S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 14, стр. 473–492 (2017) УДК 517.958, 532.5
DOI 10.17377/semi.2017.14.039 MSC 35A21, 35A22, 35D30, 76L05

LINEAR PROBLEM OF SHOCK WAVE DISTURBANCE ANALYSIS. PART 2: REFRACTION AND REFLECTION OF PLANE WAVES IN THE STABILITY CASE

E. V. SEMENKO, T. I. SEMENKO

Abstract. This part is devoted to the propagation of plane waves in the stability case.

First the fact that each post-shock plane wave is accompanied with damped wave, and so the plane waves refraction/reflection quantity characteristics are determined up to the damped waves, is established.

The correspondence between angles of incidence and angles of refraction/reflection, i.e. Snell's laws, is obtained.

The matrix of generation coefficients as a whole is calculated. Its behaviour for an ideal gas when the pre-shock Mach number tends to infinity, i.e. coefficients' amplification, is investigated. The degree of amplification for different kinds of incident waves is found. Furthermore, numerical calculations of generation coefficients for an ideal gas are performed, in particular, the coefficients' amplification is investigated numerically and the results are found to confirm analytical conclusions.

For reflection, all four reflection coefficients are calculated and some of their properties are established. In particular, vanishing of the reflected entropy-vorticity plane waves and mutual suppression of the incident and reflected acoustic plane waves at critical angles of incidence are established. The numerical calculations of reflection coefficients are also performed.

A comparison is carried out between the obtained results and the already-known ones. It is found that the known formulas for the refraction/reflection angles and for the generation/reflection coefficients

Semenko, E.V., Semenko, T.I., Linear problem of shock wave disturbance analysis. Part 2: refraction and reflection of plane waves.

Received December, 13, 2016, published May, 23, 2017.

should be specified and corrected. In particular, the existence of so-called abnormal amplification is disproved.

Keywords: Shock wave, shock disturbance, entropy-vorticity wave, acoustic wave, incident wave, refraction, transmitted wave, reflection, reflected wave, stability, neutral stability, spontaneous emission, Fourier transform.

1. Introduction

This part is organized as follows. In Section 2, the form of plane waves is investigated, in particular the fact, that any post-shock plane wave is accompanied with a damped wave, is established. In having technical nature Section 3, the transition to the dimensionless parameters and variables is performed. In short Section 4, the form of transmitted and reflected waves in the stability case is indicated. Section 5 is devoted to the refraction: in subsection 5.1, the general form of incident and refracted waves is considered; in subsection 5.2, we establish the correspondence between angles of incidence and refraction, i.e. Snell's laws; in subsection 5.3, we deduce the matrix of generation coefficients on the whole; at last in subsection 5.4, we consider the generation coefficients for an ideal gas, in particular amplification, and give the results of numerical calculations. In Section 6, the reflection is considered.

2. Pre-shock and post-shock plane waves

2.1. **General form of plane waves.** As the sought quantity in our case is the vector G, then the plane wave has the form $G = G^{\pm} = H_0^{\pm} e^{i(\xi_0 x + \eta_0 y - \omega_0 t)}$, where $\xi_0, \eta_0, \omega_0 = \text{const}$ and $H_0^{\pm}$ is a constant vector. Now we recall some well-known (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) or obvious information about plane waves.

The system of differential equations $BG = 0$ (Part 1, Eq. (1)) for plane waves has the form

$$\hat{B}(\xi_0, \eta_0, \omega_0) H_0 e^{i(\xi_0 x + \eta_0 y - \omega_0 t)} = 0\,,$$

i.e. H_0 is the eigenvector of the matrix $\hat{B}(\xi_0, \eta_0, \omega_0)$ with a zero eigenvalue. Due to the complete basis $\{\, e_{0j},\ j = \overline{1,4}\}$ of the eigenvectors of matrix $\hat{B}$ and their eigenvalues $\lambda_{1,2} = Q \pm c\sqrt{\xi^2 + \eta^2}$, $\lambda_{3,4} = Q$ (Part 1, Section 3), there are four solutions of the equation $BG = 0$ in the form of plane waves for any wave vector $\boldsymbol{k} = (\xi_0, \eta_0)$:

$$G_j = \beta_j e_{0j}(\xi_0, \eta_0) e^{i(\xi_0 x + \eta_0 y - \omega_j(\xi_0,\, \eta_0) t)}\,, \quad \beta_j = \text{const}\,, \quad j = \overline{1,4}\,,$$

$\omega_1 = \omega_-$, $\omega_2 = \omega_+$. Here $G_{1,2}$ are the acoustic plane waves, G_3 is a vorticity plane wave, and G_4 is an entropy plane wave.

The group velocity of entropy-vorticity waves is $v_{3,4} = U^0 = (u_x, u_y)$, and the phase velocity is $(U^0, \boldsymbol{k})/|\boldsymbol{k}|$. The group velocities of acoustic waves are $v_1 = U^0 - c\boldsymbol{k}/|\boldsymbol{k}|$, $v_2 = U^0 + c\boldsymbol{k}/|\boldsymbol{k}|$, and the phase velocities are $(U^0, \boldsymbol{k})/|\boldsymbol{k}| \mp c$ (and that is why, G_1 is called a slow wave and G_2 is a fast wave).

2.2. **Pre-shock plane waves.** In the pre-shock zone (superscript plus, $u_x > c$), all group velocities of plane waves are directed toward the shock, i.e. formally speaking, all plane waves are incident upon the shock (and it is actually so, as we saw in Part 1). Moreover, due to the properties of the pre-shock waves (Part 1) we conclude: any pre-shock initial data in form of plane wave $G_0 = \beta e_{0j}(\xi_0, \eta_0) e^{i(\xi_0 x + \eta_0 y)}$, $j = 1$ or 2 or 3 or 4, generates pre-shock plane wave $G = \beta e_{0j}(\xi_0, \eta_0) e^{i(\xi_0 x + \eta_0 y - \omega_j t)}$,

$\omega_j = \omega_j(\xi_0, \eta_0)$ with "plane" boundary value $G_\Gamma = \beta e_j(\eta_0, \omega_j) e^{i(\eta_0 y - \omega_j t)}$. At that for any boundary wavenumbers (η_0, ω_0), there exist four plane waves, cooperating during their incidence upon the shock, i.e. with the same (η_0, ω_0). Common form of them is

$$G = \sum_{j=1}^{4} \beta_j e_j(\eta_0, \omega_0) e^{i(\xi_j(\eta_0,\omega_0)x + \eta_0 y - \omega_0 t)}\,, \quad \beta_j = \text{const}\,,$$

the wave vectors of summands are

$$\boldsymbol{k}_j = (\xi_j(\eta_0, \omega_0), \eta_0)\,, \quad j = \overline{1,4}\,,$$

and common form of a boundary value is

$$G_\Gamma = \sum_{j=1}^{4} \beta_j e_j(\eta_0, \omega_0) e^{i(\eta_0 y - \omega_0 t)}\,.$$

Further we treat pre-shock boundary value G_Γ^+ as given.

2.3. **Post-shock plane waves. Damped waves.** Let's consider post-shock zone: superscript minus on default, $u_x < c$. Here the entropy-vorticity wave has a form

$$G = (\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4) e^{i(\xi_0 x + \eta_0 y - \omega_0 t)}\,, \quad \omega_0 = \omega_3(\xi_0, \eta_0)\,, \quad \beta_{3,4} = \text{const}\,.$$

Its group velocity is directed away from the shock, i.e. formally, the entropy-vorticity wave isn't incident upon the shock (and it is so in fact, Part 1, Subsection 5.4). The initial entropy-vorticity data are

$$G_0 = (\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4) e^{i(\xi_0 x + \eta_0 y)}$$

and the boundary value is

$$G_\Gamma = (\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4) e^{i(\eta_0 y - \omega_0 t)}$$

or in spectral variables (Appendix, Part 3, Subsection 6.1)

$$\begin{aligned} \hat{G}_0 &= (\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4)\delta(\eta - \eta_0) I_1(\xi_0 - \xi)\,, \\ \hat{G}_\Gamma &= (\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4)\delta(\eta - \eta_0) I_1(\omega - \omega_0)\,. \end{aligned} \tag{1}$$

Let us consider initial wave with the initial data $\hat{G}_0$ (1). According to Part 1, Subsection 5.1, Eq. (5.1), it has a form

$$\hat{G}_{00} = \frac{\hat{G}_0}{2\pi i Q} = \frac{(\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4)}{2\pi i Q} \delta(\eta - \eta_0) I_1(\xi_0 - \xi)\,.$$

Separating singular and regular terms (Part 1, Subsection 4.2), we get

$$\hat{G}_{00} = \frac{(\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4)}{2\pi i Q(\xi_0, \eta_0, \omega)} \delta(\eta - \eta_0) I_1(\xi_0 - \xi) + \hat{\varphi}\,,$$

where $\hat{\varphi}$ is regular with respect to ξ, ω, or after inverse Fourier transform

$$G_{00} = e^{i(\xi_0 x + \eta_0 y - \omega_0 t)} \theta(x - u_x t)(\beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4) + \varphi\,, \quad \omega_0 = \omega_3(\xi_0, \eta_0)\,,$$

where $\theta(z)$ is a Heaviside function, and $\varphi \to 0$ when $x \to \infty$ or $t \to \infty$, i.e. φ is damped wave. So initial entropy-vorticity wave cannot be a plane wave, it has a plane principal part at $x > u_x t$, is accompanied by damped wave and is a damped wave in the large.

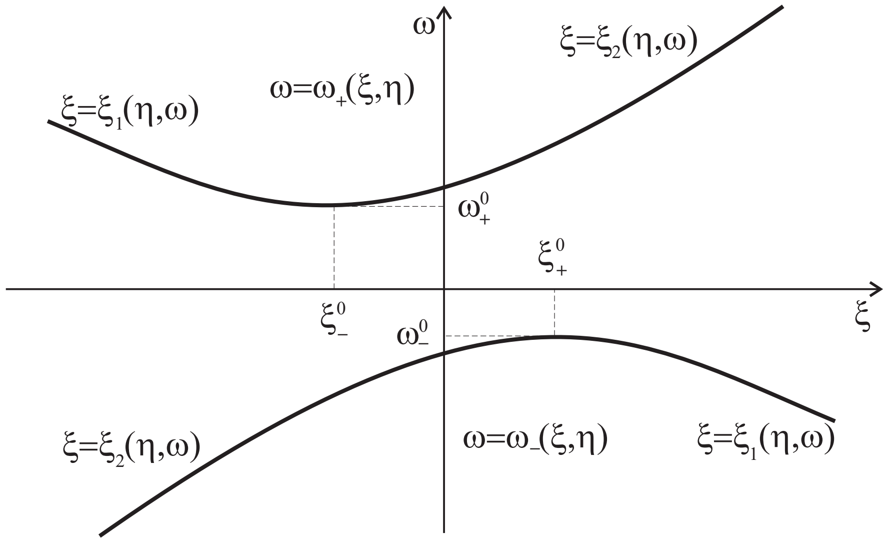

Fig. 1: Section of post-shock characteristic cone by plane $\eta = \text{const}$.

Absolutely similar, the boundary (transmitted or reflected) wave, generated by the plane entropy-vorticity boundary value $\hat{G}_\Gamma$ (1), is accompanied by damped wave, and its principal part has a form

$$G = e^{i(\xi_0 x + \eta_0 y - \omega_0 t)} \theta(u_x t - x) H\,, \quad H = \beta_3 e_{03}(\xi_0, \eta_0) + \beta_4 e_4\,, \quad \xi_0 = \xi_3(\eta_0, \omega_0)\,,$$

i.e. it is a plane wave at $x < u_x t$, with the same wave amplitude H as its boundary value.

The situation with the acoustic plane wave is somewhat more complicated. It has a form

$$G = \beta_1 e_{01}(\xi_0, \eta_0) e^{i(\xi_0 x + \eta_0 y - \omega_- t)} + \beta_2 e_{02}(\xi_0, \eta_0) e^{i(\xi_0 x + \eta_0 y - \omega_+ t)}\,,$$

$$\omega_\pm = \omega_\pm(\xi_0, \eta_0)\,, \quad \beta_{1,2} = \text{const}\,.$$

We recall the form of curves $\omega = \omega_\pm(\xi, \eta)$ at fixed η, these curves are depicted in fig.1.

Here the vector of group velocity $v = \nabla_{\boldsymbol{k}} \omega$ is directed away from the shock/toward the shock, when tangent vector to curve is directed up/down with respect to ω-axis. So the group velocity of slow acoustic wave $G = \beta e_{01} e^{i(\xi_0 x + \eta_0 y - \omega_- t)}$ is directed away from the shock at $\xi_0 < \xi_+^0$, is directed toward the shock at $\xi_0 > \xi_+^0$, and is parallel to the shock at $\xi_0 = \xi_+^0$. Similarly, the group velocity of fast acoustic wave $G = \beta e_{02} e^{i(\xi_0 x + \eta_0 y - \omega_+ t)}$ is directed toward the shock at $\xi_0 < \xi_-^0$, is directed away from the shock at $\xi_0 > \xi_-^0$, and is parallel to the shock at $\xi_0 = \xi_-^0$. Finally, due to the definition of roots $\xi_{1,2}(\eta, \omega)$ (Part 1, Section 3), we have: at given wavenumbers (ξ_0, η_0), the acoustic plane wave has the form $G_a = G_1 + G_2$, where group velocity of G_1 is directed towards the shock, group velocity of G_2 is directed away from the shock and these waves have a form

$$(2)\qquad G_1 = \begin{cases} \beta_2 e_1(\eta_0,\omega_+)e^{i(\xi_1(\eta_0,\omega_+)x+\eta_0 y-\omega_+ t)}, & \xi \leqslant \xi_-^0, \\ 0, & \xi \in (\xi_-^0,\xi_+^0), \\ \beta_1 e_1(\eta_0,\omega_-)e^{i(\xi_1(\eta_0,\omega_-)x+\eta_0 y-\omega_- t)}, & \xi \geqslant \xi_+^0, \end{cases}$$

$$G_2 = \begin{cases} \beta_1 e_2(\eta_0,\omega_-)e^{i(\xi_2(\eta_0,\omega_-)x+\eta_0 y-\omega_- t)}, & \xi \leqslant \xi_-^0, \\ \beta_1 e_2(\eta_0,\omega_-)e^{i(\xi_2(\eta_0,\,\omega_-)x+\eta_0 y-\omega_- t)} + \beta_2 e_2(\eta_0,\omega_+)e^{i(\xi_2(\eta_0,\,\omega_+)x+\eta_0 y-\omega_+ t)}, & \\ & \xi \in (\xi_-^0,\xi_+^0), \\ \beta_2 e_2(\eta_0,\omega_+)e^{i(\xi_2(\eta_0,\omega_+)x+\eta_0 y-\omega_+ t)}, & \xi \geqslant \xi_+^0. \end{cases}$$

At $\xi_0 = \xi_\pm^0(\eta_0)$ the group velocity of G_1 is parallel to the shock, and at $\xi_0 \in (\xi_-^0, \xi_+^0)$ there are no acoustic plane waves directed towards the shock. Nevertheless in fact, as we know (Part 1), any acoustic wave, and G_2 as well, is incident upon the shock. The reason of paradox is that in this cases the agent of incidence is not plane wave, it is damped wave, see below.

The initial value of plane acoustic wave is

$$G_0 = (\beta_1 e_{01}(\xi_0,\eta_0) + \beta_2 e_{02}(\xi_0,\eta_0))e^{i(\xi_0 x+\eta_0 y)},$$

or in spectral variables $\hat{G}_0 = (\beta_1 e_{01}(\xi_0,\eta_0) + \beta_2 e_{02}(\xi_0,\eta_0))\delta(\eta-\eta_0)I_1(\xi_0-\xi)$. Similarly to above, it may be shown, that initial acoustic wave has a form $G_{00} = G_1 + \varphi$, where φ is damped wave, and G_1 has a form (2), i.e. it is just a summand with group velocity, directed towards the shock.

In turn, by virtue of change of variable formula (Appendix, Part 3, Subsection 6.3, Eq. (6.12)), and formula $(\xi_1)'_\omega = 1/s_1$ (Part 1, section 3, Eq. (3.2)), the agent of incidence $\Phi(\eta,\omega) = g_1(\eta,\omega)\hat{G}_0(\xi_1(\eta,\omega),\eta)$ has a form $\Phi = \Phi_1 + \hat{\varphi}$ where $\hat{\varphi}$ is regular with respect to ω,

$$\Phi_1 = \begin{cases} -\beta_2 s_1(\eta_0,\omega_+)\delta(\eta-\eta_0)I_1(\omega-\omega_+), & \xi \leqslant \xi_-^0, \\ 0, & \xi \in (\xi_-^0,\xi_+^0), \\ -\beta_1 s_1(\eta_0,\omega_-)\delta(\eta-\eta_0)I_1(\omega-\omega_-), & \xi \geqslant \xi_+^0. \end{cases}$$

In terms of boundary wavenumbers (η_0,ω_0), we have

$$(3)\qquad G_1 = \begin{cases} \beta e_1(\eta_0,\omega_0)e^{i(\xi_1(\eta_0,\omega_0)x+\eta_0 y-\omega_0 t)}, & \omega_0 \notin (\omega_-^0,\omega_+^0), \\ 0, & \omega_0 \in (\omega_-^0,\omega_+^0), \end{cases}$$

$$(4)\qquad \Phi_1 = \begin{cases} -\beta s_1(\eta_0,\omega_0)\delta(\eta-\eta_0)I_1(\omega-\omega_0), & \omega_0 \notin (\omega_-^0,\omega_+^0), \\ 0, & \omega_0 \in (\omega_-^0,\omega_+^0), \end{cases}$$

$\omega_\pm^0 = \omega_\pm^0(\eta_0)$, $\beta = \text{const}$. So at $\omega_0 \in (\omega_-^0, \omega_+^0)$, the agent of incidence Φ is damped wave (although $\Phi \not\equiv 0$ for any initial acoustic wave, see Part 1, Subsection 5.4).

Concerning the boundary post-shock wave (transmitted or reflected), generated by the plane boundary value $G_\Gamma = \beta e_2(\eta_0,\omega_0)e^{i(\eta_0 y-\omega_0 t)}$, it has (due to Part 1, Eq. (5.2)) a form $G = G_2 + \varphi$, where φ is damped wave and

$$(5)\qquad G_2 = \beta e_2(\eta_0,\omega_0)e^{i(\xi_2(\eta_0,\omega_0)x+\eta_0 y-\omega_0 t)}.$$

General conclusion reads: both post-shock initial and boundary (transmitted or reflected) waves cannot have a form of plane wave, they are plane waves up to damped wave, i.e., up to vanishing at $t \to \infty$ agent. This "asymptotic" plane waves we call principal parts of waves. The principal parts of initial waves, generated by the initial plane entropy-vorticity data, are zero. The principal part of initial wave, generated by the initial plane acoustic data, is just a plane acoustic wave G_1 (3), with group velocity directed towards the shock; the principal part of agent of incidence Φ has a form Φ_1 (4). In particular, if the initial data generate a wave with group velocity, directed away from the shock, then the initial acoustic wave as well as the agent of incidence Φ are damped waves. At the same time, the principal part of boundary (transmitted or reflected) acoustic wave G_2 has a form (5), its group velocity is directed away from the shock. At critical values $\omega_0 = \omega^0_\pm(\eta_0)$, waves G_1 and G_2 coalesce and their group velocity is parallel to the shock. Finally, principal parts of transmitted and reflected waves, generated by the plane boundary value $G_\Gamma = H_j e^{i(\eta_0 y - \omega_0 t)}$, $H_j = \beta_j e_j(\eta_0, \omega_0)$, has a form $G = H_j e^{i(\xi_j x + \eta_0 y - \omega_0 t)}$, $\xi_j = \xi_j(\eta_0, \omega_0)$, $j = \overline{2,4}$, i.e. they are plane post-shock waves with the same wave amplitude H_j, as boundary value.

All investigations of refraction/reflection of plane waves are based on the comparison of wave vectors' directions and wave amplitudes of incident and transmitted/reflected waves. So all results we obtain for principal parts only, up to damped summands.

3. Transition to dimensionless parameters and variables

To describe basic solution, we introduce the following dimensionless parameters:

$$M^\pm = \frac{u_x^\pm}{c^\pm} \text{ (Mach numbers)}\,, \quad R = \frac{c^-}{c^+}\,, \quad D_0^\pm = \frac{rc^2}{\rho^0 T}\,, \quad B_0^\pm = \frac{c^2\rho^0}{p^0}$$

with relations

$$M^+ > 1\,, \quad M^- < 1\,, \quad R > 0\,, \quad D_0^\pm < 0\,, \quad B_0^\pm > 0\,.$$

Instead of ω, we introduce dimensionless variable

$$\alpha = \frac{\omega - \eta u_y}{|\eta|\sqrt{(c^-)^2 - (u_x^-)^2}}\,.$$

Further we denote $s = \operatorname{sgn}(\eta) = \eta/|\eta|$ and M^- just as M. Now we rewrite formulas of Part 1 in new terms.

Roots and connecting quantities. $\xi_j^\pm(\eta,\omega) = |\eta|\xi_j^\pm(\alpha)$, $j = \overline{1,4}$, where

$$\xi_1^+(\alpha) = \frac{M^+\sqrt{1-M^2}R\alpha + \sqrt{(1-M^2)R^2\alpha^2 + (M^+)^2 - 1}}{(M^+)^2 - 1}\,,$$

$$\xi_2^+(\alpha) = \frac{M^+\sqrt{1-M^2}R\alpha - \sqrt{(1-M^2)R^2\alpha^2 + (M^+)^2 - 1}}{(M^+)^2 - 1}\,,$$

$$\xi_{3,4}^+(\alpha) = \xi_0^+(\alpha) = \frac{\sqrt{1-M^2}\,R\alpha}{M^+}\,;$$

$$\xi_1^-(\alpha) = -\frac{M\alpha + \sqrt{\alpha^2-1}}{\sqrt{1-M^2}}\,, \quad \xi_2^-(\alpha) = -\frac{M\alpha - \sqrt{\alpha^2-1}}{\sqrt{1-M^2}}\,,$$

$$\xi_{3,4}^-(\alpha) = \xi_0^-(\alpha) = \frac{\alpha\sqrt{1-M^2}}{M}\,;$$

$Q_j^{\pm}(\eta,\omega) = c^{\pm}|\eta|Q_j^{\pm}(\alpha)$, $j=\overline{1,2}$, where

$$Q_1^+(\alpha) = \frac{\sqrt{1-M^2}R\alpha + M^+\sqrt{(1-M^2)R^2\alpha^2+(M^+)^2-1}}{R((M^+)^2-1)}\,,$$

$$Q_2^+(\alpha) = \frac{\sqrt{1-M^2}R\alpha - M^+\sqrt{(1-M^2)R^2\alpha^2+(M^+)^2-1}}{R((M^+)^2-1)}\,,$$

$$Q_1^-(\alpha) = -\frac{\alpha+M\sqrt{\alpha^2-1}}{\sqrt{1-M^2}}\,,\quad Q_2^-(\alpha) = -\frac{\alpha-M\sqrt{\alpha^2-1}}{\sqrt{1-M^2}}\,.$$

Critical values

$$\omega=\omega_{\pm}^0(\eta) \Longleftrightarrow \alpha=\pm1\,;\quad \xi_{\pm}^0(\eta) = \mp|\eta|\frac{M}{\sqrt{1-M^2}}\,.$$

Here the function $\sqrt{\alpha^2-1}$ is analytic in upper half-plane $\operatorname{Im}\alpha>0$ and its branch is chosen under the condition $\sqrt{(i\beta)^2-1} = i\sqrt{\beta^2+1}$. So at real α and $|\alpha|\geqslant1$ in fact $\sqrt{\alpha^2-1} = \operatorname{sgn}(\alpha)\sqrt{\alpha^2-1}$.

Boundary bases (superscript $\pm$ on default).

$$e_1(\eta,\omega) = e_1(\alpha,s) = \begin{pmatrix} 1 \\ -\xi_1(\alpha)/Q_1(\alpha) \\ -s/Q_1(\alpha) \\ 0 \end{pmatrix},$$

$$e_2(\eta,\omega) = e_2(\alpha,s) = \begin{pmatrix} 1 \\ -\xi_2(\alpha)/Q_2(\alpha) \\ -s/Q_2(\alpha) \\ 0 \end{pmatrix},$$

$$e_3(\eta,\omega) = e_3(\alpha,s) = \frac{1}{\sqrt{(\xi_0(\alpha))^2+1}}\begin{pmatrix} 0 \\ s \\ -\xi_0(\alpha) \\ 0 \end{pmatrix},\quad e_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

$$g_1(\eta,\omega) = g_1(\alpha,s) = \frac{1}{2}\left(1\,;\,-\xi_1(\alpha)/Q_1(\alpha)\,;\,-s/Q_1(\alpha);\,-1\right),$$

$$g_2(\eta,\omega) = g_2(\alpha,s) = \frac{1}{2}\left(1\,;\,-\xi_2(\alpha)/Q_2(\alpha)\,;\,-s/Q_2(\alpha)\,;\,-1\right),$$

$$g_3(\eta,\omega) = g_3(\alpha,s) = \frac{1}{\sqrt{(\xi_0(\alpha))^2+1}}\left(0\,;\,s\,;\,-\xi_0(\alpha)\,;\,0\right),\ g_4 = g_{04} = (0\,;\,0\,;\,0\,;\,1);$$

$s_j(\eta,\omega) = cs_j(\alpha)$, $j=\overline{1,4}$, where $s_j^{\pm}(\alpha) = M^{\pm}$, $j=\overline{3,4}$,

$$s_1^+(\alpha) = \frac{\sqrt{(1-M^2)R^2\alpha^2+(M^+)^2-1}}{RQ_1^+(\alpha)},\ s_2^+(\alpha) = -\frac{\sqrt{(1-M^2)R^2\alpha^2+(M^+)^2-1}}{RQ_2^+(\alpha)},$$

$$s_1^-(\alpha) = \frac{\sqrt{1-M^2}\,\sqrt{\alpha^2-1}}{Q_1^-(\alpha)},\quad s_2^-(\alpha) = -\frac{\sqrt{1-M^2}\,\sqrt{\alpha^2-1}}{Q_2^-(\alpha)}.$$

Boundary matrices and different functions.

For matrices A_0, A, we mean superscript $\pm$.

$$A_0 = cA_{01} = c\begin{pmatrix} M & 1 & 0 & 0 \\ 1 & M & 0 & -1 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{pmatrix}$$

$$A = A_1 \cdot A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ M & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & M & 0 & 1/D_0 \end{pmatrix}.$$

For functions $\hat{F}_0$ and Y_j, $j = \overline{1,4}$, we mean $\rho = \rho^-$, $c = c^-$, $M = M^-$.

$$\hat{F}_0(\eta,\omega) = |\eta|\rho c \left(\frac{M^+}{R} - M\right) A_1 \hat{F}_0(\alpha, s)$$

$$= |\eta|\rho c \left(\frac{M^+}{R} - M\right) A_1 \begin{pmatrix} R\sqrt{1-M^2}\alpha/M^+ \\ 0 \\ -Ms \\ -M\sqrt{1-M^2}\,\alpha \end{pmatrix};$$

$$Y_j(\eta,\omega) = |\eta|\rho c \left(\frac{M^+}{R} - M\right) Y_j(\alpha, s),$$

$$Y_j(\alpha, s) = g_j^-(\alpha, s) A_2^{-1} \hat{F}_0(\alpha, s), \quad j = \overline{1,4}.$$

By means of direct calculations, we find

$$Y_1(\alpha, s) = \frac{Y_{11}^0 \alpha^2 + Y_{12}^0 \alpha\sqrt{\alpha^2 - 1} + Y_{13}^0}{2Q_1^-(\alpha)} = Y_1(\alpha),$$

$$Y_2(\alpha, s) = \frac{Y_{11}^0 \alpha^2 - Y_{12}^0 \alpha\sqrt{\alpha^2 - 1} + Y_{13}^0}{2Q_2^-(\alpha)} = Y_2(\alpha),$$

where

$$Y_{11}^0 = \frac{R}{M^+}\left(MD_0\left(M - \frac{M^+}{R}\right) - 1 - M^2\right),$$

(6)

$$Y_{12}^0 = \frac{RM}{M^+}\left(MD_0\left(M - \frac{M^+}{R}\right) - 2\right), \quad Y_{13}^0 = M;$$

$$Y_3(\alpha, s) = \frac{\alpha s\sqrt{1-M^2}(1 - MR/M^+)}{\sqrt{(\xi_0^-(\alpha))^2 + 1}} = sY_3(\alpha),$$

$$Y_4(\alpha, s) = MD_0(MR/M^+ - 1)\sqrt{1-M^2}\,\alpha = Y_4(\alpha).$$

4. Common form of refracted and reflected waves in the stability case

Since $Y_1(\eta,\omega) \neq 0$, then the formulas for generated by pre-shock wave shock disturbance and boundary values of transmitted waves (Part 1, subsection 5.3, Eq. (5.5)) assume the form

$$\hat{f}^+ = -i\frac{g_1^-(\eta,\omega)}{Y_1(\eta,\omega)}(A^-)^{-1}A^+A_0^+\hat{G}_\Gamma^+(\eta,\omega), \tag{7}$$

$$\hat{G}_\Gamma^- = (A_0^-)^{-1}\left[E - (A^-)^{-1}\hat{F}_0\frac{g_1^-(\eta,\omega)}{Y_1(\eta,\omega)}\right](A^-)^{-1}A^+A_0^+\hat{G}_\Gamma^+(\eta,\omega).$$

The decomposition of transmitted wave into the sum of acoustic $\hat{G}^-_{2\Gamma} = \gamma_2 e^-_2$, vorticity $\hat{G}^-_{3\Gamma} = \gamma_3 e^-_3$ and entropy $\hat{G}^-_{4\Gamma} = \gamma_4 e^-_4$ waves, where $\gamma_j = g^-_j A^-_0 \hat{G}^-_\Gamma / s^-_j$, leads to the formulas

(8)

$$\gamma_j = \left[\frac{g^-_j(\eta,\omega)}{s^-_j(\eta,\omega)} - \frac{Y_j(\eta,\omega)}{s^-_j(\eta,\omega)Y_1(\eta,\omega)} g^-_1(\eta,\omega)\right](A^-)^{-1}A^+A^+_0\hat{G}^+_\Gamma(\eta,\omega),\ j=\overline{2,4}\,.$$

In turn, the formulas for generated by post-shock acoustic wave shock disturbance and boundary values for reflected waves (Part 1, Subsection 5.4, Eqs. (5.8)-(5.9)), assume the form

$$\text{(9)}\qquad \hat{f}^-(\eta,\omega) = -i\frac{\Phi(\eta,\omega)}{Y_1(\eta,\omega)},\ \hat{G}^-_{j\Gamma}(\eta,\omega) = \gamma_j e^-_j,\ \gamma_j = -\frac{Y_j\Phi(\eta,\omega)}{s^-_j Y_1(\eta,\omega)},\ j=\overline{2,4},$$

where the agent of incidence $\Phi(\eta,\omega) = g^-_1(\eta,\omega)\hat{G}^-_0(\xi^-_1(\eta,\omega),\eta)$.

5. Pre-shock plane waves and their refraction

5.1. **General form of incident and transmitted waves.** Here we treat as given the incident pre-shock plane wave

$$\text{(10)}\qquad G^+ = \sum_{j=1}^{4}\beta_j e^+_j(\eta_0,\omega_0)e^{i(\xi^+_j(\eta_0,\omega_0)x+\eta_0 y-\omega_0 t)},\quad \beta_j = \text{const}\,.$$

Its boundary value is

$$G^+_\Gamma = \sum_{j=1}^{4}\beta_j e^+_j(\eta_0,\omega_0)e^{i(\eta_0 y-\omega_0 t)},$$

or in spectral variables

$$\text{(11)}\qquad \hat{G}^+_\Gamma = \sum_{j=1}^{4}\beta_j e^+_j(\eta_0,\omega_0)\delta(\eta-\eta_0)I_1(\omega-\omega_0)\,.$$

Then, separating the singular and regular parts in Eqs. (7), (8), we obtain the general form of the principal (up to regular with respect to ω term $\hat{\varphi}$) parts of shock disturbance

$$\hat{f}^+ = \gamma_0\delta(\eta-\eta_0)I_1(\omega-\omega_0)+\hat{\varphi}\,,\quad \gamma_0 = \gamma_0(\beta_1,\beta_2,\beta_3,\beta_4)\,,$$

and boundary value for transmitted post-shock wave

$$\hat{G}^-_\Gamma = \sum_{k=2}^{4}\gamma_k e^-_k(\eta_0,\omega_0)\delta(\eta-\eta_0)I_1(\omega-\omega_0)+\hat{\varphi}\,,\quad \gamma_k = \gamma_k(\beta_1,\beta_2,\beta_3,\beta_4)\,,\quad k=\overline{2,4}\,.$$

In physical variables, we obtain shock disturbance in a form $f^+ = \gamma_0 e^{i(\eta_0 y-\omega_0 t)}+\varphi$, and, by virtue of common refracted/reflected waves representation (Part 1, Eq. (5.2)), the transmitted wave has a form

$$\text{(12)}\qquad G^- = \sum_{k=2}^{4}\gamma_k e^-_k(\eta_0,\omega_0)e^{i(\xi^-_k(\eta_0,\omega_0)x+\eta_0 y-\omega_0 t)}+\varphi$$

where φ is damped wave.

We consider coefficients γ_0, γ_k, $k=\overline{2,4}$ a bit later, in subsection 5.3.

Hereafter we denote

$$\alpha_0 = \frac{\omega_0 - \eta_0 u_y}{|\eta_0|\sqrt{(c^-)^2 - (u_x^-)^2}}, \quad s_0 = \mathrm{sgn}(\eta_0) = \frac{\eta_0}{|\eta_0|}.$$

Between critical values, i.e. at $|\alpha_0| < 1$, wavenumber ξ_2^- is complex, $\xi_2^- = r_1 + ir_2$, and $r_2 = |\eta_0|\sqrt{1-\alpha_0^2}/\sqrt{1-M^2} > 0$ (Section 3), and so the principal part of transmitted acoustic wave has a form

$$G_2^- = \gamma_2 e_2^-(\alpha_0, s_0) e^{-r_2 x} e^{i(r_1 x + \theta_0 y - \omega_0 t)}.$$

Before it was stated the non-existence of transmitted acoustic wave for this situation (e.g. [14]). In fact, as we see, the transmitted acoustic wave exists, but its principal part is the evanescent wave with respect to x. It strongly reminds the case of total internal reflection in optics. Anyway, usual refraction's description, such as comparison of the incident and transmitted wave vectors and amplitudes, is hardly useful here, and physical sense of this case needs further specification. Further we consider the refracted acoustic wave only out of critical values, at $|\alpha_0| \geqslant 1$.

5.2. **Correspondence between angles of incidence and refraction.** The wave vectors of incident G^+ (10) and transmitted G^- (12) waves in dimensionless variables are $\boldsymbol{k}_j^\pm = |\eta_0|(\xi_j^\pm(\alpha_0), s_0)$ $j = \overline{1,3}$ (wave vectors for vorticity wave $\boldsymbol{k}_3^\pm$ and entropy wave $\boldsymbol{k}_4^\pm$ are the same). Following to previous works (e.g. [5]), we introduce the angle of incidence θ^+ and the angle of refraction θ^-, $\theta^\pm \in [-\pi, \pi]$, as the angles between wave vector $\boldsymbol{k}^\pm$ and x-axis (and so $\theta^+ = 0$ is normal incidence): $\theta^\pm = s_0 \cdot \mathrm{arccotan}(\xi^\pm/|\eta_0|)$. For the incident/transmitted waves in dimensionless variables we have $\theta_j^\pm = s_0 \cdot \mathrm{arccotan}(\xi_j^\pm(\alpha_0))$, $j = \overline{1,3}$.

In the equations $\theta_j^+ = s_0 \cdot \mathrm{arccotan}(\xi_j^+(\alpha_0))$, $j = \overline{1,3}$, using the formulas of Section 3, we may express α_0 via any angle of incidence:

$$\alpha_0 = \frac{M^+\cos\theta_1^+ - 1}{|\sin\theta_1^+|R\sqrt{1-M^2}} = \frac{M^+\cos\theta_2^+ + 1}{|\sin\theta_2^+|R\sqrt{1-M^2}} = \frac{M^+\cos\theta_3^+}{|\sin\theta_3^+|R\sqrt{1-M^2}}.$$

Then we substitute these expressions in formulas for angles of refraction $\theta_k^- = s_0 \cdot \mathrm{arccotan}(\xi_k^-(\alpha_0))$, $k = \overline{1,3}$ and so obtain the connections between incident and refracted angles (Snell's laws):
for transmitted entropy-vorticity wave

$$\cos\theta_3^- = \begin{cases} \dfrac{M^+\cos\theta_3^+}{\sqrt{(M^+\cos\theta_3^+)^2 + \sin^2\theta_3^+ R^2 M^2}}, & \text{at incident entropy-vorticity wave,} \\ \dfrac{M^+\cos\theta_1^+ - 1}{\sqrt{(M^+\cos\theta_1^+ - 1)^2 + \sin^2\theta_1^+ R^2 M^2}}, & \text{at incident slow acoustic wave,} \\ \dfrac{M^+\cos\theta_2^+ + 1}{\sqrt{(M^+\cos\theta_2^+ + 1)^2 + \sin^2\theta_2^+ R^2 M^2}}, & \text{at incident fast acoustic wave;} \end{cases}$$

for transmitted acoustic wave we should take into account critical values: condition $|\alpha_0| \geqslant 1$ means $|\theta_j^+| \notin [\theta_-^j, \theta_+^j]$, where $\theta_\pm^j$ are different for different kinds of incident waves:

at incident entropy-vorticity wave $\cos\theta_2^- =$

$$= \operatorname{sgn}(\cos\theta_3^+)\frac{-MR^2\sin^2\theta_3^+ + M^+|\cos\theta_3^+|\sqrt{(M^+\cos\theta_3^+)^2 - R^2(1-M^2)\sin^2\theta_3^+}}{(M^+\cos\theta_3^+)^2+\sin^2\theta_3^+R^2}\,,$$

$$\theta_\pm^3 = \operatorname{arccotan}\frac{\mp\sqrt{1-M^2}R}{M^+}\,;$$

at incident slow acoustic wave

$$\cos\theta_2^- = \operatorname{sgn}(M^+\cos\theta_1^+ - 1)\times$$

$$\times\,\frac{-MR^2\sin^2\theta_1^+ + |M^+\cos\theta_1^+ - 1|\sqrt{(M^+\cos\theta_1^+ - 1)^2 - R^2(1-M^2)\sin^2\theta_1^+}}{(M^+\cos\theta_1^+ - 1)^2+\sin^2\theta_1^+R^2}\,,$$

$$\theta_\pm^1 = \operatorname{arccotan}\frac{\mp M^+\sqrt{1-M^2}R + \sqrt{(1-M^2)R^2+(M^+)^2-1}}{(M^+)^2-1}\,;$$

at incident fast acoustic wave

$$\cos\theta_2^- = \operatorname{sgn}(M^+\cos\theta_2^+ + 1)\times$$

$$\times\,\frac{-MR^2\sin^2\theta_2^+ + |M^+\cos\theta_2^+ + 1|\sqrt{(M^+\cos\theta_2^+ + 1)^2 - R^2(1-M^2)\sin^2\theta_2^+}}{(M^+\cos\theta_2^+ + 1)^2+\sin^2\theta_2^+R^2M^2}\,,$$

$$\theta_\pm^2 = \operatorname{arccotan}\frac{\mp M^+\sqrt{1-M^2}R - \sqrt{(1-M^2)R^2+(M^+)^2-1}}{(M^+)^2-1}\,.$$

The comparison with formulas from the classical work by [5] shows: our formulas coincides with formulas of [5] for transmitted entropy-vorticity wave and for transmitted acoustic wave at $|\theta_j^+| < \theta_-^j$ ($\alpha_0 > 1$) and differ at $|\theta_j^+| > \theta_+^j$ ($\alpha_0 < -1$).

5.3. **Matrix of generation coefficients.** Now let's turn to the generation coefficients, i.e. to the ratio of wave amplitudes of transmitted and incident waves. Boundary value of the incident wave in spectral variables has a form (11)

$$\hat{G}_\Gamma^+(\eta,\omega) = \sum_{j=1}^{4}\beta_j e_j^+(\eta_0,\omega_0)\delta(\eta-\eta_0)I_1(\omega-\omega_0)\,.$$

We write it in a form $\hat{G}_\Gamma^+ = (\Psi^+\beta)\delta(\eta-\eta_0)I_1(\omega-\omega_0)$, where

$$\beta = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{pmatrix}$$

is vector of density dimension, $\Psi^+ = \Psi^+(\alpha_0, s_0)$ is dimensionless matrix:

$$\Psi^+(\alpha,s) = \big(e_1^+(\alpha,s)\ ;\ e_2^+(\alpha,s)\ ;\ e_3^+(\alpha,s)\ ;\ e_4^+(\alpha,s)\big) =$$

$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ -\dfrac{\xi_1^+(\alpha)}{Q_1^+(\alpha)} & -\dfrac{\xi_2^+(\alpha)}{Q_2^+(\alpha)} & \dfrac{s}{\sqrt{(\xi_0^+(\alpha))^2+1}} & 0 \\ -\dfrac{s}{Q_1^+(\alpha)} & -\dfrac{s}{Q_2^+(\alpha)} & -\dfrac{\xi_0^+(\alpha)}{\sqrt{(\xi_0^+(\alpha))^2+1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s = \pm 1 .$$

As before in subsection 5.1, we separate in Eqs. (7), (8) singular and regular terms (Part 1, Subsection 4.2), but write the result more concretely (we save for principal parts the same notations f^+ and $G^-_{j\Gamma}$):

$$f^+ = \gamma_0 e^{i(\eta_0 y - \omega_0 t)}, \quad \gamma_0 = -i\frac{g_1^-(\eta_0,\omega_0)}{Y_1(\eta_0,\omega_0)}(A^-)^{-1}A^+A_0^+(\Psi^+\beta);$$

$$G^- = \sum_{j=2}^{4}\gamma_j e_j^-(\eta_0,\omega_0)e^{i(\xi_j^-(\eta_0,\omega_0)x+\eta_0 y-\omega_0 t)},$$

$$\gamma_j = \left[\frac{g_j^-(\eta_0,\omega_0)}{s_j^-(\eta_0,\omega_0)} - \frac{Y_j(\eta_0,\omega_0)}{s_j^-(\eta_0,\omega_0)Y_1(\eta_0,\omega_0)}g_1^-(\eta_0,\omega_0)\right](A^-)^{-1}A^+A_0^+(\Psi^+\beta),\ j=\overline{2,4}.$$

Here wave amplitudes γ_j, $j = \overline{2,4}$ have density dimension, while γ_0 has other dimension (length). To make the same dimension (for taking ratio), we suppose to consider quantity
$f_1^+ = \rho^-((f^+)'_t + u_y(f^+)'_y)/c^-$ or in spectral variables $\hat{f}_1^+ = -i\rho^-(\omega-\eta u_y)\hat{f}^+/c^-$.
Then its principal part (under the same notation f_1^+) is

$$f_1^+ = \gamma_1 e^{i(\eta_0 y-\omega_0 t)}, \quad \gamma_1 = -\frac{\rho^-(\omega_0-u_y\eta_0)g_1^-(\eta_0,\omega_0)}{c^-Y_1(\eta_0,\omega_0)}(A^-)^{-1}A^+A_0^+(\Psi^+\beta).$$

Going to the dimensionless variables, we obtain

$$\gamma_1 = -\frac{\alpha_0\sqrt{1-M^2}\,g_1^-(\alpha_0,s_0)}{(M^+/R-M)Y_1(\alpha_0)}(A_2^-)^{-1}A_RA_2^+A_{01}^+(\Psi^+\beta),$$

$$\gamma_j = \left[\frac{g_j^-(\alpha_0,s_0)}{s_j^-(\alpha_0)} - \frac{Y_j(\alpha_0,s_0)}{s_j^-(\alpha_0)Y_1(\alpha_0)}g_1^-(\alpha_0,s_0)\right](A_2^-)^{-1}A_RA_2^+A_{01}^+(\Psi^+\beta), \quad j=\overline{2,4},$$

where $\rho=\rho^-$, $c=c^-$, $M=M^-$,

$$A_R = \frac{1}{R}(A_1^-)^{-1}A_1^+ = \begin{pmatrix} 1/R & 0 & 0 & 0 \\ 0 & 1/R^2 & 0 & 0 \\ 0 & 0 & 1/R^2 & 0 \\ 0 & 0 & 0 & 1/R^3 \end{pmatrix},$$

or finally in matrix form

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \Psi\beta, \quad \Psi = \Psi^-(A_2^-)^{-1}A_RA_2^+A_{01}^+\Psi^+,$$

where

$$\Psi^- = \begin{pmatrix} -\dfrac{\alpha_0\sqrt{1-M^2}\, g_1^-(\alpha_0, s_0)}{(M^+/R - M)Y_1(\alpha_0)} \\ \dfrac{g_2^-(\alpha_0, s_0)}{s_2^-(\alpha_0)} - \dfrac{Y_2(\alpha_0)}{s_2^-(\alpha_0)Y_1(\alpha_0)} g_1^-(\alpha_0, s_0) \\ \dfrac{g_3^-(\alpha_0, s_0)}{M} - \dfrac{s_0 Y_3(\alpha_0)}{MY_1(\alpha_0)} g_1^-(\alpha_0, s_0) \\ \dfrac{g_4^-}{M} - \dfrac{Y_4(\alpha_0)}{MY_1(\alpha_0)} g_1^-(\alpha_0, s_0) \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{\alpha_0\sqrt{1-M^2}}{(M^+/R - M)Y_1(\alpha_0)} & 0 & 0 & 0 \\ -\dfrac{Y_2(\alpha_0)}{s_2^-(\alpha_0)Y_1(\alpha_0)} & \dfrac{1}{s_2^-(\alpha_0)} & 0 & 0 \\ -\dfrac{s_0 Y_3(\alpha_0)}{MY_1(\alpha_0)} & 0 & \dfrac{1}{M} & 0 \\ -\dfrac{Y_4}{MY_1(\alpha_0)} & 0 & 0 & \dfrac{1}{M} \end{pmatrix}$$

$$\times \begin{pmatrix} \dfrac{1}{2} & -\dfrac{\xi_1^-(\alpha_0)}{2Q_1^-(\alpha_0)} & -\dfrac{s_0}{2Q_1^-(\alpha_0)} & -\dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{\xi_2^-(\alpha_0)}{2Q_2^-(\alpha_0)} & -\dfrac{s_0}{2Q_2^-(\alpha_0)} & -\dfrac{1}{2} \\ 0 & \dfrac{s_0}{\sqrt{(\xi_0^-(\alpha_0))^2+1}} & -\dfrac{\xi_0^-(\alpha_0)}{\sqrt{(\xi_0^-(\alpha_0))^2+1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As vectors β and γ have the same dimension (density), then elements of matrix Ψ are dimensionless ratios of wave amplitudes of transmitted (principal parts) and incident plane waves. So matrix Ψ is actually the matrix of generation (transmission) coefficients, see e.g. [5].

When the angles of incidence lie between critical values, i.e. $|\alpha_0| < 1$, then second row of matrix Ψ^- (and, consequently, of matrix Ψ) became complex. It is just the case of evanescent transmitted acoustic wave, see above. So we consider the generation coefficients for transmitted acoustic wave (second row of Ψ) only at $|\alpha_0| \geqslant 1$.

The second problem is that at critical values $\alpha_0 = \pm 1$, quantity $s_2^-(\alpha_0) = 0$, and so we have a zero denominator in second row of matrix Ψ_0^-. In particular, this singularity is a reason of so-called abnormal amplification near the critical angles of incidence, see [5, 14]. But if we introduce the vector

$$\widetilde{g}_2(\alpha, s) = \frac{1}{Q_1^-(\alpha)} \left(0\,;\, \sqrt{1-M^2}\alpha\,;\, Ms\,;\, 0\right), \quad s = \pm 1,$$

and function

$$\widetilde{Y}_2(\alpha) = -\frac{MR(1-M^2)\alpha^2/M^+ + M^2}{Q_1^-(\alpha)}\,,$$

then

$$g_2^- = g_1^- - \frac{s_2^-}{1-M^2}\widetilde{g}_2\,, \quad Y_2(\alpha) = Y_1(\alpha) - \frac{s_2^-}{1-M^2}\widetilde{Y}_2\,,$$

and second row of matrix Ψ^- assume the form

$$\frac{g_1^- - \dfrac{s_2^-}{1-M^2}\widetilde{g}_2}{s_2^-} - \frac{Y_1 - \dfrac{s_2^-}{1-M^2}\widetilde{Y}_2}{s_2^- Y_1} g_1^- = \frac{1}{1-M^2}\left[\frac{\widetilde{Y}_2}{Y_1}g_1^- - \widetilde{g}_2\right],$$

so finally we obtain

$$\Psi^- = \begin{pmatrix} -\dfrac{\alpha_0\sqrt{1-M^2}}{(M^+/R-M)Y_1(\alpha_0)} & 0 & 0 & 0 \\ \dfrac{\widetilde{Y}_2(\alpha_0)}{(1-M^2)Y_1(\alpha_0)} & -\dfrac{1}{(1-M^2)} & 0 & 0 \\ -\dfrac{s_0 Y_3(\alpha_0)}{MY_1(\alpha_0)} & 0 & \dfrac{1}{M} & 0 \\ -\dfrac{Y_4}{MY_1(\alpha_0)} & 0 & 0 & \dfrac{1}{M} \end{pmatrix}$$
$$\times \begin{pmatrix} \dfrac{1}{2} & -\dfrac{\xi_1^-(\alpha_0)}{2Q_1^-(\alpha_0)} & -\dfrac{s_0}{2Q_1^-(\alpha_0)} & -\dfrac{1}{2} \\ 0 & \dfrac{\alpha_0\sqrt{1-M^2}}{Q_1^-(\alpha_0)} & \dfrac{Ms_0}{Q_1^-(\alpha_0)} & 0 \\ 0 & \dfrac{s_0}{\sqrt{(\xi_0^-(\alpha_0))^2+1}} & -\dfrac{\xi_0^-(\alpha_0)}{\sqrt{(\xi_0^-(\alpha_0))^2+1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and this representation shows in fact matrix Ψ has not singularity at critical values.

So we have a representation of matrix of transmission coefficients Ψ and may calculate all sixteen coefficients, using some mathematical programmes. The final formulas are rather complicated and unwieldy.

5.4. **Generation coefficients for an ideal gas, amplification.** Let's consider an ideal gas. It is well known (see e.g. [1, 5]) that equation of state for ideal gas together with Rankine-Hugoniot conditions allow to express all dimensionless parameters through polytropic coefficient γ and pre-shock Mach number M^+:

$$B_0^\pm = \gamma\,, \quad D_0^\pm = 1-\gamma\,, \quad R = \frac{\sqrt{(2\gamma(M^+)^2-(\gamma-1))((\gamma-1)(M^+)^2+2)}}{(\gamma+1)M^+}\,,$$

$$M^- = \sqrt{\frac{(\gamma-1)(M^+)^2+2}{2\gamma(M^+)^2-(\gamma-1)}}\,.$$

We assume $\gamma = 5/3$, carry out the numerical calculations of generation coefficients, and compare them to the results of classical work [5]. Only two small remarks should

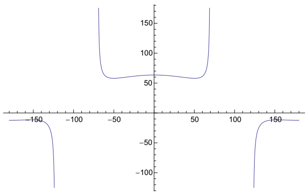

Fig. 2: The angular dependence of the transmission coefficient for acoustic wave caused by incident fast acoustic wave at $M^+ = 8$.

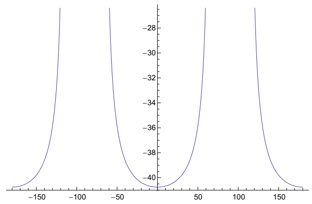

Fig. 3: The angular dependence of the transmission coefficient for acoustic wave caused by incident entropy wave at $M^+ = 8$.

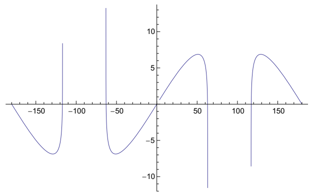

Fig. 4: The angular dependence of the transmission coefficient for acoustic wave caused by incident vorticity wave at $M^+ = 8$.

be made for correct comparison: first, we need to express generation coefficients not through the dimensionless variable α, but through the angles of incidence θ_j^+, $j = \overline{1,3}$, measured in degrees; second, in work [5] they consider the ratios of quantities of pressure dimension, so to be more exact, we need actually calculate matrix

$$\widetilde{\Psi} = (H_0^-)^{-1}\Psi^-(A_2^-)^{-1}\widetilde{A}_R A_2^+ A_{01}^+ \Psi^+ H_0^+ \,, \tag{13}$$

where

$$H_0^{\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & B_0^{\pm} & 0 \\ 0 & 0 & 0 & -B_0^{\pm} \end{pmatrix}, \quad \widetilde{A}_R = R^2 A_R = \begin{pmatrix} R & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/R \end{pmatrix}.$$

For comparison we show only plots of coefficients given in mentioned paper, they are depicted in fig. 2, 3, 4.

Compare these plots with plots of [5], we can see the principal distinction: in work [5] they differ cases of different tangent velocity of basic flow u_y: case $u_y = 0$ (normal incidence of basic flow), and two more cases of oblique incidence, when post-shock basic flow is subsonic $u_y^2 + (u_x^-)^2 < (c^-)^2$, and supersonic $u_y^2 + (u_x^-)^2 > (c^-)^2$, and give different plots for these cases. But as we see, generation coefficients depend only on dimensionless variable α, and don't depend on u_y, in particular don't depend on cases of subsonic/supersonic basic flow behind the shock. In other respects, our plots are more or less like plots of [5], with some differences at $|\theta| > \theta_+^j$.

Apart we consider the amplification, i.e. the degree of growth of transmission coefficients when the pre-shock Mach number $M^+ \to \infty$. In representation (13), matrix $\Psi_0^- = (H_0^-)^{-1}\Psi^-(A_2^-)^{-1}$ has finite limit at $M^+ \to \infty$, and matrix $\Psi_0^+ =$

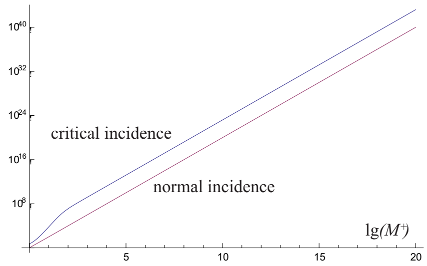

Fig. 5: Generation coefficients for transmitted sound wave caused by fast sound wave at normal and critical incidence.

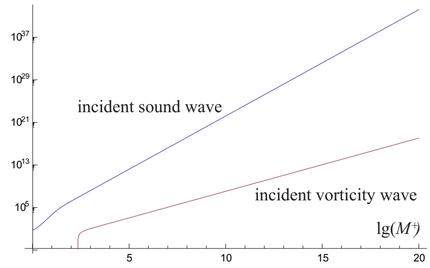

Fig. 6: Generation coefficients for transmitted sound wave caused by fast sound wave and by vorticity wave.

$= \widetilde{A}_R A_2^+ A_{01}^+ \Psi^+ H_0^+$ has a representation $\Psi_0^+ = (M^+)^2 \Psi_2 + O(M^+)$, where

$$\Psi_2 = \begin{pmatrix} R_0 - \dfrac{\sqrt{1-M_0^2}R_0^3\alpha}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & R_0 + \dfrac{\sqrt{1-M_0^2}R_0^3\alpha}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & 0 & -\gamma R_0 \\ 1 - \dfrac{2\sqrt{1-M_0^2}R_0^2\alpha}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & 1 + \dfrac{2\sqrt{1-M_0^2}R_0^2\alpha}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & 0 & -\gamma \\ -\dfrac{R_0 \mathrm{sgn}(\eta)}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & \dfrac{R_0 \mathrm{sgn}(\eta)}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & 0 & 0 \\ -\dfrac{\sqrt{1-M_0^2}R_0\alpha}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & \dfrac{\sqrt{1-M_0^2}R_0\alpha}{\sqrt{(1-M_0^2)R_0^2\alpha^2+1}} & 0 & 0 \end{pmatrix},$$

$$R_0 = \frac{\sqrt{2\gamma(\gamma-1)}}{\gamma+1}, \quad M_0 = \sqrt{\frac{\gamma-1}{2\gamma}}\,.$$

Matrix Ψ_2 has rank two and

$$\Psi_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \Psi_2 \begin{pmatrix} \gamma/2 \\ \gamma/2 \\ 0 \\ 1 \end{pmatrix} = 0\,.$$

So general conclusion reads: amplification has second degree on the whole, but for vorticity wave and for linear combination: sum of two acoustic waves multiply by $\gamma/2$ plus entropy wave, amplification has a first degree and these degrees of amplification don't depend on angle of incidence.

In works [5, 14], the amplification of third degree $O((M^+)^3)$ (abnormal amplification) for close to critical angles of incidence is stated. The seeming singularity of transmission coefficients at critical values misleads authors. In fact, as we see, there is no any singularity in transmission coefficients and so abnormal amplification doesn't exist.

We illustrate last conclusion by the numerical calculations. In fig.5, we depict in logarithmic scale for both axes the generation coefficient for transmitted acoustic wave, caused by incident fast acoustic wave, as a function of M^+ at critical angle of incidence, and zero angle of incidence (normal incidence). As we see, asymptotically both plots become parallel lines, i.e. they have the same degree of amplification (two). In fig.6, we depict plots of generation coefficient for transmitted acoustic wave, caused by incident fast acoustic wave, and caused by incident vorticity wave. We clearly see the difference between degrees of amplification: second for incident sound wave and first for incident vorticity wave.

6. Post-shock plane waves and reflection

Here we mean superscript minus on default. As we see above (Subsection 2.3), initial post-shock wave cannot be a plane wave, so as incident plane wave we may consider only the principal part of initial post-shock wave, generated by the initial plane data, it is wave G_1 in form (3). Since $G_1 \equiv 0$ at wavenumbers (η_0, ω_0), located between critical values (i.e. at $|\alpha_0| < 1$ in dimensionless variables), then we consider only wavenumbers out of critical values ($|\alpha_0| \geqslant 1$).

Using the principal part Φ_1 in form (4) of the agent of incidence Φ, we separate singular and regular parts in the formulas (9) for the shock disturbance $\hat{f}$ and for the boundary values of reflected waves $\hat{G}_{j\Gamma}$, $j = \overline{2,4}$, and get expressions for their principal parts (under the same notations $\hat{f}$, $\hat{G}_{j\Gamma}$):

$$\hat{f} = i\beta \frac{s_1(\eta_0,\omega_0)}{Y_1(\eta_0,\omega_0)} \delta(\eta-\eta_0) I_1(\omega-\omega_0)\,, \tag{14}$$

(15)

$$\hat{G}_{j\Gamma} = \beta\gamma_j e_j(\eta_0,\omega_0)\,, \quad \gamma_j = \frac{s_1(\eta_0,\omega_0)Y_j(\eta_0,\omega_0)}{s_j(\eta_0,\omega_0)Y_1(\eta_0,\omega_0)} \delta(\eta-\eta_0) I_1(\omega-\omega_0)\,, \quad j=\overline{2,4}\,.$$

Hence the principal parts of reflected waves themselves have a form (see Subsection 2.3)

$$G_j = \beta\gamma_j e_j(\eta_0,\omega_0) e^{i(\xi_j x + \eta_0 y - \omega_0 t)}\,, \quad \xi_j = \xi_j(\eta_0,\omega_0)\,, \quad j=\overline{2,4}\,.$$

First we compare angles of incidence and reflection. The wave vector for incident wave G_1 in form (3) is $\boldsymbol{k}_1 = (\xi_1(\eta_0,\omega_0),\eta_0)$, and the wave vectors for reflected waves are $\boldsymbol{k}_j = (\xi_j(\eta_0,\omega_0),\eta_0)$, $j=\overline{2,4}$, or in dimensionless variables $\boldsymbol{k}_j = |\eta_0|(\xi_j(\alpha_0), s_0)$, $j=\overline{1,3}$ ($\boldsymbol{k}_3 = \boldsymbol{k}_4$). Like above, we introduce the angle of incidence θ_1, and the angles of reflection θ_j, $j=\overline{2,3}$ in form $\theta_j = s_0 \cdot \operatorname{arccotan}(\xi_j(\alpha_0))$, $j=\overline{1,3}$. Then we express α_0 through angle of incidence θ_1: condition $|\alpha_0| \geqslant 1$ means $|\theta_1| \notin (\theta_-, \theta_+)$, where $\theta_\pm = \operatorname{arccotan}(\mp M/\sqrt{1-M^2})$ are critical post-shock angles of incidence, and

$$\alpha_0 = \begin{cases} \dfrac{M\cos\theta_1 + 1}{|\sin\theta_1|\sqrt{1-M^2}}, & |\theta_1| > \theta_+, \\[2mm] \dfrac{M\cos\theta_1 - 1}{|\sin\theta_1|\sqrt{1-M^2}}, & |\theta_1| < \theta_-. \end{cases}$$

We substitute α_0 into the formulas for angles of reflection and obtain the correspondence between angles of incidence and reflection (laws of reflection):
for the reflected sound wave

$$\cos\theta_2 = \begin{cases} \dfrac{2M - \cos\theta_1(1+M^2)}{1+M^2-2M\cos\theta_1}, & |\theta_1| < \theta_-, \\[2mm] -\dfrac{2M + \cos\theta_1(1+M^2)}{1+M^2+2M\cos\theta_1}, & |\theta_1| > \theta_+\,; \end{cases}$$

for the reflected entropy-vorticity wave

$$\cos\theta_3 = \begin{cases} \dfrac{M\cos\theta_1 - 1}{\sqrt{2(1+M^2-2M\cos\theta_1)}}, & |\theta_1| < \theta_-, \\ \dfrac{M\cos\theta_1 + 1}{\sqrt{2(1+M^2+2M\cos\theta_1)}}, & |\theta_1| > \theta_+ . \end{cases}$$

These formulas strongly differ from those of [5], but it is evident misunderstanding: in mentioned work they simply make inaccurate conclusion from absolutely correct equation.

Now let's turn to reflection coefficients. We naturally take the quantity β (dimension of density) as wave amplitude for incident wave G_1 in form (3). In order to reduce shock disturbance f to the same dimension we, like for refraction, take principal part of $f_1 = \rho(f'_t + u_y f'_y)/c$ or in spectral variables due to (14)

$$\hat{f}_1 = -i\frac{\rho(\omega_0 - \eta_0 u_y)}{c}\hat{f} = \beta\frac{\rho(\omega_0 - \eta_0 u_y)s_1(\eta_0,\omega_0)}{cY_1(\eta_0,\omega_0)}\delta(\eta-\eta_0)I_1(\omega-\omega_0)\,,$$

so reflection coefficient for shock disturbance is

$$\gamma_1 = \frac{\rho(\omega_0 - \eta_0 u_y)s_1(\eta_0,\omega_0)}{cY_1(\eta_0,\omega_0)} = \frac{\sqrt{1-M^2}\,\alpha_0 s_1(\alpha_0)}{(M^+/R - M)Y_1(\alpha_0)}\,.$$

The formulas for other reflection coefficients follow from (15):

$$\gamma_j = \frac{s_1(\alpha_0)Y_j(\alpha_0,s_0)}{s_j(\alpha_0)Y_1(\alpha_0)}\,, \quad j = \overline{2,4}\,.$$

Finally, by means of direct calculations and keeping in mind the branch of $\sqrt{\alpha^2-1}$ (see Section 3), we obtain

(16)
$$\begin{aligned} \gamma_1 &= \frac{2(1-M^2)|\alpha_0|\sqrt{\alpha_0^2-1}}{(M^+/R-M)(Y_{11}^0\alpha_0^2 + Y_{12}^0|\alpha_0|\sqrt{\alpha_0^2-1} + Y_{13}^0)} = \gamma_1(\alpha_0)\,, \\ \gamma_2 &= -\frac{Y_{11}^0\alpha_0^2 - Y_{12}^0|\alpha_0|\sqrt{\alpha_0^2-1} + Y_{13}^0}{Y_{11}^0\alpha_0^2 + Y_{12}^0|\alpha_0|\sqrt{\alpha_0^2-1} + Y_{13}^0} = \gamma_2(\alpha_0)\,, \\ \gamma_3 &= \frac{2(1-M^2)(1-MR/M^+)|\alpha_0|\sqrt{\alpha_0^2-1}s_0}{(Y_{11}^0\alpha_0^2 + Y_{12}^0|\alpha_0|\sqrt{\alpha_0^2-1} + Y_{13}^0)\sqrt{M^2+\alpha_0^2(1-M^2)}} = s_0\gamma_3(\alpha_0)\,, \\ \gamma_4 &= \frac{2D_0(1-MR/M^+)(1-M^2)|\alpha_0|\sqrt{\alpha_0^2-1}}{(Y_{11}^0\alpha_0^2 + Y_{12}^0|\alpha_0|\sqrt{\alpha_0^2-1} + Y_{13}^0)} = \gamma_4(\alpha_0)\,, \end{aligned}$$

where Y_{11}^0, Y_{12}^0, Y_{13}^0 are defined in section 3, Eq. (6).

In fig.7, we depict the result of numerical calculations of reflection coefficient for acoustic wave γ_2 as a function of angle of incidence θ_1 for an ideal gas.

The comparison with corresponding plot of [5] shows the same principal distinction as for refraction: in work [5] they give different plots at different values of u_y, although in fact reflection coefficients don't depend on u_y. There is one strange difference more: critical angles of [5] are located near zero $\theta_1 = 0$, while in fact they are located near right angle $\theta_1 = 90°$, and plot in mentioned work looks like plot in fig.7 shifted along θ-axis.

Now we establish some properties of reflection. First let's consider critical values $\alpha_0 = \pm 1$. Here wave vectors for incident and reflected sound waves coincide $\boldsymbol{k}_1 = \boldsymbol{k}_2$, and group velocities of these waves are parallel to the shock. At that $\gamma_1(\pm 1) =$

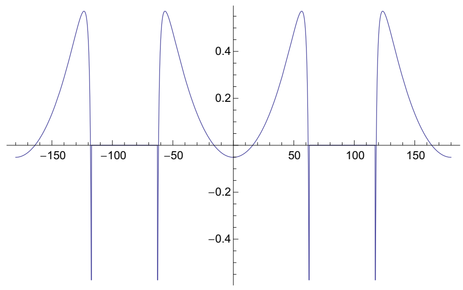

Fig. 7: The angular dependence of the reflection coefficient for acoustic wave at $M^+ = 8$.

$= \gamma_3(\pm 1) = \gamma_4(\pm 1) = 0$, i.e. the shock disturbance and reflected entropy-vorticity waves vanish, and $\gamma_2(\pm 1) = -1$, i.e. $G_1 + G_2 = 0$, and reflected sound wave suppresses (eliminates) the incident wave, so that the total principal part of the wave behind the shock vanishes. So the picture of reflection at critical values ensures the continuous transition to the case between critical values $|\alpha_0| < 1$, when incident acoustic wave, shock deformation and all reflected waves are damped and their principal parts are zero.

The equality of the reflection coefficient to unity at the critical angles was noted before (see e.g. [13]), but mutual suppression of the incident and reflected acoustic waves was not discussed before.

Dynamics (behaviours) of reflection coefficients may be investigated comprehensively. In particular, in is easy to show $|\gamma_2| \leqslant 1$, i.e. the wave amplitude of reflected sound wave is less then the wave amplitude of incident wave. Further, it can be shown for an ideal gas, that the coefficient γ_2 always has eight zero independently on M^+, and just this situation we see in fig.7. But $\gamma_2 = 0$ means principal part of reflected sound wave vanish, and it is this vanishing that creates an impression of a three-wave configuration at reflection, see [11, 25]. In fact, $\gamma_2 = 0$ only means reflected sound wave is damped, but four-wave configuration is maintained. The obvious reason for this misunderstanding is the neglect of the damped waves.

References

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Vol. 6, Fluid Mechanics*, Elsiver, 2nd edition, 1987. MR0961259

[2] S.P. D'yakov, *On the stability of shock waves*, Sov. Phys. JETP, **27** (1954), 288–295. MR0066173

[3] S.P. D'yakov, *The interaction of shock waves with small perturbations*, Sov. Phys. JETP, **33** (1957), 948–974.

[4] V. M. Kontorovitch, *Reflection and refraction of sound by shock waves*, Sov. Phys. JETP, **33** (1957), 1527–1528.

[5] J.F. McKenzie, K.O. Westphal, *Interaction of linear waves with oblique shock waves*, Phys. Fluids, **11** (1968), 2350–2362. Zbl 0172.53203

[6] W.R. Johnson, O. Laporte, *Interaction of Cylindrical Sound Waves with a Stationary Shock Wave*, Phys. Fluids, **1** (1958), 82–94. MR0112501

[7] G.R. Fowles, G.W. Swan, *Stability of plane shock waves*, Phys.Rev.Lett., **30** (1973), 1023–1025.

[8] W.K. Van Moorhem, A.R.George, *On the stability of plane shocks*, Journal of Fluid Mechanics, **68** (1975), 97–108. Zbl 0303.76038
[9] G. W. Swan, G. R. Fowles, *Shock wave stability*, Phys. Fluids, **18** (1975), 28–35. Zbl 0321.76022
[10] G.R. Fowles, *Conditional stability of shock waves – a criterion for detonation*, Phys. Fluids, **19** (1976), 227–238. Zbl 0336.76027
[11] G.R. Fowles, *Stimulated and spontaneous emission of acoustic waves from shock fronts*, Phys. Fluids, **24** (1981), 220–227. Zbl 0464.76049
[12] M. Mond, I.M. Rutkevich, *Spontaneous acoustic emission from strong ionizing shocks*, Journal of Fluid Mechanics, **275** (1994), 121–146. MR1297865
[13] J.-C. Robinet, G. Casalis, *Critical interaction of a shock wave with an acoustic wave*, Phys. Fluids, **13** (2001), 1047–1059. MR1830406
[14] I. Men'shov, Y. Nakamura, *Abnormal amplification of sound waves refracted by an oblique shock wave*, JAXA Special Publication, **SP-03-002** (2004), 23–28.
[15] S.V. Iordanski, *On stability of a plane shock wave*, Journal of Applied Mathematics and Mechanics, **21** (1957), 465–472. MR0092523
[16] J.J. Erpenbeck, *Stability of Steady-State Equilibrium Detonations*, Phys.Fluids, **5** (1962), 604–614.
[17] J.J. Erpenbeck, *Stability of Step Shocks*, Phys.Fluids, **5** (1962), 1181–1187. MR0155515
[18] R.M. Zaidel, *Shock wave from a slightly curved piston*, Journal of Applied Mathematics and Mechanics, **24** (1960), 316–320. MR0129707
[19] R.M. Zaidel, *The perturbations propagation in plane shock waves*, Journal of Applied Mechanics and Technical Physics, **4** (1967), 30–39.
[20] J.W.Bates, *Initial-value-problem solution for isolated rippled shock fronts in arbitrary fluid media*, Phys. Rev. E, **69** (2004), 056313-1–056313-16. MR2096556
[21] J.W.Bates, *Instability of isolated planar shock waves*, Phys. Fluids, **19** (2007), 1–15. Zbl 1182.76053
[22] A.Tumin, *Initial-value problem for small disturbances in an idealized one-dimensional detonation*, Phys. Fluids, **19** (2007), 106105-1–106105-12. Zbl 1182.76778
[23] G.R. Fowles, A.F.P. Houwing, *Instabilities of shock and detonation waves*, Phys. Fluids, **27** (1984), 1982–1990. MR0758730
[24] N. M. Kuznetsov, *Contribution to shock-wave stability theory*, Zh. Eksp. Teor. Fiz., **88** (1985), 470–486.
[25] N. M. Kuznetsov, *Stability of shock waves*, Usp. Fiz. Nauk, **159** (1989), 493–527.
[26] J.W. Bates, D.C. Montgomery, *The D'yakov-Kontorovich Instability of Shock Waves in Real Gases*, Phys. Rev. Lett., **84** (2000), 1180–1183.
[27] A.V. Konyukhov, A.P. Likhachev, V.E. Fortov, S.I. Anisimov, A.M. Oparin, *Stability and ambiguous representation of shock wave discontinuity in thermodynamically nonideal media*, Pis'ma Zh. Eksp. Teor. Fiz., **90** (2009), 28–34.
[28] G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a complex variable*, Hod Books, Ithaka, N. Y., 1983. Zbl 0548.30001
[29] E. Zauderer, *Partial Differential Equations*, Wiley, 3rd edition, 2006. MR2244913

Evgeny Veniaminovich Semenko, Tatiana Ivanovna Semenko
Novosibirsk State Technical University,
Prospekt K. Marksa 20,
630073, Novosibirsk, Russia
E-mail address: semenko54@gmail.com