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ON IMAGES OF PARTIAL COMPUTABLE FUNCTIONS OVER COMPUTABLE POLISH SPACES

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Abstract. This paper is a part of the ongoing program on analysing the complexity of various problems in computable analysis in terms of the effective Borel and Lusin hierarchies. We give an answer to the question by A. Morozov and K. Weihrauch that concerns a characterisation of image complexity of partial computable functions over computable Polish spaces.

Keywords: computable Polish space, partial computable function, computable analysis.

1. Introduction

In this paper we work with an effectivisation of Polish spaces (see [12, 4, 17] among others) which is compatible with the notion of a computable (recursive) metric space [13].

We assume that a computable Polish space is a complete computable metric space without isolated points. In this paper we consider the computable Polish spaces as a proper subclass of the effectively enumerable topological spaces [9].

Computability theory has a long term tradition to study partial computable functions. While the class of computable Polish spaces is one of the main objects for investigation in the Effective Descriptive Set Theory [17] the class of partial computable functions over computable Polish spaces has not been deeply investigated yet. In this paper we address natural problems related to partial computable functions over computable Polish spaces:

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- Does the class of partial computable functions have a universal partial computable function?
- What descriptive complexity do images of partial computable functions have?

We give positive answers to these questions. To construct a universal function we establish the correspondence between partial computable functions and the classical enumeration operators [14]. In order to study images of partial computable functions we use the effective Borel and Lusin hierarchies on computable Polish spaces [12, 17]. In particular our proofs are based on the following properties of Borel and analytic subsets of a computable Polish space $\mathcal{X}$:

- A set B is a Π^0_2–set in the effective Borel hierarchy on $\mathcal{X}$ (a Π^0_2–subset of X) if and only if $B = \bigcap_{n\in\omega} A_n$ for a computable sequence of effectively open sets $\{A_n\}_{n\in\omega}$.
- A set $A \in$ is a Σ^1_1–set in the effective Lusin hierarchy on $\mathcal{X}$ (a Σ^1_1–subset of X) if and only if $A = \{y \mid (\exists x \in X)B(x, y)\}$, where B is a Π^0_2–subset of $X \times X$.

The paper is organised as follows. Section 2 and Section 3 contain preliminaries and basic background.

In Section 4 we introduce the class of effectively enumerable T_0–spaces with point recovering which contains computable Polish spaces among others and plays an important role in the description of the images of surjective partial computable functions.

In Section 5 we propose the notion of a partial computable function in the settings of effectively enumerable spaces. On the computable Polish spaces this definition agrees with the definition of a computable function introduced by K. Weihrauch for computable metric spaces [20]. We show that this class is closed under composition over effectively enumerable spaces.

Section 6 contains main results, where we work with computable Polish spaces. After showing the correspondence between the partial computable functions and the classical enumeration operators we prove the existence of a universal partial computable function. Then we turn to our main goal that is an investigation of images of partial computable functions. First we show the existence of a partial computable surjection between any computable Polish space and any effectively enumerable topological space with point recovering. Using this result we prove that for any computable Polish spaces $\mathcal{X}$ and $\mathcal{Y}$, the images of partial computable functions $f : \mathcal{X} \to \mathcal{Y}$ are precisely Σ^1_1–subsets of Y. We conclude with the future work.

2. Preliminaries

We refer the reader to [14] and [15] for basic definitions and fundamental concepts of recursion theory. We recall that, in particular, φ_e denotes the partial computable (recursive) function with an index e in the Kleene numbering, φ^s_e denotes the computation of φ_e for s steps such that the function φ^s_e is uniformly primitive recursive. In this paper we also use notations $W_e = \mathrm{dom}(\varphi_\mathrm{e})$, $W^s_e = \mathrm{dom}(\varphi^\mathrm{s}_\mathrm{e})$, and $\pi_e = \mathrm{im}(\varphi_\mathrm{e})$. A sequence $\{V_i\}_{i\in\omega}$ of computably enumerable (c.e.) sets is *computable* (or uniformly computably enumerable) if $\{(n, i) | \, n \in V_i\}$ is computably enumerable. It is worth noting that this is equivalent to existence of a computable function $f : \omega \to \omega$ such that $V_i = W_{f(i)}$.

In the major part of our paper we work with the following notion of a computable Polish space. A computable Polish space is a complete separable metric space $\mathcal{X}$ without isolated points and with a metric d such that there is a countable dense set $\mathcal{B} = \{b_1, b_2, \dots\}$ called a *basis of* X that makes the following two relations: $\{(n,m,i) \mid d(b_n,b_m) < q_i,\ q_i \in \mathbb{Q}\}$ and $\{(n,m,i) \mid d(b_n,b_m) > q_i,\ q_i \in \mathbb{Q}\}$ computably enumerable (c.f. [13]). The standard notations $B(x,y)$ and $\overline{B}(x,y)$ are used for open and closed balls with the center x and the radius y. We consider this concept in the framework of effectively enumerable spaces (see Section 3.1).

We use the Baire space $\mathcal{N} = (\omega^\omega, \tau_\mathcal{N})$ defined as follows.

$$\omega^0 = \{\bot\},\ \text{where } \bot \text{ is the empty word},$$
$$\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n,$$
$$\omega^\omega = \{f \mid f : \omega \to \omega\}$$
$$\text{(informally, the set of all paths in the tree } \omega^{<\omega}\text{)}.$$

The standard topology $\tau_\mathcal{N}$ on ω^ω is generated by the base that contains all clopen sets of the type

$$\mathfrak{A}_w = \{f \in \mathcal{N} \mid f[s] = w,\ s = \text{length}(w)\},$$

where $w \in \omega^{<\omega}$ and the interpretation of $f[s]$ is as follows:

$$f[0] = \bot,$$
$$f[s] = \langle f(0), \dots, f(s-1)\rangle .$$

We take a standard agreement [14] that a downward closed nonempty subset $T \subseteq \omega^{<\omega}$ is a tree and $[T]$ denotes the set $\{f \in \mathcal{N} \mid (\forall s \in \omega) f[s] \in T\}$. Further on we use the Cantor space $\mathcal{C} = (2^\omega, \tau_\mathcal{C})$ with the standard topology $\tau_\mathcal{C}$ defined similar to $\tau_\mathcal{N}$.

3. Basic Background

3.1. **Effectively Enumerable Topological Spaces.** Now we recall the notion of an effectively enumerable topological space. Let (X,τ,α) be a topological space, where X is a non-empty set, $B_\tau \subseteq 2^X$ is a base of the topology τ and $\alpha : \omega \to B_\tau$ is a numbering.

Definition 1. [9] *A topological space (X,τ,α) is effectively enumerable if the following conditions hold.*

(1) *There exists a computable function $g : \omega \times \omega \times \omega \to \omega$ such that*

$$\alpha(i) \cap \alpha(j) = \bigcup_{n\in\omega} \alpha(g(i,j,n)).$$

(2) *The set $\{i \mid \alpha(i) \neq \emptyset\}$ is computably enumerable.*

For a computable Polish space $(X, \mathcal{B}, d)$ in a natural way we define the numbering of the base of the standard topology as follows. First we fix a computable numbering $\alpha^* : \omega \setminus \{0\} \to (\omega \setminus \{0\}) \times \mathbb{Q}^+$. Then,

$$\alpha(0) = \emptyset,$$
$$\alpha(i) = B(b_n, r) \text{ if } i > 0 \text{ and } \alpha^*(i) = (n,r).$$

For $\alpha^*(i) = (n,r)$ later we use notation $n = u(i)$ and $r = r_i$.

It is easy to see that (X,τ,α) is an effectively enumerable topological space. Therefore we consider the computable Polish spaces as a proper subclass of the effectively enumerable topological spaces. For details we refer to [9]. In this paper for such effectively enumerable topological space (X,τ,α) we use the standard relations on the indices of basic balls defined as follows:

$$i \prec_X j \leftrightharpoons d(b_{u(i)}, b_{u(j)}) + r_i < r_j,$$
$$i \,|_X\, j \leftrightharpoons d(b_{u(i)}, b_{u(j)}) > r_i + r_j,$$

for details see, e.g., [16]. The relation $\prec_X$ is irreflexive and transitive and if $i \prec_X j$ then $\mathrm{cl}(\alpha(i)) \subseteq \alpha(j)$. It is easy to see that these relations are computably enumerable on the indices of basic balls. Below we use the following properties of computable Polish spaces and these relations.

Theorem 1 (Nested Sphere principle)**.** [5] *A necessary and sufficient condition that the metric space $\mathcal{X}$ be complete is that every sequence of closed nested spheres in $\mathcal{X}$ with radii tending to zero have nonvoid intersection, moreover the intersection is a one point set.*

Lemma 1. *Let (X,τ,α) be a computable Polish space. Suppose $y \notin \overline{B}(b,q)$ and $\{y\} = \bigcap_{n\in\omega} B(a_n,r_n)$, where $r_n \to \infty$ and $\bar{B}(a_{n+1},r_{n+1}) \subseteq B(a_n,r_n)$. Then, there exists $n \in \omega$ such that $B(a_n,r_n) \,|_X\, B(b,q)$.*

Proof. Let us find $n \in \omega$ such that $2 \cdot r_n < d(y,b) - q$. Then, $d(a_n,b) \geq d(y,b) - d(a_n,y) > d(y,b) - r_n > q + r_n$. By definition, $B(a_n,r_n) \,|_X\, B(b,q)$. □

Lemma 2. *Let (X,τ,α) be a computable Polish space. Suppose $y \in B(b,q)$ and $\{y\} = \bigcap_{n\in\omega} B(a_n,r_n)$, where $r_n \to \infty$ and $\bar{B}(a_{n+1},r_{n+1}) \subseteq B(a_n,r_n)$. Then, there exists $n \in \omega$ such that $B(a_n,r_n) \prec_X B(b,q)$.*

Proof. The proof is similar to the proof of Lemma 1. □

We recall the notion of an effectively open set.

Definition 2. [9] *Let (X,τ,α) be an effectively enumerable topological space. A set $A \subseteq X$ is effectively open if there exists a computably enumerable set V such that*

$$A = \bigcup_{n\in V} \alpha(n).$$

It is worth noting the set of all effectively open subsets of X is closed under intersection and union since the class of effectively enumerable sets is a lattice.

4. Effectively Enumerable T_0–spaces with Point Recovering

In this section we introduce effectively enumerable T_0–spaces with point recovering. Further on we will see that they play an important role in the description of images of surjective partial computable functions.

Definition 3. *Let $\mathcal{X} = (X,\tau,\alpha)$ be an effectively enumerable T_0–space. We say that $\mathcal{X}$ admits point recovering if $\{A_x \mid x \in X\}$ is a Σ^1_1-subset of $\mathcal{P}(\omega)$, where $A_x = \{n \mid x \in \beta(n)\}$. Here $\mathcal{P}(\omega)$ is considered as the Cantor space $\mathcal{C}$.*

Proposition 1. *Every computable Polish space $\mathcal{X} = (X,\tau,\alpha)$ admits point recovering. Moreover, $\{A_x \mid x \in X\}$ is a Π^0_2–subset of $\mathcal{C}$.*

Proof. Let $\mathcal{X} = (X, \tau, \alpha)$ be a computable Polish space. We prove that the set $\{A_x \mid x \in X\}$ is a Π^0_2-set in the effective Borel hierarchy on $\mathcal{C}$. For that let us show that, for $I \subseteq \omega$, $(\exists x \in X)\, I = A_x$ if and only if the following conditions hold.

Cond 1: $(\forall k \in \omega)(\exists m \in \omega)(\exists n \in \omega)(\exists r \in \mathbb{Q}^+)(\exists l \in \omega)(\exists r' \in \mathbb{Q}^+)\Big(k \in I \to$
$\big(\alpha^*(k) = (n, r) \wedge \alpha^*(m) = (l, r') \wedge r' < \dfrac{r}{2} \wedge m \prec_X k \wedge m \in I)\Big)$
where α^* is defined on the page 2.

Cond 2: $(\forall k \in \omega)(\forall m \in \omega)\Big(\big(k \in I \wedge m \in I\big) \to \alpha(k) \cap \alpha(m) \neq \emptyset\Big)$.

Cond 3: $I \neq \emptyset$.

Cond 4: $(\forall k \in \omega)(\forall m \in \omega)\Big(\big(k \in I \wedge k \prec_X m\big) \to m \in I\Big)$.

Let us denote $\Psi(I) =$ Cond $1(I) \wedge$Cond $2(I) \wedge$Cond $3(I) \wedge$Cond $4(I)$. By definition, Ψ is in Π^0_2-form.

If there exists $x \in X$ such that $I = A_x$ then $\Psi(A_x)$ holds by the definition of A_x. Assume now that, for $I \subseteq \omega$, $\Psi(I)$ holds. We are going to show that there exists $y \in X$ such that $\bigcap_{l \in I} \alpha(l) = \{y\}$. To construct y we start with some $k \in I$ since, by Cond 3, I is nonempty. Using Cond 1 we choose a chain of elements of I such that $k = k_1 \succ_X k_2 \succ_X \cdots \succ_X k_n \succ_X \ldots$ and, for all $s \in \omega$, $r_{k_{s+1}} < \frac{r_{k_s}}{2}$.

By the property of the relation $\succ_X$, $\alpha(k_s) \supset \overline{\alpha(k_{s+1})}$ for all $s \in \omega$. From Theorem 1 it follows that their intersection is one point set. Put $\bigcap_{s \in \omega} \alpha(k_s) = \{y\}$.

To show that $I \subseteq A_y$ assume $k \in I$ and $\alpha(k) = B(a, r)$. We check that $y \in B(a, r)$. Suppose contrary that $y \notin B(a, r)$, i.e., $d(y, a) > r$. Using Cond 1 we can find $m \in I$ such that $\alpha(m) = B(b, q)$, where $d(a, b) < r - q$ and $q < \frac{r}{2}$. It is clear that $\bar{B}(b, q) \subset B(a, r)$ therefore $y \notin \bar{B}(b, q)$. By Lemma 1, there exists $s \in \omega$ such that $\alpha(k_s) \cap B(b, q) = \emptyset$. This contradicts to Cond 2. Therefore $y \in B(a, r)$.

To show that $A_y \supseteq I$, assume $n \in A_y$. Then by Lemma 2 there exists $s \in \omega$ such that $k_s \prec_X n$. By the condition Cond 4, $n \in I$. Therefore, $\{A_x \mid x \in X\}$ is a Π^0_2-set in the effective Borel hierarchy on $\mathcal{C}$. □

Remark 1. *It is easy to see than Cond 1*(I) *– Cond 4*(I) *in Proposition 1 can be rewritten in the special form*

$$\forall \bar{k}\big(\eta(\bar{k}, I) \vee \Phi(\bar{k}, I)\big),$$

where η *is a disjunction of formulas of the kind* $k_i \notin I$ *and* Φ *is a computable disjunction (possible infinite) of* $\exists$*–formulas with positive occurrences of* I *i.e.* Φ *does not contain formulas of the kind* $k_i \notin I$*. Indeed, for example, Cond 4*(I) *can be rewritten as follows:*

$$(\forall k \in \omega)(\forall m \in \omega)\Big(k \notin I \vee m \in I \vee \neg k \prec_X m\Big).$$

Since $\neg k \prec_X m \risingdotseq (\forall l \in \omega) Q(m, k, l)$*, where* $Q(m, k, l)$ *defines computable subset of* ω^3*, we have*

$$\textit{Cond 4 } (I) \leftrightarrow (\forall k \in \omega)(\forall m \in \omega)(\forall l \in \omega)\Big(k \notin I \vee m \in I \vee Q(m, k, l)\Big).$$

Later we use this form in the proof of Theorem 3.

Proposition 2. *There exists an effectively enumerable topological space that does not admit point recovering.*

Proof. Let us consider $\mathcal{C}$ as a subset of $\mathbb{R}$ and take $Y \subseteq \mathcal{C}$ such that $\{A_x \mid x \in Y\}$ is non-analytic. It is possible to do this since the number of subsets $Y \subseteq \mathcal{C}$ with analytic $\{A_x \mid x \in Y\}$ is no more than continuum. Then put $X = \mathbb{R} \setminus Y$ and $\mathcal{X} = (X, \tau_X)$, where the topology τ_X is induced by $\tau_{\mathbb{R}}$.

It is clear that $\mathcal{X}$ is an effectively enumerable topological space since $\mathcal{C}$ is nowhere dense in $\mathbb{R}$. Taking into account that $\{A_x \mid x \in X\} = \{A_x \mid x \in \mathbb{R}\} \setminus \{A_x \mid x \in Y\}$ we conclude that $\mathcal{X}$ does not admit point recovering.

□

5. Partial Computable Functions over Effectively Enumerable Topological Spaces

In this section we introduce the notion of a partial computable function $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} = (X, \tau_X, \alpha)$ is an effectively enumerable topological space and $\mathcal{Y} = (Y, \tau_Y, \beta)$ is an effectively enumerable T_0–space.

Definition 4. *Let $\mathcal{X} = (X, \tau_X, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \tau_Y, \beta)$ be an effectively enumerable T_0–space. A function $f : \mathcal{X} \to \mathcal{Y}$ is called partial computable if the following properties hold. There exist a computable sequence of effectively open sets $\{O_n\}_{n\in\omega}$ and a computable function $H : \omega^2 \to \omega$ such that*

(1) $\mathrm{dom}(f) = \bigcap_{n\in\omega} O_n$ *and*
(2) $f^{-1}(\beta(m)) = \bigcup_{i\in\omega} \alpha(H(m,i)) \cap \mathrm{dom}(f)$.

In the following if a partial computable function f is everywhere defined we say f is a *total computable function.*

Proposition 3. *Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space.*

1. *If $f : \mathcal{X} \to \mathcal{Y}$ is a computable function, then f is continuous at every points of* $\mathrm{dom}(f)$.
2. *A total function $f : \mathcal{X} \to \mathcal{Y}$ is computable if and only if f is effectively continuous.*

Proof. The claims are straightforward form Definition 4. □

The following theorem shows that, for effectively enumerable T_0–spaces, the set of partial computable functions is closed under composition.

Theorem 2. *Let $\mathcal{X} = (X, \tau_X, \alpha)$, $\mathcal{Y} = (Y, \tau_Y, \beta)$ and $\mathcal{Z} = (Y, \tau_Z, \varrho)$ be effectively enumerable T_0–spaces. If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are partial computable functions then the composition $h = g \circ f$ is a partial computable function.*

Proof. In the following, we check (1) and (2) properties of partial computability.

(1).

$$\begin{aligned}\mathrm{dom}(h) &= \mathrm{dom}(f)\cap f^{-1}(\mathrm{dom}(g)) =\\ &= \mathrm{dom}(f)\cap f^{-1}(\bigcap_{m\in\omega} B_m) =\\ &= \mathrm{dom}(f)\cap \left(\bigcap_{m\in\omega} f^{-1}(B_m)\right) =\\ &= \mathrm{dom}(f)\cap \left(\bigcap_{m\in\omega} O_m\right),\end{aligned}$$

where $\{O_m\}_{m\in\omega}$ is a computable sequence of effectively open subsets of X and $\{B_m\}_{m\in\omega}$ is a computable sequence of effectively open subsets of Y. So h satisfies the property (1).
(2). It is easy to see that

$$h^{-1}(\varrho(n)) = f^{-1}(g^{-1}(\varrho(n))) = f^{-1}(\mathrm{dom}(g)\cap B_n) = \mathrm{dom}(h)\cap f^{-1}(B_n),$$

where B_n ia an effectively open subset of Y. Therefore h satisfies the property (2). □

Proposition 4. *Let $\mathcal{X} = (X, \tau_X, \alpha)$, $\mathcal{Y} = (Y, \tau_Y, \beta)$ be computable Polish spaces, $f : \mathcal{X} \to \mathcal{Y}$ be a partial computable function and B be a Π^0_2-subset of Y. Then the preimage $f^{-1}(B)$ is a Π^0_2-subset of X.*

Proof. Assume Y_0 is a Π^0_2–subset of $\mathcal{Y}$. By definition,

$$y \in Y_0 \leftrightarrow U(y),$$
$$\text{where } U = \bigcap_{n\in\omega} A_n \text{ and } A_n = \bigcup_{j\in T_n} \beta(j) \text{ for a c.e. set } T_n.$$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a partial computable function. Then

$$\begin{aligned}&x \in f^{-1}(B) \leftrightarrow\\ &x \in \mathrm{dom}(f) \wedge U(f(x)) \leftrightarrow\\ &x \in \mathrm{dom}(f) \wedge (\forall n \in \omega)(\exists j \in T_n)\Big(f(x) \in \beta(j)\Big).\end{aligned}$$

This is a Π^0_2-condition on $\mathcal{X}$ since, for $x \in \mathrm{dom}(f)$, $f(x) \in \beta(j) \leftrightarrow x \in f^{-1}(\beta(j)) \leftrightarrow x \in \bigcup_{i\in\omega} \alpha(H(j, i))$ for a computable function H. □

6. Partial Computable Functions over Computable Polish Spaces

In this section we consider the partial computable functions over the subclass of effectively enumerable topological spaces which is the class of computable Polish spaces. We give a characterisation of partial computability in terms of classical enumeration operators (see e.g. [14]). Then based on this characterisation we show the existence of a universal partial computable function for the partial computable functions $f : X \to Y$, where $\mathcal{X}$ and $\mathcal{Y}$ are computable Polish spaces.

6.1. **Characterisation.**

Definition 5. [14] *A function $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is called enumeration operator if*

$$\Gamma_e(A) = B \leftrightarrow B = \{j | \exists i\, c(i,j) \in W_e,\ D_i \subseteq A\},$$

where W_e is the e-th computably enumerable set, and D_i is the i-th finite set. A function $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is called reduced enumeration operator if

$$\Gamma_e(A) = B \leftrightarrow B = \{j | (\exists i \in A)\, c(i,j) \in W_e\},$$

where W_e is the e-th computably enumerable set.

Now we recall the notion of a computable function introduced in [9].

Definition 6. [9] *Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space.*
A partial function $f : \mathcal{X} \to \mathcal{Y}$ is called computable if there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for every $x \in X$,

(1) *If $x \in dom(f)$ then*

$$\Gamma_e(\{i \in \omega | x \in \alpha(i)\}) = \{j \in \omega \mid f(x) \in \beta(j)\}.$$

(2) *If $x \notin dom(f)$ then, for all $y \in Y$,*

$$\bigcap_{j\in\omega}\{\beta(j)|j \in \Gamma_e(A_x)\} \neq \bigcap_{j\in\omega}\{\beta(j)|j \in B_y\},$$

where $A_x = \{i \in \omega | x \in \alpha(i)\},\ B_y = \{j \in \omega | y \in \beta(j)\}$.

In this case we say that Γ_e completely defines the function f.

Remark 2. *It is easy to see that if we work with effectively enumerable topological spaces then a function is computable if and only if there exists a reduced enumeration operator satisfying the requirements of Definition 6.*

Proposition 5. *Let Γ_e be an enumeration operator and $\mathcal{X} = (X, \tau, \alpha)$ and $\mathcal{Y} = (Y, \lambda, \beta)$ be computable Polish spaces. Then $E = \{x \mid \Psi(\Gamma_e(A_x))\}$ is a Π^0_2-subset of X, where Ψ is a Π^0_2-condition from Proposition 1. Moreover, the function $f : \mathcal{X} \to \mathcal{Y}$ defined as follows: $\mathrm{dom}(f) = E$ and, for $x \in \mathrm{dom}(f)$, $f(x) = y \leftrightarrow \Gamma_e(A_x) = B_y$ is a partial computable function.*

Proof. Let us show that the condition $\Psi(\Gamma_e(A_x))$ defines a Π^0_2-subset of X. From Remark 1 it follows that $\Psi(I)$ is a conjunction of some Π^0_2-formulas in the form

$$\forall \bar{k}\big(\bigvee_{i\in D} k_i \notin I \vee \Phi(\bar{k}, I)\big),$$

where D is a finite subset of the indices of $\bar{k}$ and Φ is a computable disjunction of $\exists$–formulas with positive occurrences of I. Therefore, for $i = 1, \ldots, 4$ every $\mathrm{Cond}\, i(\Gamma_e(A_x))$ defines the set

$$\{x \mid \forall \bar{k}\, x \in A^i_{\bar{k}}\},\ \text{where}$$

$$A^i_{\bar{k}} = \{x \mid \bigvee_{j\in D} k_j \notin \Gamma_e(A_x) \vee \Phi_i(\bar{k}, \Gamma_e(A_x))\} =$$

$$\{x \mid \bigvee_{j\in D} k_j \notin \Gamma_e(A_x)\} \cup \{x \mid \Phi_i(\bar{k}, \Gamma_e(A_x))\}.$$

Let us make a close look at $A^i_{\bar{k}}$. The first element of the union is a Π^0_1-subset of $\mathcal{X}$ and the second one is a Σ^0_1-subset of $\mathcal{X}$ according to the following observation.

By the definition of the enumeration operator Γ_e,

$$\{x \mid k \in \Gamma_e(A_x)\} = \bigcup_{<k,j>\in W_e} \bigcap_{j\in D_k} \alpha(j).$$

Therefore it is effectively open and its complement is co-effectively closed. As a corollary every set $\{x \mid \mathrm{Cond}\, i(\Gamma_e(A_x))\}$ is a Π^0_2-subset of $\mathcal{X}$.

Let us show that f is a partial computable function. It is worth noting that $x \in \mathrm{dom}(f) \leftrightarrow \Gamma_e(A_x) \in \{B_y \mid y \in Y\} \leftrightarrow \Psi(\Gamma_e(A_x))$. So $\mathrm{dom}(f)$ is a Π^0_2-subset of X. For $x \in \mathrm{dom}(f)$,

$$x \in f^{-1}(\beta(j)) \leftrightarrow f(x) \in \beta(j) \leftrightarrow \exists k\, (k \in \{i | x \in \alpha(i)\} \wedge c(k,j) \in W_e) \leftrightarrow$$
$$\bigvee_{c(k,j)\in W_e} x \in \alpha(k) \leftrightarrow x \in \bigcup_{m\in\omega} \alpha(H(j,m))$$

for a computable function $H : \omega \times \omega \to \omega$. Therefore f is a partial computable function.

□

Further on if Γ_e and f satisfy the conditions of Proposition 5 we say that Γ_e *defines* f.

Theorem 3. *Let $\mathcal{X} = (X, \tau, \alpha)$ and $\mathcal{Y} = (Y, \lambda, \beta)$ be computable Polish spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is computable if and only if it is partial computable.*

Proof. $\to$) The claim follows from Proposition 5.

$\leftarrow$) Now suppose, $\mathrm{dom}(f) = \bigcap_{n\in\omega} O_n$ and, for $x \in \mathrm{dom}(f)$, $f(x) \in \beta(n) \leftrightarrow x \in \bigcup_{i\in\omega} \alpha(H(n,i))$, where $\{O_n\}_{n\in\omega}$ is a computable sequence of effectively open sets such that $O_{n+1} \subseteq O_n$ and $H : \omega^2 \to \omega$ is a computable function. It is worth noting that, for all $n \in \omega$ and $i \in \omega$, $O_n \cap \alpha(H(n,i))$ is an effectively open set. So, $O_n \cap \alpha(H(n,i)) = \bigcup_{t\in T_{ni}} \alpha(t)$, where $n \in \omega$, $i \in \omega$ and $\{T_{ni}\}_{n,\, i\in\omega}$ is a computable sequence of c.e. sets. Put

$$W_e = \{c(t,n) \mid (\exists\, i \in \omega)\, t \in T_{ni}\}.$$

Let Γ_e be a reduced enumeration operator that corresponds to W_e. By Proposition 5 this operator defines a function f_{Γ_e}. Let us show that $f = f_{\Gamma_e}$. We first prove that $\mathrm{dom}(f) = \mathrm{dom}(f_{\Gamma_e})$. If $x \in \bigcap_{n\in\omega} O_n$ then, by construction, $\Gamma_e(A_x) = B_{f(x)}$. Indeed, let $n \in \omega$ be such that $f(x) \in \beta(n)$. By definition, $(\exists i \in \omega)\, x \in \alpha(H(n,i))$. This means that $x \in \alpha(t)$ for $t \in T_{ni}$, therefore $n \in \Gamma_e(A_x)$. Conversely, if $n \in \Gamma_e(A_x)$ then $(\exists i \in \omega)(\exists t \in T_{ni})\, f(x) \in \beta(n)$. So $\Gamma_e(A_x) = B_{f(x)}$ and $x \in \mathrm{dom}(f_{\Gamma_e})$. So $\mathrm{dom}(f) \subseteq \mathrm{dom}(f_{\Gamma_e})$.

If $x \notin \bigcap_{n\in\omega} O_n$ then there exists $k \in \omega$ such that $x \notin O_n$ for all $n \geq k$. In other words, $x \notin \bigcup_{t\in T_{ni}} \alpha(t)$ for all $n \geq k$. This means that, for all $n \geq k$, $\neg(\exists\, t \in T_{ni})\, x \in \alpha(t)$, i.e., $n \notin \Gamma_e(A_x)$. Therefore $\Gamma_e(A_x)$ is finite and $B = \cap\{\beta(j) \mid j \in \Gamma_e(A_x)\}$ is a finite intersection of basic open balls. Since we consider spaces without isolated points, $B \neq \cap\{\beta(j) \mid j \in B_y\} = \{y\}$ for any $y \in Y$. In particular, $x \notin \mathrm{dom}(f_{\Gamma_e})$.

Now if $x \in \mathrm{dom}(f) = \mathrm{dom}(f_{\Gamma_e})$ then, by the definitions of f and Γ_e,

$$\Gamma_e(A_x) = \{j | \exists s\, H(j,s) \in A_x\} = \{j | x \in f^{-1}(\beta(j)\} = \{j | f(x) \in \beta(j)\}.$$

Therefore $B_{f(x)} = B_{f_{\Gamma_e}(x)}$. Since any point y is uniquely defined by the set of basic neighborhoods, $f(x) = f_{\Gamma_e}(x)$. So Γ_e completely defines f.

□

6.2. **Universal Partial Computable Function.** For computable Polish spaces $\mathcal{X}$ and $\mathcal{Y}$ we denote the set of partial computable functions $f : \mathcal{X} \to \mathcal{Y}$ as $\mathcal{PCF}_{\mathcal{XY}}$.

A partial computable function $F : \omega \times \mathcal{X} \to \mathcal{Y}$ is called *universal* for $\mathcal{PCF}_{\mathcal{XY}}$ if $\{F(n) : \mathcal{X} \to \mathcal{Y} \mid n \in \omega\} = \mathcal{PCF}_{\mathcal{XY}}$.

Theorem 4. *There exists a universal partial computable function for* $\mathcal{PCF}_{\mathcal{XY}}$.

Proof. The claim follows from Proposition 5 and Theorem 3. Indeed, for every partial computable function $f : \mathcal{X} \to \mathcal{Y}$ we can effectively construct an enumeration operator Γ_e which defines this function. Therefore e is one of the indices of f. Any enumeration operator Γ_e defines the function f_e. So, define $F(e, x) = y$ iff $f_e(x) = y$, where f_e is defined by Γ_e. From uniformity of the constructions in Proposition 5 and Theorem 3 it follows that F is a partial computable function. □

7. On Images of Partial computable Functions

7.1. **Images of Partial Computable Surjections.** In this section we propose a characterisation of effectively enumerable topological spaces that are images of partial computable surjections from computable Polish spaces.

Theorem 5. *Let $\mathcal{X} = (X, \tau, \alpha)$ be a computable Polish space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space. Then the following assertions are equivalent.*

(1) *There exists a partial computable surjection $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$.*
(2) *The space $\mathcal{Y}$ admits point recovering.*

Proof. 1) → 2). Assume $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is a partial computable surjection. It means that $\mathrm{dom}(f) = \bigcap_{n\in\omega} O_n = \bigcap_{n\in\omega} \bigcup_{s\in\omega} \alpha(g(n, s))$ and, for all $x \in \mathrm{dom}(f)$, $f(x) \in \beta(n) \leftrightarrow x \in \bigcup_{i\in\omega} \alpha(H(n, i))$, where $g : \omega^2 \to \omega$ and $H : \omega^2 \to \omega$ are computable functions. Recall that $A_x = \{n \mid x \in \alpha(n)\}$ and $B_y = \{m \mid y \in \beta(m)\}$. In order to show that $\mathcal{Y}$ admits recovering let us prove that $\{B_y \mid y \in Y\}$ is a Σ^1_1-subset of $\mathcal{P}(\omega)$ considered as the Cantor space. Since f is a surjection, for $I \subseteq \omega$, $(\exists y \in Y)\, I = B_y$ if and only if $(\exists x \in \mathrm{dom}(f))\, I = B_{f(x)}$. Let us make analysis. If $I = B_{f(x)}$ then

$$n \in I \leftrightarrow \quad (\exists x \in \mathrm{dom}(f)) f(x) \in \beta(n) \leftrightarrow (\exists x \in \mathrm{dom}(f))\, x \in \bigcup_{i\in\omega} \beta(H(n, i)) \leftrightarrow$$
$$(\exists x \in \mathrm{dom}(f))(\exists i \in \omega)\, H(n, i) \in A_x.$$

From Proposition 1 it follows that $J \in \{A_x \mid x \in X\} \leftrightarrow \Psi(J)$, where $\Psi(J)$ is a Π^0_2-subset of $\mathcal{C}$. It is easy to see that $x \in \mathrm{dom}(f) \leftrightarrow (\forall n \in \omega)\, x \in O_n \leftrightarrow (\forall n \in \omega)(\exists s \in \omega) g(n, s) \in A_x$. Finally, we have

$$(\exists y \in Y)\, I = B_y \leftrightarrow (\exists J \subseteq \omega) \quad \Big(\Psi(J) \wedge (\forall n \in \omega)\Big(n \in I \leftrightarrow \Big((\exists i \in \omega)\, H(n, i) \in J \wedge$$
$$(\forall m \in \omega)(\exists s \in \omega)\, g(n, s) \in J\Big)\Big)\Big).$$

Now we can see that $\{B_y \mid y \in Y\}$ is a Σ^1_1-subset of $\mathcal{C}$.

2) → 1). Let $\mathcal{Y}$ admit point recovering. We construct a required partial computable surjection in few steps:

$$\mathcal{X} \twoheadrightarrow \mathcal{N} \twoheadrightarrow \mathcal{C} \twoheadrightarrow \mathcal{C}^2 \twoheadrightarrow \mathcal{Y}.$$

Step 1. First we construct a homeomorphism $F : \mathcal{N} \to \mathcal{X}$ such that $F^{-1} : \mathcal{X} \to \mathcal{N}$ is a partial computable surjection (see e.g. [6]). It is worth noting that for any ball $B(x,r)$ one can choose two nonempty balls $\overline{B}(x_1, r_1)$ and $\overline{B}(x_2, r_2)$ which are inside of $B(x,r)$ and don't intersect each other. Continuing the process one can effectively generate infinitely many disjoint nonempty balls which are inside of $B(x,r)$ therefore one can produce a computable infinite sequence of such balls. Without loss of generality we could assume that, for all $n \in \omega$, $r_n < 1$. Now we construct F by stages.
Stage 1. Put $B(x^1_n, r^1_n) = B(x_n, r_n)$ and make the correspondence between each $\langle n\rangle \in \omega^{<\omega}$ and the ball $B(x^1_n, r^1_n)$.
Stage s+1. Assume that on the step s we already constructed balls $B(x^s_w, r^s_w)$, where $w \in \omega^{<\omega}$ and $\mathrm{length}(w) = s$, and made the correspondence between each w and the ball $B(x^s_w, r^s_w)$. Next we proceed as follows. Inside of every ball $B(x^s_w, r^s_w)$ we construct an effective sequence of disjoint nonempty balls $B(x^{s+1}_{w\sqcup\langle n\rangle}, r^{s+1}_{w\sqcup\langle n\rangle})$ such that, for all $n \in \omega$, $r^{s+1}_{w\sqcup\langle n\rangle} \leq \frac{1}{2} r^s_w$. Then we make the correspondence between each $w \sqcup \langle n\rangle$ and the ball $B(x^{s+1}_{w\sqcup\langle n\rangle}, r^{s+1}_{w\sqcup\langle n\rangle})$.

Finally, we define

$$\{F(v)\} = \bigcap_{s>0} \overline{B}(x^s_{v[s]}, r^s_{v[s]}).$$

By the Nested Sphere principle (see Theorem 1), F is correctly defined on $\mathcal{N}$ since $r^s_{v[s]} \to 0$. Moreover $\{F(v)\} = \bigcap_{s>0} B(x^s_{v[s]}, r^s_{v[s]})$ since, by construction, $B(x^s_{v[s]}, r^s_{v[s]}) \supset \overline{B}(x^{s+1}_{v[s]\sqcup\langle n\rangle}, r^{s+1}_{v[s]\sqcup\langle n\rangle})$. Using this strict inclusion we show that $\mathrm{im}(F)$ is a Π^0_2-set in the effective Borel hierarchy on $\mathcal{X}$.

Indeed, assume, for all $s \in \omega$, there exists a word w of the length s, $x \in B(x^s_w, r^s_w)$. By assumption,

$$x \in \bigcup_{w:\mathrm{length}(w)=s+1} B(x^{s+1}_w, x^{s+1}_w)$$

and, by construction, $B(x^s_u, r^s_u) \cap B(x^s_v, r^s_v) = \emptyset$ for any different words u and v of the length s. Therefore if $x \in B(x^s_v, r^s_v)$ then there exist n and a word $w = v \sqcup \langle n\rangle$ of the length $s+1$ such that $x \in B(x^{s+1}_w, r^{s+1}_w)$. So we construct a chain of words $w_1 \sqsubseteq w_2 \sqsubseteq \dots$ such that $v[s] = w_s$ for some $v \in \mathcal{N}$ and $x = F(v)$. Therefore $x \in \mathrm{im}(F)$ if and only if, for all $s \in \omega$, there exists a word w of the length s such that $x \in B(x^s_w, r^s_w)$. This is a Π^0_2-condition. Therefore $\mathrm{im}(F)$ is a Π^0_2-set in the effective Borel hierarchy on $\mathcal{X}$. By construction, $F(\mathfrak{A}_w) = B(x^s_w, r^s_w) \bigcap \mathrm{im}(F)$, where $s = \mathrm{length}(w)$. Since $\mathcal{X}$ is a computable Polish space, F^{-1} is a partial computable function.
Step 2. A partial computable surjection $g : \mathcal{N} \twoheadrightarrow \mathcal{C}$ is defined in the following standard way $g(f) = \lambda n.f(n) \,mod\, 2$.
Step 3. A partial computable bijection $\lambda : \mathcal{C}^2 \twoheadrightarrow \mathcal{C}$ is defined in the following standard way $\lambda(I,J) = \{2n \mid n \in I\} \cup \{2n+1 \mid n \in J\}$.
Step 4. Let us construct a partial computable surjection $h : \mathcal{C}^2 \twoheadrightarrow \mathcal{Y}$. Assume Θ is a Σ^1_1-condition that certifies point recovering of $\mathcal{Y}$ i.e. $I = B_y$ for some $y \in Y$ iff $\Theta(I) \leftrightharpoons (\exists J \subseteq \omega)\Phi(I,J)$, where $\Phi(I,J)$ is a Π^0_2-condition (see e.g. [12]). Put $D = \{(I,J) \mid \Phi(I,J)\} \subseteq \mathcal{C}^2$.

If $(I,J) \in D$ then $I = B_y$ for some $y \in Y$. Since Y is a T_0-space, this y is uniquely defined by I. Define $h(I,J) = y$. We have $(I,J) \in h^{-1}(\beta(n)) \leftrightarrow I = B_z$ for some $z \in \beta(n) \leftrightarrow \Phi(I,J) \wedge n \in I$. So, $\mathrm{dom}(h) = D$ is a Π^0_2-subset of $\mathcal{C}^2$

and $h^{-1}(\beta(n)) = D \cap (\{I \subseteq \omega \mid n \in I\} \times \mathcal{C})$. Therefore h is a partial computable surjection.
Step 5. A required partial computable surjection is the composition $f = h \circ \lambda^{-1} \circ g \circ F^{-1}$ i.e.

$$\mathcal{X} \overset{F^{-1}}{\twoheadrightarrow} \mathcal{N} \overset{g}{\twoheadrightarrow} \mathcal{C} \overset{\lambda^{-1}}{\twoheadrightarrow} \mathcal{C}^2 \overset{h}{\twoheadrightarrow} \mathcal{Y}.$$

□

7.2. **Complexity of Images of Partial Computable Functions.** In this section we characterise the images of the partial computable functions over computable Polish spaces in terms of the effective Lusin hierarchy.

Proposition 6. *Let $\mathcal{X}$ be a computable Polish space, $\mathcal{Y}$ be an effectively enumerable T_0-space and $Y_0 \subseteq Y$. Then the following assertions are equivalent.*

(1) *Y_0 is the image of a partial computable function $f : \mathcal{X} \to \mathcal{Y}$.*
(2) *$\{B_y \mid y \in Y_0\}$ is a Σ^1_1–subset of $\mathcal{C}$.*

Proof. 1) → 2). Assume Y_0 is the image of a partial computable function $f : \mathcal{X} \to \mathcal{Y}$. For $x \in \mathrm{dom}(f)$ and $y = f(x) \in Y_0$, we have

$$n \in B_{f(x)} \leftrightarrow f(x) \in \beta(n) \leftrightarrow x \in \bigcup_{i\in\omega} \alpha(H(n,i)) \leftrightarrow \{H(n,i) \mid i \in \omega\} \cap A_x \neq \emptyset.$$

Then, for $I \subseteq \omega$,

$$I \in \{B_y \mid y \in Y_0\} \leftrightarrow$$
$$\exists J \in \{A_x \mid x \in \mathrm{dom}(f)\}(\forall n \in \omega)\Big(n \in I \leftrightarrow \{H(n,i) \mid i \in \omega\} \cap A_x \neq \emptyset\Big).$$

This is a Σ^1_1-condition since

$$J \in \{A_x \mid x \in \mathrm{dom}(f)\} \leftrightarrow (\forall m \in \omega) J \cap J_m \neq \emptyset,$$

where $\{J_m\}_{m\in\omega}$ is a computable sequence of c.e. sets such that $\mathrm{dom}(f) = \bigcap_{m\in\omega} O_m$ and $O_m = \bigcup_{i\in J_m} \alpha(i)$.
2) → 1). Let $\{B_y \mid y \in Y_0\} \in \Sigma^1_1$. This means that $J \in \{B_y \mid y \in Y_0\} \leftrightarrow (\exists I \subseteq \omega)\, Q(I,J)$, where $Q(I,J)$ is a Π^0_2–condition on $\mathcal{C}$ (see e.g. [12]). Put $D = \{(I,J) \mid Q(I,J)\} \subseteq \mathcal{C}^2$. Let us construct a partial computable function $h : \mathcal{C}^2 \to \mathcal{Y}$ such that $\mathrm{dom}(h) = D$ and $\mathrm{im}(h) = Y_0$.

If $Q(I,J)$ then $J = B_y$ for some $y \in Y$. Since $\mathcal{Y}$ is a T_0-space, y is uniquely defined by I. Define $h(I,J) = y$. We have $(I,J) \in h^{-1}(\beta(n)) \leftrightarrow J = B_z$ for some $z \in \beta(n) \leftrightarrow Q(I,J) \wedge n \in J$. So, $\mathrm{dom}(h) = D$ is a Π^0_2-condition on $\mathcal{C}^2$ and $h^{-1}(\beta(n)) = D \cap (\{I \subseteq \omega \mid n \in I\} \times \mathcal{C})$. Therefore h is a partial computable function. Using Theorem 5 we construct the composition of partial computable surjections f, g and h as follows:

$$\mathcal{X} \overset{f}{\twoheadrightarrow} \mathcal{C} \overset{g}{\twoheadrightarrow} \mathcal{C}^2 \overset{h}{\to} \mathcal{Y}.$$

This is the required function. □

Proposition 7. *Let $\mathcal{Y}$ be a computable Polish space, $Y_0 \subseteq Y$ and $\widetilde{Y_0} = \{B_y \mid y \in Y_0\}$. Then Y_0 is a Σ^1_1–subset of $\mathcal{Y}$ if and only if $\widetilde{Y_0}$ is a Σ^1_1–subset of $\mathcal{C}$.*

Proof. $\rightarrow$). Assume Y_0 is a Σ^1_1–subset of $\mathcal{Y}$. By definition,

$$y \in Y_0 \leftrightarrow (\exists z \in Y)\, U(y, z),$$

where U is a Π^0_2-subset of Y. Let $F : \mathcal{C} \twoheadrightarrow \mathcal{Y}$ be a partial computable surjection. Then

$$J \in \{B_y \mid y \in Y_0\} \leftrightarrow$$
$$(\exists y \in Y)\Big(J = B_y \wedge (\exists K \in \mathrm{dom}(F))\, U(y, F(K))\Big) \leftrightarrow$$
$$(\exists I \in \mathrm{dom}(F))(\exists K \in \mathrm{dom}(F))\Big(U(F(I), F(K)) \wedge (\forall n \in \omega)\big(F(I) \in \beta(n) \leftrightarrow n \in J\big)\Big).$$

From Proposition 4 and the note that, for $z \in \mathrm{dom}(F)$, $F(z) \in \beta(j) \leftrightarrow z \in F^{-1}(\beta(j)) \leftrightarrow z \in \bigcup_{i\in\omega} \beta_{\mathcal{C}}(H(j, i))$ for a computable function H it follows that this is a Σ^1_1-condition on $\mathcal{C}$. So $\widetilde{Y_0}$ is a Σ^1_1-subset of $\mathcal{C}$.
$\leftarrow$). Let $\widetilde{Y_0}$ be a Σ^1_1-subset of $\mathcal{C}$. By definition, $J \in \widetilde{Y_0} \leftrightarrow (\exists I \in \mathcal{C})V(I, J)$. Let $G : \mathcal{Y} \twoheadrightarrow \mathcal{C}$ be a partial computable surjection.

Then,

$$y \in Y_0 \leftrightarrow$$
$$(\exists I \in \{B_y \mid y \in Y_0\})(\forall n \in \omega)\Big(y \in \beta(n) \leftrightarrow n \in I\Big) \leftrightarrow$$
$$(\exists z \in \mathrm{dom}(G))G(z) \in \{B_y \mid y \in Y_0\} \wedge (\forall n \in \omega)\Big(y \in \beta(n) \leftrightarrow n \in G(z)\Big) \leftrightarrow$$
$$(\exists z \in \mathrm{dom}(G))(\exists b \in \mathrm{dom}(G))\Big(V(G(b), G(z)) \wedge (\forall n \in \omega)\Big(y \in \beta(n) \leftrightarrow n \in G(z)\Big)\Big).$$

By analogy, Y_0 is a Σ^1_1-subset of Y.

□

Theorem 6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be computable Polish spaces and $Y_0 \subseteq Y$. Then the following assertions are equivalent.*

(1) *Y_0 is the image of a partial computable function $f : \mathcal{X} \to \mathcal{Y}$.*
(2) *Y_0 is a Σ^1_1–subset of Y.*

Proof. The claim follows from Proposition 6 and Proposition 7.

□

8. Conclusions and Future Work

We presented several results in the framework of the effective descriptive set theory (EDST) on computable Polish spaces. Informally, for $\mathcal{PCF}_{\mathcal{XY}}$ we showed the following:

- the existence of a universal partial computable function;
- the existence of a partial computable surjection between any computable Polish space and any effectively enumerable topological space with point recovering;
- the descriptive complexity of images of partial computable functions between computable Polish spaces.

These results give a rise on new research directions:

- Investigations of bounds on the descriptive complexity of the images of total computable functions over computable Polish spaces. We make a conjecture that bounds will be different for particular classes of computable Polish spaces. For example, it is easy to see that for the total computable real functions, the images range over intervals of special kind.
- Characterisations of complexity of index sets for important problems on $\mathcal{PCF}_{\mathcal{XY}}$. In the previous papers [8, 6] we already did few steps in this direction. For the real-valued partial computable functions $\mathcal{PCF}_{\mathcal{X}\mathbb{R}}$ defined on the computable Polish space $\mathcal{X}$ we characterised the complexity of important problems such as totality and root verification. We also showed that for some problems the corresponding complexity does not depend on the choice of a computable Polish space while for other ones the corresponding choice plays a crucial role. It will be challenging to get similar results for the general class $\mathcal{PCF}_{\mathcal{XY}}$.
- Generalisations of EDST on computable Polish spaces to EDST on the wider class of effective topological spaces. One of the promising candidates could be effectively enumerable topological spaces with point recovering.

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