and determinants in [6] (a short overview of this work can be found also in [3]).

5. Conclusion

In this work we have presented the method of enumeration of restricted permutations belonging to two arbitrary disjoint complementary classes. This method is based on combinatorial properties of the permanent and the generalized determinant. The main part of the work is devoted to application of this method to the enumeration of restricted permutations with special character of inversions. The method has a simple algorithmic realization. But from a computational point of view it is firstly limited by the complexity of the computing the permanent. Most likely in general case the generalized determinant as well as the permanent does not allow a good way of calculating. In this regard the question of the estimate becomes actual, and the presented in this paper connection between the generalized determinant and associative finite dimensional algebras turns out useful.

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References

[1] V.S. Shevelev, *Some problems of the theory of enumerating the permutations with restricted position*, Journal of Soviet Mathematics, **61**:4 (1992), 2272–2317 (English translation). MR1172614

[2] V. Shevelev, *Combinatorial minors for matrix functions and their applications*, Zeszyty Naukowe Politechniki Śląskiej. Seria: Matematyka Stosowana, **z.4** (2014), 5–16.

[3] H. Minc, *Permanents*, Reading, MA: Addison-Wesley, 1978. MR0504978

[4] B. Kuzma, *A note on immanant preservers*, Fundamentalnaya i prikladnaya matematika, **13**:4 (2007), 113–120.

[5] E.B. Vinberg, *A Course in Algebra*, AMS, 2003. MR1974508

[6] W.B. Jurkat, H.J. Ryser, *Matrix Factorizations of Determinants and Permanents*, Journal of Algebra, **3**:1 (1966), 1–27. MR0195879

Dmitry Borisovich Efimov
Department of Mathematics Komi Science Centre,
Chernova st., 3a,
167000, Syktyvkar, Russia
E-mail address: dmefim@mail.ru