LINEAR PROBLEM OF SHOCK WAVE DISTURBANCE ANALYSIS. PART 2: REFRACTION AND REFLECTION OF PLANE WAVES IN THE STABILITY CASE

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Abstract. This part is devoted to the propagation of plane waves in the stability case.

First the fact that each post-shock plane wave is accompanied with damped wave, and so the plane waves refraction/reflection quantity characteristics are determined up to the damped waves, is established.

The correspondence between angles of incidence and angles of refraction/reflection, i.e. Snell’s laws, is obtained.

The matrix of generation coefficients as a whole is calculated. Its behaviour for an ideal gas when the pre-shock Mach number tends to infinity, i.e. coefficients’ amplification, is investigated. The degree of amplification for different kinds of incident waves is found. Furthermore, numerical calculations of generation coefficients for an ideal gas are performed, in particular, the coefficients’ amplification is investigated numerically and the results are found to confirm analytical conclusions.

For reflection, all four reflection coefficients are calculated and some of their properties are established. In particular, vanishing of the reflected entropy-vorticity plane waves and mutual suppression of the incident and reflected acoustic plane waves at critical angles of incidence are established. The numerical calculations of reflection coefficients are also performed.

A comparison is carried out between the obtained results and the already-known ones. It is found that the known formulas for the refraction/reflection angles and for the generation/reflection coefficients...
should be specified and corrected. In particular, the existence of so-called abnormal amplification is disproved.

**Keywords:** Shock wave, shock disturbance, entropy-vorticity wave, acoustic wave, incident wave, refraction, transmitted wave, reflection, reflected wave, stability, neutral stability, spontaneous emission, Fourier transform.

1. **Introduction**

This part is organized as follows. In Section 2, the form of plane waves is investigated, in particular the fact, that any post-shock plane wave is accompanied with a damped wave, is established. In having technical nature Section 3, the transition to the dimensionless parameters and variables is performed. In short Section 4, the form of transmitted and reflected waves in the stability case is indicated. Section 5 is devoted to the refraction: in subsection 5.1, the general form of incident and refracted waves is considered; in subsection 5.2, we establish the correspondence between angles of incidence and refraction, i.e. Snell’s laws; in subsection 5.3, we deduce the matrix of generation coefficients on the whole; at last in subsection 5.4, we consider the generation coefficients for an ideal gas, in particular amplification, and give the results of numerical calculations. In Section 6, the reflection is considered.

2. **Pre-shock and post-shock plane waves**

2.1. **General form of plane waves.** As the sought quantity in our case is the vector $G$, then the plane wave has the form $G = G^e H_0 e^{i(ξ_0x + η_0y - ω_0 t)}$, where $ξ_0, η_0, ω_0 = \text{const}$ and $H_0^e$ is a constant vector. Now we recall some well-known (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) or obvious information about plane waves.

The system of differential equations $BG = 0$ (Part 1, Eq. (1)) for plane waves has the form

$$
B(ξ_0, η_0, ω_0)H_0 e^{i(ξ_0x + η_0y - ω_0 t)} = 0,
$$

i.e. $H_0$ is the eigenvector of the matrix $B(ξ_0, η_0, ω_0)$ with a zero eigenvalue. Due to the complete basis $\{e_0j, j = 1, 4\}$ of the eigenvectors of matrix $B$ and their eigenvalues $λ_{1,2} = Q ± c\sqrt{ξ_0^2 + η_0^2}$, $λ_{3,4} = Q$ (Part 1, Section 3), there are four solutions of the equation $BG = 0$ in the form of plane waves for any wave vector $k = (ξ_0, η_0)$:

$$
G_j = β_j e_0j(ξ_0, η_0)e^{i(ξ_0x + η_0y - ω_j t)}, \quad β_j = \text{const}, \quad j = 1, 4,
$$

$ω_1 = ω_-, ω_2 = ω_+$. Here $G_{1,2}$ are the acoustic plane waves, $G_3$ is a vorticity plane wave, and $G_4$ is an entropy plane wave.

The group velocity of entropy-vorticity waves is $v_{3,4} = U_0 = (u_x, u_y)$, and the phase velocity is $(U_0, k)/|k|$. The group velocities of acoustic waves are $v_1 = U_0 - ck/|k|$, $v_2 = U_0 + ck/|k|$, and the phase velocities are $(U_0, k)/|k| = c$ (and that is why, $G_1$ is called a slow wave and $G_2$ is a fast wave).

2.2. **Pre-shock plane waves.** In the pre-shock zone (superscript plus, $u_x > c$), all group velocities of plane waves are directed toward the shock, i.e. formally speaking, all plane waves are incident upon the shock (and it is actually so, as we saw in Part 1). Moreover, due to the properties of the pre-shock waves (Part 1) we conclude: any pre-shock initial data in form of plane wave $G_0 = β e_0j(ξ_0, η_0)e^{i(ξ_0x + η_0y)}$, $j = 1$ or 2 or 3 or 4, generates pre-shock plane wave $G = β e_0j(ξ_0, η_0)e^{i(ξ_0x + η_0y - ω_j t)}$. 


Let's consider post-shock zone: during their incidence upon the shock, i.e. with the same initial entropy-vorticity data are \( G_0 = (\beta_3 e_{03}(\xi, \eta) + \beta_4 e_{43}) e^{i(\xi, \eta, \omega_0) t} \), \( \omega_0 = \omega_3(\xi, \eta) \), \( \beta_3, \beta_4 = \text{const} \).

Further we treat pre-shock boundary value \( G_\Gamma^+ \) as given.

### 2.3. Post-shock plane waves. Damped waves.

Let's consider post-shock zone: superscript minus on default, \( u_x < c \). Here the entropy-vorticity wave has a form

\[
G = (\beta_3 e_{03}(\xi, \eta) + \beta_4 e_{43}) e^{i(\xi, \eta, \omega_0 - \omega t)} , \quad \omega_0 = \omega_3(\xi, \eta) , \quad \beta_3, \beta_4 = \text{const} .
\]

Its group velocity is directed away from the shock, i.e. formally, the entropy-vorticity wave isn't incident upon the shock (and it is so in fact, Part 1, Subsection 5.4). The initial entropy-vorticity data are

\[
G_0 = (\beta_3 e_{03}(\xi, \eta) + \beta_4 e_{43}) e^{i(\xi, \eta, \omega_0)}
\]

and the boundary value is

\[
G_\Gamma = (\beta_3 e_{03}(\xi, \eta) + \beta_4 e_{43}) e^{i(\eta, \omega_0 - \omega t)}.
\]

Let us consider initial wave with the initial data \( G_0 \). According to Part 1, Subsection 5.1, Eq. (5.1), it has a form

\[
\hat{G}_0 = \frac{\hat{G}_0}{2\pi i Q} = \frac{(\beta_3 e_{03}(\xi, \eta) + \beta_4 e_{43})}{2\pi i Q} \delta(\eta - \omega_0) I_1(\xi - \xi) .
\]

Separating singular and regular terms (Part 1, Subsection 4.2), we get

\[
\hat{G}_0 = \frac{(\beta_3 e_{03}(\xi, \eta)) + \beta_4 e_{43}}{2\pi i Q(\xi, \eta, \omega)} \delta(\eta - \omega_0) I_1(\xi - \xi) + \hat{\phi} ,
\]

where \( \hat{\phi} \) is regular with respect to \( \xi, \omega \), or after inverse Fourier transform

\[
G_0 = e^{i(\xi, \eta, \omega_0 - \omega t)} \theta(x - u_x t)(\beta_3 e_{03}(\xi, \eta) + \beta_4 e_{43}) + \varphi , \quad \omega_0 = \omega_3(\xi, \eta) ,
\]

where \( \theta(z) \) is a Heaviside function, and \( \varphi \to 0 \) when \( x \to \infty \) or \( t \to \infty \), i.e. \( \varphi \) is damped wave. So initial entropy-vorticity wave cannot be a plane wave, it has a plane principal part at \( x > u_x t \), is accompanied by damped wave and is a damped wave in the large.
Absolutely similar, the boundary (transmitted or reflected) wave, generated by the plane entropy-vorticity boundary value $G_T$ (1), is accompanied by damped wave, and its principal part has a form

$$G = e^{i(\xi_0 x + \eta_0 y - \omega_0 t)}\theta(u_x t - x)H, \quad H = \beta_3 e^{\xi_3 (\eta_0, \omega_0)} + \beta_4 e^{\xi_4}, \quad \xi_0 = \xi_3 (\eta_0, \omega_0),$$

i.e. it is a plane wave at $x < u_x t$, with the same wave amplitude $H$ as its boundary value.

The situation with the acoustic plane wave is somewhat more complicated. It has a form

$$G = \beta_1 e^{\xi_1 (\eta_0, \omega_0)}e^{i(\xi_0 x + \eta_0 y - \omega_- t)} + \beta_2 e^{\xi_2 (\eta_0, \omega_0)}e^{i(\xi_0 x + \eta_0 y - \omega_+ t)},$$

$$\omega_{\pm} = \omega_{\pm}(\xi_0, \eta_0), \quad \beta_{1,2} = \text{const}.$$

We recall the form of curves $\omega = \omega_{\pm}(\xi, \eta)$ at fixed $\eta$, these curves are depicted in fig.1.

Here the vector of group velocity $v = \nabla_k \omega$ is directed away from the shock/toward the shock, when tangent vector to curve is directed up/down with respect to $\omega$-axis. So the group velocity of slow acoustic wave $G = \beta_1 e^{i(\xi_0 x + \eta_0 y - \omega_- t)}$ is directed away from the shock at $\xi_0 < \xi^0_-$, is directed toward the shock at $\xi_0 > \xi^0_-$, and is parallel to the shock at $\xi_0 = \xi^0_-$.

Similarly, the group velocity of fast acoustic wave $G = \beta_2 e^{i(\xi_0 x + \eta_0 y - \omega_+ t)}$ is directed toward the shock at $\xi_0 < \xi^0_+$, is directed away from the shock at $\xi_0 > \xi^0_+$, and is parallel to the shock at $\xi_0 = \xi^0_+$.

Finally, due to the definition of roots $\xi_{1,2}(\eta, \omega)$ (Part 1, Section 3), we have: at given wavenumbers ($\xi_0, \eta_0$), the acoustic plane wave has the form $G_a = G_1 + G_2$, where group velocity of $G_1$ is directed towards the shock, group velocity of $G_2$ is
directed away from the shock and these waves have a form

\[ G_1 = \begin{cases} 
\beta_2 e_1(\eta_0, \omega_+) e^{i(\xi_1(\eta_0, \omega_+) x + \eta_0 y - \omega_+ t)}, & \xi \leq \xi_0^0, \\
0, & \xi \in (\xi_0^0, \xi_+^0), \\
\beta_1 e_1(\eta_0, \omega_-) e^{i(\xi_1(\eta_0, \omega_-) x + \eta_0 y - \omega_- t)}, & \xi \geq \xi_+^0, 
\end{cases} \]

(2)

\[ G_2 = \begin{cases} 
\beta_1 e_2(\eta_0, \omega_-) e^{i(\xi_2(\eta_0, \omega_-) x + \eta_0 y - \omega_- t)}, & \xi \leq \xi_-, \\
\beta_1 e_2(\eta_0, \omega_-) e^{i(\xi_2(\eta_0, \omega_-) x + \eta_0 y - \omega_- t)} + \beta_2 e_2(\eta_0, \omega_+) e^{i(\xi_2(\eta_0, \omega_+) x + \eta_0 y - \omega_+ t)}, & \xi \in (\xi_-, \xi_0^0), \\
\beta_2 e_2(\eta_0, \omega_+) e^{i(\xi_2(\eta_0, \omega_+) x + \eta_0 y - \omega_+ t)}, & \xi \geq \xi_0^0. 
\end{cases} \]

At \( \xi_0 = \xi_+^0(\eta_0) \) the group velocity of \( G_1 \) is parallel to the shock, and at \( \xi_0 \in (\xi_0^0, \xi_+^0) \) there are no acoustic plane waves directed towards the shock. Nevertheless in fact, as we know (Part 1), any acoustic wave, and \( G_2 \) as well, is incident upon the shock. The reason of paradox is that in this cases the agent of incidence is not plane wave, it is damped wave, see below.

The initial value of plane acoustic wave is

\[ G_0 = (\beta_1 e_{01}(\xi_0, \eta_0) + \beta_2 e_{02}(\xi_0, \eta_0)) e^{i(\xi_0 x + \eta_0 y)}, \]

or in spectral variables \( \hat{G}_0 = (\beta_1 e_{01}(\xi_0, \eta_0) + \beta_2 e_{02}(\xi_0, \eta_0)) \delta(\eta - \eta_0) I_1(\xi_0 - \xi) \).

Similarly to above, it may be shown, that initial acoustic wave has a form

\[ G_{0} = G_1 + \varphi, \]

where \( \varphi \) is damped wave, and \( G_1 \) has a form (2), i.e. it is just a summand with group velocity, directed towards the shock.

In turn, by virtue of change of variable formula (Appendix, Part 3, Subsection 6.3, Eq. (6.12)), and formula (\( \xi_+^0 \) = 1/s1 (Part 1, section 3, Eq. (3.2)), the agent of incidence \( \Phi(\eta, \omega) = q_1(\eta, \omega) \hat{G}_0(\xi_1(\eta, \omega), \eta) \) has a form \( \Phi = \Phi_1 + \hat{\varphi} \) where \( \hat{\varphi} \) is regular with respect to \( \omega \),

\[ \Phi_1 = \begin{cases} 
-\beta_2 s_1(\eta_0, \omega_+) \delta(\eta - \eta_0) I_1(\omega - \omega_+), & \xi \leq \xi_0^0, \\
0, & \xi \in (\xi_0^0, \xi_+^0), \\
-\beta_1 s_1(\eta_0, \omega_-) \delta(\eta - \eta_0) I_1(\omega - \omega_-), & \xi \geq \xi_+^0. 
\end{cases} \]

In terms of boundary wavenumbers \( (\eta_0, \omega_0) \), we have

\[ G_1 = \begin{cases} 
\beta_1 e_1(\eta_0, \omega_0) e^{i(\xi_1(\eta_0, \omega_0) x + \eta_0 y - \omega_0 t)}, & \omega_0 \notin (\omega_0^0, \omega_+^0), \\
0, & \omega_0 \in (\omega_0^0, \omega_+^0), 
\end{cases} \]

(3)

\[ G_1 = \begin{cases} 
-\beta_1 s_1(\eta_0, \omega_0) \delta(\eta - \eta_0) I_1(\omega - \omega_0), & \omega_0 \notin (\omega_0^0, \omega_+^0), \\
0, & \omega_0 \in (\omega_0^0, \omega_+^0). 
\end{cases} \]

(4)

\( \omega_0^0 = \omega_0^0(\eta_0), \beta = \text{const.} \) So at \( \omega_0 \in (\omega_0^0, \omega_+^0) \), the agent of incidence \( \Phi \) is damped wave (although \( \Phi \neq 0 \) for any initial acoustic wave, see Part 1, Subsection 5.4).

Concerning the boundary post-shock wave (transmitted or reflected), generated by the plane boundary value \( G_T = \beta e_2(\eta_0, \omega_0) e^{i(\eta_0 y - \omega_0 t)} \), it has (due to Part 1, Eq. (5.2)) a form \( G = G_2 + \varphi \), where \( \varphi \) is damped wave and

\[ G_2 = \beta e_2(\eta_0, \omega_0) e^{i(\xi_2(\eta_0, \omega_0) x + \eta_0 y - \omega_0 t)}. \]

(5)

General conclusion reads: both post-shock initial and boundary (transmitted or reflected) waves cannot have a form of plane wave, they are plane waves up to damped wave, i.e., up to vanishing at \( t \to \infty \) agent. This "asymptotic" plane waves
we call principal parts of waves. The principal parts of initial waves, generated by the initial plane entropy-vorticity data, are zero. The principal part of initial wave, generated by the initial plane acoustic data, is just a plane acoustic wave \( G_1 \) (3), with group velocity directed towards the shock; the principal part of agent of incidence \( \Phi \) has a form \( \Phi_1 \) (4). In particular, if the initial data generate a wave with group velocity, directed away from the shock, then the initial acoustic wave as well as the agent of incidence \( \Phi \) are damped waves. At the same time, the principal part of boundary (transmitted or reflected) acoustic wave \( G_2 \) has a form (5), its group velocity is directed away from the shock. At critical values \( \omega_0 = \omega_{0\pm} \) (\( \eta_\theta \)), waves \( G_1 \) and \( G_2 \) coalesce and their group velocity is parallel to the shock. Finally, principal parts of transmitted and reflected waves, generated by the plane boundary value \( G_T = H_j e^{i(\eta_\theta y - \omega_0 t)} \), \( H_j = \beta_j c_j (\eta_\theta, \omega_0) \), has a form \( G = H_j e^{i(\xi_j x + \eta_\theta y - \omega_0 t)} \), \( \xi_j = \xi_j (\eta_\theta, \omega_0) \), \( j = 2, 4 \), i.e. they are plane post-shock waves with the same wave amplitude \( H_j \), as boundary value.

All investigations of refraction/reflection of plane waves are based on the comparison of wave vectors' directions and wave amplitudes of incident and transmitted/reflected waves. So all results we obtain for principal parts only, up to damped summands.

3. TRANSITION TO DIMENSIONLESS PARAMETERS AND VARIABLES

To describe basic solution, we introduce the following dimensionless parameters:

\[
M^\pm = \frac{u^\pm}{c^\pm} \quad \text{(Mach numbers)}, \quad R = \frac{c^-}{c^+}, \quad D_0^\pm = \frac{rc^2}{\rho^0 T}, \quad B_0^\pm = \frac{c^2 \rho^0}{p^0}
\]

with relations

\[
M^+ > 1, \quad M^- < 1, \quad R > 0, \quad D_0^+ < 0, \quad B_0^+ > 0.
\]

Instead of \( \omega \), we introduce dimensionless variable

\[
\alpha = \frac{\omega - \eta u_y}{|\eta| \sqrt{(c^-)^2 - (u_x^2)^2}}.
\]

Further we denote \( s = \text{sng}(\eta) = \eta/|\eta| \) and \( M^- \) just as \( M \). Now we rewrite formulas of Part 1 in new terms.

**Roots and connecting quantities.** \( \xi_j^\pm (\eta, \omega) = |\eta| \xi_j^\pm (\alpha) \), \( j = 1, 4 \), where

\[
\xi_1^+ (\alpha) = \frac{M^+ \sqrt{1 - M^2} R \alpha + \sqrt{(1 - M^2) R^2 \alpha^2 + (M^+)^2 - 1}}{(M^+)^2 - 1},
\]

\[
\xi_2^+ (\alpha) = \frac{M^+ \sqrt{1 - M^2} R \alpha - \sqrt{(1 - M^2) R^2 \alpha^2 + (M^+)^2 - 1}}{(M^+)^2 - 1},
\]

\[
\xi_3,4^+ (\alpha) = \xi_0^+ (\alpha) = \frac{\sqrt{1 - M^2} R \alpha}{M^+};
\]

\[
\xi_1^- (\alpha) = -\frac{M \alpha + \sqrt{\alpha^2 - 1}}{\sqrt{1 - M^2}}, \quad \xi_2^- (\alpha) = -\frac{M \alpha - \sqrt{\alpha^2 - 1}}{\sqrt{1 - M^2}},
\]

\[
\xi_3,4^- (\alpha) = \xi_0^- (\alpha) = \frac{\alpha \sqrt{1 - M^2}}{M}.
\]
$Q_j^±(\eta, \omega) = c^±|\eta|Q_j^±(\alpha)$, $j = 1, 2$, where

\begin{align*}
Q_1^+(\alpha) &= \frac{\sqrt{1 - M^2} R\alpha + M^+ \sqrt{(1 - M^2) R^2 \alpha^2 + (M^+)^2 - 1}}{R((M^+)^2 - 1)}, \\
Q_2^+(\alpha) &= \frac{\sqrt{1 - M^2} R\alpha - M^+ \sqrt{(1 - M^2) R^2 \alpha^2 + (M^+)^2 - 1}}{R((M^+)^2 - 1)}, \\
Q_1^-(\alpha) &= -\frac{\alpha + M \sqrt{\alpha^2 - 1}}{\sqrt{1 - M^2}}, \quad Q_2^-(\alpha) = -\frac{\alpha - M \sqrt{\alpha^2 - 1}}{\sqrt{1 - M^2}}.
\end{align*}

Critical values

\[ \omega = \omega^0_\pm(\eta) \iff \alpha = \pm 1; \quad \xi^0_\pm(\eta) = \mp|\eta| \frac{M}{\sqrt{1 - M^2}}. \]

Here the function $\sqrt{\alpha^2 - 1}$ is analytic in upper half-plane $\text{Im} \alpha > 0$ and its branch is chosen under the condition $\sqrt{(i\beta)^2 - 1} = i\sqrt{\beta^2 + 1}$. So at real $\alpha$ and $|\alpha| \geq 1$ in fact $\sqrt{\alpha^2 - 1} = \text{sgn}(\alpha) \sqrt{\alpha^2 - 1}$.

**Boundary bases** (superscript $\pm$ on default).

\begin{align*}
e_1(\eta, \omega) &= e_1(\alpha, s) = \begin{pmatrix} 1 \\ -\xi_1(\alpha)/Q_1(\alpha) \\ -s/Q_1(\alpha) \\ 0 \end{pmatrix}, \\
e_2(\eta, \omega) &= e_2(\alpha, s) = \begin{pmatrix} 1 \\ -\xi_2(\alpha)/Q_2(\alpha) \\ -s/Q_2(\alpha) \\ 0 \end{pmatrix}, \\
e_3(\eta, \omega) &= e_3(\alpha, s) = \frac{1}{\sqrt{(\xi_0(\alpha))^2 + 1}} \begin{pmatrix} 0 \\ s \\ -\xi_0(\alpha) \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \\
g_1(\eta, \omega) &= g_1(\alpha, s) = \frac{1}{2} \begin{pmatrix} 1; -\xi_1(\alpha)/Q_1(\alpha); -s/Q_1(\alpha); -1 \end{pmatrix}, \\
g_2(\eta, \omega) &= g_2(\alpha, s) = \frac{1}{2} \begin{pmatrix} 1; -\xi_2(\alpha)/Q_2(\alpha); -s/Q_2(\alpha); -1 \end{pmatrix}, \\
g_3(\eta, \omega) &= g_3(\alpha, s) = \frac{1}{\sqrt{(\xi_0(\alpha))^2 + 1}} \begin{pmatrix} 0; s; -\xi_0(\alpha); 0 \end{pmatrix}, \quad g_4 = g_0_4 = \begin{pmatrix} 0; 0; 0; 1 \end{pmatrix}; \\
s_j(\eta, \omega) &= cs_j(\alpha), \quad j = 1, 4, \quad \text{where } s_j^\pm(\alpha) = M^\pm, \quad j = 3, 4,
\end{align*}

\begin{align*}
s_1^+(\alpha) &= \frac{\sqrt{(1 - M^2) R^2 \alpha^2 + (M^+)^2 - 1}}{RQ_1^+(\alpha)}, \quad s_2^+(\alpha) = -\frac{\sqrt{(1 - M^2) R^2 \alpha^2 + (M^+)^2 - 1}}{RQ_2^+(\alpha)}, \\
s_1^-(\alpha) &= \frac{\sqrt{1 - M^2} \sqrt{\alpha^2 - 1}}{Q_1^-(\alpha)}, \quad s_2^-(\alpha) = -\frac{\sqrt{1 - M^2} \sqrt{\alpha^2 - 1}}{Q_2^-(\alpha)}.
\end{align*}

**Boundary matrices and different functions.**

For matrices $A_0$, $A$, we mean superscript $\pm$.

\[ A_0 = cA_0_1 = c \begin{pmatrix} M & 1 & 0 & 0 \\ 1 & M & 0 & -1 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{pmatrix} \]
\[ A = A_1 \cdot A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ M & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & M & 0 & 1/D_0 \end{pmatrix} . \]

For functions \( \hat{F}_0 \) and \( Y_j, j = 1, 4 \), we mean \( \rho = \rho^-, \, c = c^- \), \( M = M^- \).

\[ \hat{F}_0(\eta, \omega) = |\eta|\rho c \left( \frac{M^+}{R} - M \right) A_1 \hat{F}_0(\alpha, s) \]

\[ = |\eta|\rho c \left( \frac{M^+}{R} - M \right) A_1 \begin{pmatrix} R\sqrt{1 - M^2\alpha/M^+} \\ 0 \\ -Ms \\ -M\sqrt{1 - M^2} \end{pmatrix} ; \]

\[ Y_j(\eta, \omega) = |\eta|\rho c \left( \frac{M^+}{R} - M \right) Y_j(\alpha, s) , \]

\[ Y_j(\alpha, s) = g_{ij}^- (\alpha, s) A_2^{-1} \hat{F}_0(\alpha, s) , \quad j = 1, 4 \]

By means of direct calculations, we find

\[ Y_1(\alpha, s) = \frac{Y_{11}^0 a^2 + Y_{12}^0 \alpha \sqrt{\alpha^2 - 1} + Y_{13}^0}{2Q_1(\alpha)} = Y_1(\alpha) , \]

\[ Y_2(\alpha, s) = \frac{Y_{11}^0 a^2 - Y_{12}^0 \alpha \sqrt{\alpha^2 - 1} + Y_{13}^0}{2Q_2(\alpha)} = Y_2(\alpha) , \]

where

\[ Y_{11}^0 = \frac{R}{M^+} \left( M D_0 \left( M - \frac{M^+}{R} \right) - 1 - M^2 \right) , \]

\[ Y_{12}^0 = \frac{RM}{M^+} \left( M D_0 \left( M - \frac{M^+}{R} \right) - 2 \right) , \quad Y_{13}^0 = M ; \]

\[ Y_3(\alpha, s) = \frac{\alpha s \sqrt{1 - M^2(1 - MR/M^+)} \sqrt{\xi_0^- (\alpha)^2 + 1}}{a^2} \]

\[ Y_4(\alpha, s) = M D_0 (MR/M^+ - 1) \sqrt{1 - M^2} \alpha = Y_4(\alpha) . \]

4. **Common form of refracted and reflected waves in the stability case**

Since \( Y_1(\eta, \omega) \neq 0 \), then the formulas for generated by pre-shock wave shock disturbance and boundary values of transmitted waves (Part 1, subsection 5.3, Eq. (5.5)) assume the form

\[ \hat{f}^+ = -i g_{1}^- (\eta, \omega) Y_1(\eta, \omega) (A^-)^{-1} A_0^+ \hat{G}_T^+(\eta, \omega) , \]

\[ \hat{G}_T^+ = (A_0^+)^{-1} \left[ E - (A^-)^{-1} \hat{F}_0 \frac{g_{1}^- (\eta, \omega)}{Y_1(\eta, \omega)} (A^-)^{-1} A_0^+ \hat{G}_T^+(\eta, \omega) . \]
The decomposition of transmitted wave into the sum of acoustic \( \hat{G}_{2T}^+ = \gamma_2 e_2^+ \), vorticity \( \hat{G}_{3T}^+ = \gamma_3 e_3^- \) and entropy \( \hat{G}_{4T}^+ = \gamma_4 e_4^- \) waves, where \( \gamma_j = g_j A_0 \hat{G}_j^* / s_j \), leads to the formulas

\[
\gamma_j = \frac{g_j (\eta, \omega)}{s_j (\eta, \omega)} - \frac{Y_j (\eta, \omega)}{s_j (\eta, \omega) Y_1 (\eta, \omega)} \left( A^- \right)^{-1} A^+_0 \hat{G}_j^+ (\eta, \omega), \quad j = 2, 3, 4.
\]

In turn, the formulas for generated by post-shock acoustic wave shock disturbance and boundary values for reflected waves (Part 1, Subsection 5.4, Eqs. (5.8)-(5.9)), assume the form

\[
\hat{f}^- (\eta, \omega) = -i \Phi (\eta, \omega) Y_1 (\eta, \omega), \quad \hat{G}_{jT}^- (\eta, \omega) = \gamma_j e_j^- - \frac{Y_j (\eta, \omega)}{s_j (\eta, \omega)}, \quad j = 2, 3, 4,
\]

where the agent of incidence \( \Phi (\eta, \omega) = g_1^- (\eta, \omega) \hat{G}_0^- (\xi_1^- (\eta, \omega), \eta) \).

### 5. Pre-shock plane waves and their refraction

#### 5.1. General form of incident and transmitted waves.

Here we treat as given the incident pre-shock plane wave

\[
G^+ = \sum_{j=1}^4 \beta_j e_j^+ (\eta_0, \omega_0) e^{i (\xi_j^+ (\eta_0, \omega_0) x + \eta_0 y - \omega_0 t)}, \quad \beta_j = \text{const}.
\]

Its boundary value is

\[
G_T^+ = \sum_{j=1}^4 \beta_j e_j^+ (\eta_0, \omega_0) e^{i (\eta_0 y - \omega_0 t)},
\]
or in spectral variables

\[
\hat{G}_T^+ = \sum_{j=1}^4 \beta_j e_j^+ (\eta_0, \omega_0) \delta (\eta - \eta_0) I_1 (\omega - \omega_0).
\]

Then, separating the singular and regular parts in Eqs. (7), (8), we obtain the general form of the principal (up to regular with respect to \( \omega \) term \( \hat{f} \)) parts of shock disturbance

\[
\hat{f}^+ = \gamma_0 \delta (\eta - \eta_0) I_1 (\omega - \omega_0) + \hat{f}^+, \quad \gamma_0 = \gamma_0 (\beta_1, \beta_2, \beta_3, \beta_4),
\]
and boundary value for transmitted post-shock wave

\[
\hat{G}_{T^-}^+ = \sum_{k=2}^4 \gamma_k e_k^- (\eta_0, \omega_0) \delta (\eta - \eta_0) I_1 (\omega - \omega_0) + \hat{f}^+, \quad \gamma_k = \gamma_k (\beta_1, \beta_2, \beta_3, \beta_4), \quad k = 2, 3, 4.
\]

In physical variables, we obtain shock disturbance in a form \( f^+ = \gamma_0 e^{i (\eta_0 y - \omega_0 t)} + \hat{f}^+ \), and, by virtue of common refracted/reflected waves representation (Part 1, Eq. (5.2)), the transmitted wave has a form

\[
G^- = \sum_{k=2}^4 \gamma_k e_k^- (\eta_0, \omega_0) e^{i (\xi_k^- (\eta_0, \omega_0) x + \eta_0 y - \omega_0 t)} + \hat{f}^+
\]

where \( \hat{f} \) is damped wave.

We consider coefficients \( \gamma_0, \gamma_k, k = 2, 3, 4 \) a bit later, in subsection 5.3.
Hereafter we denote
\[ \alpha_0 = \frac{\omega_0 - \eta \omega_y}{|\eta| \sqrt{(c')^2 - (u_x')^2}}, \quad s_0 = \text{sgn}(\eta_0) = \frac{\eta_0}{|\eta_0|}. \]

Between critical values, i.e. at \( |\alpha_0| < 1 \), wavenumber \( \xi_2^- \) is complex, \( \xi_2^- = r_1 + ir_2 \), and \( r_2 = |\eta_0| \sqrt{1 - \alpha_0^2} / \sqrt{1 - M^2} > 0 \) (Section 3), and so the principal part of transmitted acoustic wave has a form
\[ G_2^- = \gamma_2 e_2^j (\alpha_0, s_0) e^{-r_2 x} e^{i(r_1 x + \theta_0 y - \omega_0 t)}. \]

Before it was stated the non-existence of transmitted acoustic wave for this situation (e.g. [14]). In fact, as we see, the transmitted acoustic wave exists, but its principal part is the evanescent wave with respect to \( x \). It strongly reminds the case of total internal reflection in optics. Anyway, usual refraction’s description, such as comparison of the incident and transmitted wave vectors and amplitudes, is hardly useful here, and physical sense of this case needs further specification. Further we consider the refracted acoustic wave only out of critical values, at \( |\alpha_0| \geq 1 \).

### 5.2. Correspondence between angles of incidence and refraction

The wave vectors of incident \( G^+ \) (10) and transmitted \( G^- \) (12) waves in dimensionless variables are \( k_j^\pm = |\eta_0|(\xi_j^\pm(\alpha_0), s_0) \), \( j = \Gamma, \Omega \) (wave vectors for vorticity wave \( k_j^\pm \) and entropy wave \( k_j^\mp \) are the same). Following to previous works (e.g. [5]), we introduce the angle of incidence \( \theta^+ \) and the angle of refraction \( \theta^- \), \( \theta^\pm \in [-\pi, \pi] \), as the angles between wave vector \( k_j^\pm \) and \( x \)-axis (and so \( \theta^+ = 0 \) is normal incidence): \( \theta^\pm = s_0 \cdot \text{arccotan}(\xi_j^\pm / |\eta_0|) \). For the incident/transmitted waves in dimensionless variables we have \( \theta_j^\pm = s_0 \cdot \text{arccotan}(\xi_j^\pm(\alpha_0)), \ j = \Gamma, \Omega \).

In the equations \( \theta_j^+ = s_0 \cdot \text{arccotan}(\xi_j^+(\alpha_0)), \ j = \Gamma, \Omega \), using the formulas of Section 3, we may express \( \alpha_0 \) via any angle of incidence:
\[ \alpha_0 = \frac{M^+ \cos \theta_j^+ - 1}{|\sin \theta_j^+| R \sqrt{1 - M^2}} = \frac{M^+ \cos \theta_j^+ + 1}{|\sin \theta_j^+| R \sqrt{1 - M^2}} = \frac{M^+ \cos \theta_j^-}{|\sin \theta_j^-| R \sqrt{1 - M^2}}. \]

Then we substitute these expressions in formulas for angles of refraction
\[ \theta_k^\pm = s_0 \cdot \text{arccotan}(\xi_k^\pm(\alpha_0)), \ k = \Gamma, \Omega \] and so obtain the connections between incident and refracted angles (Snell’s laws):

- for transmitted entropy-vorticity wave
  \[ \cos \theta_j^- = \begin{cases} \frac{M^+ \cos \theta_j^+}{\sqrt{(M^+ \cos \theta_j^+)^2 + \sin^2 \theta_j^+ R^2 M^2}}, & \text{at incident entropy-vorticity wave}, \\ \frac{M^+ \cos \theta_j^- - 1}{\sqrt{(M^+ \cos \theta_j^- - 1)^2 + \sin^2 \theta_j^- R^2 M^2}}, & \text{at incident slow acoustic wave}, \\ \frac{M^+ \cos \theta_j^+ + 1}{\sqrt{(M^+ \cos \theta_j^+ + 1)^2 + \sin^2 \theta_j^+ R^2 M^2}}, & \text{at incident fast acoustic wave}; \end{cases} \]

for transmitted acoustic wave we should take into account critical values: condition \( |\alpha_0| \geq 1 \) means \( |\theta_j^+| \notin [\theta_j^-, \theta_j^+] \), where \( \theta_j^\pm \) are different for different kinds of incident waves.
at incident entropy-vorticity wave
\[
\cos \theta_{-} = \frac{-MR^2 \sin^2 \theta_{3}^+ + M^+ |\cos \theta_{3}^+| \sqrt{(M^+ \cos \theta_{3}^+)^2 - R^2(1 - M^2) \sin^2 \theta_{3}^+}}{(M^+ \cos \theta_{3}^+)^2 + \sin^2 \theta_{3}^+ R^2},
\]
\[
\theta_{3}^+ = \arccotan \frac{\mp \sqrt{1 - M^2} R}{M^+};
\]
at incident slow acoustic wave
\[
\cos \theta_{-} = \text{sgn}(M^+ \cos \theta_{1}^+ - 1) \times \frac{-MR^2 \sin^2 \theta_{1}^+ + |M^+ \cos \theta_{1}^+ - 1| \sqrt{(M^+ \cos \theta_{1}^+ - 1)^2 - R^2(1 - M^2) \sin^2 \theta_{1}^+}}{(M^+ \cos \theta_{1}^+ - 1)^2 + \sin^2 \theta_{1}^+ R^2},
\]
\[
\theta_{1}^\pm = \arccotan \frac{\mp M^+ \sqrt{1 - M^2} R + (1 - M^2) R^2 + (M^+)^2 - 1}{(M^+)^2 - 1};
\]
at incident fast acoustic wave
\[
\cos \theta_{-} = \text{sgn}(M^+ \cos \theta_{2}^+ + 1) \times \frac{-MR^2 \sin^2 \theta_{2}^+ + |M^+ \cos \theta_{2}^+ + 1| \sqrt{(M^+ \cos \theta_{2}^+ + 1)^2 - R^2(1 - M^2) \sin^2 \theta_{2}^+}}{(M^+ \cos \theta_{2}^+ + 1)^2 + \sin^2 \theta_{2}^+ R^2 M^2},
\]
\[
\theta_{2}^\pm = \arccotan \frac{\mp M^+ \sqrt{1 - M^2} R - (1 - M^2) R^2 + (M^+)^2 - 1}{(M^+)^2 - 1}.
\]

The comparison with formulas from the classical work by [5] shows: our formulas coincide with formulas of [5] for transmitted entropy-vorticity wave and for transmitted acoustic wave at \(|\theta_{j}^\pm| < \theta_{j}^\pm (\alpha_{0} > 1)\) and differ at \(|\theta_{j}^\pm| > \theta_{j}^\pm (\alpha_{0} < -1)\).

5.3. Matrix of generation coefficients. Now let’s turn to the generation coefficients, i.e. to the ratio of wave amplitudes of transmitted and incident waves. Boundary value of the incident wave in spectral variables has a form (11)
\[
\hat{G}_{T}^+(\eta, \omega) = \sum_{j=1}^{4} \beta_j e_j^+(\eta_0, \omega_0) \delta(\eta - \eta_0) I_1(\omega - \omega_0).
\]

We write it in a form \(\hat{G}_{T}^+ = (\Psi^+ \beta) \delta(\eta - \eta_0) I_1(\omega - \omega_0)\), where
\[
\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}
\]
is vector of density dimension, \(\Psi^+ = \Psi^+(\alpha_0, s_0)\) is dimensionless matrix:
\[
\Psi^+(\alpha, s) = (e_1^+(\alpha, s) ; e_2^+(\alpha, s) ; e_3^+(\alpha, s) ; e_4^+(\alpha, s)) =
\]
Then its principal part (under the same notation principal parts the same notations terms (Part 1, Subsection 4.2), but write the result more concretely (we save for principal parts the same notations $f^+$ and $G^-)$:

$$f^+ = \gamma_0 e^{i(\eta_0 y - \omega_0 t)} + i\gamma_0 = -\frac{g_1(\eta_0, \omega_0)}{Y_1(\eta_0, \omega_0)} (A^-)^{-1} A^+ A_0^+ (\Psi^+ \beta) ;$$

$$G^- = \sum_{j=2}^{4} \gamma_j e_j(\eta_0, \omega_0) e^{i(\xi_j(\eta_0, \omega_0) y + \eta_0 y - \omega_0 t)} ,$$

$$\gamma_j = \frac{g_j(\eta_0, \omega_0)}{s_j(\eta_0, \omega_0)} - \frac{Y_j(\eta_0, \omega_0)}{s_j(\eta_0, \omega_0) Y_1(\eta_0, \omega_0) g_1(\eta_0, \omega_0)} \left( A^- \right)^{-1} A^+ A_0^+ (\Psi^+ \beta), \quad j = 2, 4 .$$

Here wave amplitudes $\gamma_j, j = 2, 4$ have density dimension, while $\gamma_0$ has other dimension (length). To make the same dimension (for taking ratio), we suppose to consider quantity

$$f_1^+ = \rho^-((f^+)\rho_y + u_y (f^+)\rho_y) / c^-$$ or in spectral variables $\tilde{f}_1^+ = -i\rho^- (\omega - \eta u_y) \tilde{f}^+ / c^-$. Then its principal part (under the same notation $\tilde{f}_1^+$) is

$$f_1^+ = \gamma_1 e^{i(\eta_0 y - \omega_0 t)} + i\gamma_1 = -\frac{\rho^- (\omega_0 - \eta_0 \eta_0) g_1(\eta_0, \omega_0)}{c^- Y_1(\eta_0, \omega_0)} (A^-)^{-1} A^+ A_0^+ (\Psi^+ \beta) .$$

Going to the dimensionless variables, we obtain

$$\gamma_1 = -\frac{\alpha_0 \sqrt{1 - M^2} g_1(\alpha_0, s_0)}{(A_2^-)^{-1} A_R A_2^+ A_0^+ (\Psi^+ \beta) ,$$

$$\gamma_j = \frac{g_j(\alpha_0, s_0)}{s_j(\alpha_0)} - \frac{Y_j(\alpha_0, s_0)}{s_j(\alpha_0) Y_1(\alpha_0) g_1(\alpha_0, s_0)} \left( A_2^- \right)^{-1} A_R A_2^+ A_0^+ (\Psi^+ \beta) , \quad j = 2, 4 ,$$

where $\rho = \rho^-, c = c^-, M = M^-,$

$$A_R = \frac{1}{R} (A_1^-)^{-1} A_1^+ = \begin{pmatrix} 1/R & 0 & 0 & 0 \\ 0 & 1/R^2 & 0 & 0 \\ 0 & 0 & 1/R^2 & 0 \\ 0 & 0 & 0 & 1/R^3 \end{pmatrix} ,$$

or finally in matrix form

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \Psi \beta , \quad \Psi = (A_2^-)^{-1} A_R A_2^+ A_0^+ \Psi^+ ,$$
where

\[
\Psi^{-} = \begin{pmatrix}
\frac{\alpha_0 \sqrt{1 - M^2}}{(M^*/R - M) Y_1(\alpha_0)} g_1^-(\alpha_0, s_0) \\
\frac{g_2^-(\alpha_0, s_0)}{s_2^-(\alpha_0)} - \frac{Y_2(\alpha_0)}{s_2^- (\alpha_0) Y_1(\alpha_0)} g_1^-(\alpha_0, s_0) \\
\frac{g_3^-(\alpha_0, s_0)}{M} - \frac{s_0 Y_3(\alpha_0)}{M Y_1(\alpha_0)} g_1^-(\alpha_0, s_0) \\
\frac{g_4^-}{M} - \frac{Y_4(\alpha_0)}{M Y_1(\alpha_0)} g_1^-(\alpha_0, s_0)
\end{pmatrix} = \begin{pmatrix}
-\frac{\alpha_0 \sqrt{1 - M^2}}{(M^*/R - M) Y_1(\alpha_0)} & 0 & 0 & 0 \\
-\frac{Y_2(\alpha_0)}{s_2^- (\alpha_0) Y_1(\alpha_0)} & \frac{1}{s_2^- (\alpha_0)} & 0 & 0 \\
-\frac{s_0 Y_3(\alpha_0)}{M Y_1(\alpha_0)} & 0 & \frac{1}{M} & 0 \\
-\frac{Y_4}{M Y_1(\alpha_0)} & 0 & 0 & \frac{1}{M}
\end{pmatrix}
\]

As vectors \(\beta\) and \(\gamma\) have the same dimension (density), then elements of matrix \(\Psi\) are dimensionless ratios of wave amplitudes of transmitted (principal parts) and incident plane waves. So matrix \(\Psi\) is actually the matrix of generation (transmission) coefficients, see e.g. [5].

When the angles of incidence lie between critical values, i.e. \(|\alpha_0| < 1\), then second row of matrix \(\Psi^{-}\) (and, consequently, of matrix \(\Psi\)) became complex. It is just the case of evanescent transmitted acoustic wave, see above. So we consider the generation coefficients for transmitted acoustic wave (second row of \(\Psi\)) only at \(|\alpha_0| \geq 1\).

The second problem is that at critical values \(\alpha_0 = \pm 1\), quantity \(s_2^- (\alpha_0) = 0\), and so we have a zero denominator in second row of matrix \(\Psi^{-}\). In particular, this singularity is a reason of so-called abnormal amplification near the critical angles of incidence, see [5, 14]. But if we introduce the vector

\[
\tilde{g}_2(\alpha, s) = \frac{1}{Q_1(\alpha)} \left( 0; \sqrt{1 - M^2 \alpha}; M s; 0 \right), \quad s = \pm 1,
\]
and function

\[ \tilde{Y}_2(\alpha) = -\frac{MR(1 - M^2)\alpha^2/M^+ + M^2}{Q_1(\alpha)}, \]

then

\[ g_2 = g_1 - \frac{s_2}{1 - M^2} \tilde{Y}_2, \quad Y_2(\alpha) = Y_1(\alpha) - \frac{s_2}{1 - M^2} \tilde{Y}_2, \]

and second row of matrix \( \Psi^- \) assume the form

\[ \frac{g_1 - s_2}{s_2} \tilde{Y}_2 = \frac{1}{1 - M^2} \left[ \frac{\tilde{Y}_2 g_1 - \tilde{g}_2}{Y_1 g_1 - \tilde{g}_2} \right], \]

so finally we obtain

\[
\Psi^- = \begin{pmatrix}
-\frac{\alpha_0 \sqrt{1 - M^2}}{(M^+/R - M)Y_1(\alpha_0)} & 0 & 0 & 0 \\
\frac{\tilde{Y}_2(\alpha_0)}{(1 - M^2)Y_1(\alpha_0)} & -\frac{1}{1 - M^2} & 0 & 0 \\
-s_0 Y_2(\alpha_0) & 0 & 1 & 0 \\
-s_0 Y_1(\alpha_0) & 0 & 0 & \frac{1}{M} \\
\end{pmatrix}
\times
\begin{pmatrix}
\frac{1}{2} & -\frac{\xi_1(\alpha_0)}{2Q_1(\alpha_0)} & -\frac{s_0}{2Q_1(\alpha_0)} & -\frac{1}{2} \\
0 & \frac{\alpha_0 \sqrt{1 - M^2}}{Q_1(\alpha_0)} & \frac{Ms_0}{Q_1(\alpha_0)} & 0 \\
0 & \frac{s_0}{\sqrt{(\xi_0(\alpha_0))^2 + 1}} & -\frac{\xi_0(\alpha_0)}{\sqrt{(\xi_0(\alpha_0))^2 + 1}} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

and this representation shows in fact matrix \( \Psi \) has not singularity at critical values.

So we have a representation of matrix of transmission coefficients \( \Psi \) and may calculate all sixteen coefficients, using some mathematical programmes. The final formulas are rather complicated and unwieldy.

5.4. Generation coefficients for an ideal gas, amplification. Let’s consider an ideal gas. It is well known (see e.g. \([1, 5]\)) that equation of state for ideal gas together with Rankine-Hugoniot conditions allow to express all dimensionless parameters through polytropic coefficient \( \gamma \) and pre-shock Mach number \( M^+ \):

\[ B_0^\pm = \gamma, \quad D_0^\pm = 1 - \gamma, \quad R = \frac{\sqrt{2(\gamma M^+)^2 - (\gamma - 1)((\gamma - 1)(M^+)^2 + 2)}}{(\gamma + 1)M^+}, \]

\[ M^- = \sqrt{\frac{(\gamma - 1)(M^+)^2 + 2}{2(\gamma M^+)^2 - (\gamma - 1)}}. \]

We assume \( \gamma = 5/3 \), carry out the numerical calculations of generation coefficients, and compare them to the results of classical work \([3]\). Only two small remarks should
be made for correct comparison: first, we need to express generation coefficients not through the dimensionless variable \( \alpha \), but through the angles of incidence \( \theta_j \), \( j = 1, 3 \), measured in degrees; second, in work [5] they consider the ratios of quantities of pressure dimension, so to be more exact, we need actually calculate matrix

\[
\Psi = (H_0^-)^{-1} \Psi^-(A_z^-)^{-1} \tilde{A}_R A_2^+ A_0^+ \psi^+ H_0^+,
\]

where

\[
H_0^\pm = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & B_0^\pm & 0 \\
0 & 0 & 0 & -B_0^\pm \\
\end{pmatrix},
\tilde{A}_R = R^2 A_R = \begin{pmatrix}
R & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1/R \\
\end{pmatrix}.
\]

For comparison we show only plots of coefficients given in mentioned paper, they are depicted in fig. 2, 3, 4.

Compare these plots with plots of [5], we can see the principal distinction: in work [5] they differ cases of different tangent velocity of basic flow \( u_y \): case \( u_y = 0 \) (normal incidence of basic flow), and two more cases of oblique incidence, when post-shock basic flow is subsonic \( u_y^2 + (u_x^2 - c^2)^2 < (c^-)^2 \), and supersonic \( u_y^2 + (u_x^2 - c^2)^2 > (c^-)^2 \), and give different plots for these cases. But as we see, generation coefficients depend only on dimensionless variable \( \alpha \), and don’t depend on \( u_y \), in particular don’t depend on cases of subsonic/supersonic basic flow behind the shock. In other respects, our plots are more or less like plots of [5], with some differences at \( |\theta| > \theta_j^+ \).

Apart we consider the amplification, i.e. the degree of growth of transmission coefficients when the pre-shock Mach number \( M^+ \to \infty \). In representation (13), matrix \( \Psi_0^- = (H_0^-)^{-1} \Psi^-(A_z^-)^{-1} \) has finite limit at \( M^+ \to \infty \), and matrix \( \Psi_0^+ = \)
\[ = \tilde{A}_k A^+_k A^+_0 \Psi^+ \Phi_0^+ \] has a representation

\[ \Psi_0^+ = (M^+)^2 \Psi_2 + O(M^+) \]

where

\[
\Psi_2 = \begin{pmatrix}
R_0 - \frac{\sqrt{1 - M^2_0 R_0^2 \alpha^2} + 1}{2 \sqrt{1 - M^2_0 R_0^2 \alpha^2}} & R_0 + \frac{\sqrt{1 - M^2_0 R_0^2 \alpha^2} + 1}{2 \sqrt{1 - M^2_0 R_0^2 \alpha^2}} & 0 & -\gamma R_0 \\
1 - \frac{\sqrt{1 - M^2_0 R_0^2 \alpha^2} + 1}{2 \sqrt{1 - M^2_0 R_0^2 \alpha^2}} & 0 & 0 & 0 \\
0 & \frac{\sqrt{1 - M^2_0 R_0^2 \alpha^2} + 1}{2 \sqrt{1 - M^2_0 R_0^2 \alpha^2}} & \sqrt{1 - M^2_0 R_0^2 \alpha^2} & 0 \\
0 & 0 & \frac{\sqrt{1 - M^2_0 R_0^2 \alpha^2} + 1}{2 \sqrt{1 - M^2_0 R_0^2 \alpha^2}} & \sqrt{1 - M^2_0 R_0^2 \alpha^2}
\end{pmatrix}
\]

Matrix \( \Psi_2 \) has rank two and

\[ \Psi_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \Psi_2 \begin{pmatrix} \gamma/2 \\ \gamma/2 \\ 0 \\ 1 \end{pmatrix} = 0. \]

So general conclusion reads: amplification has second degree on the whole, but for vorticity wave and for linear combination: sum of two acoustic waves multiply by \( \gamma/2 \) plus entropy wave, amplification has a first degree and these degrees of amplification don’t depend on angle of incidence.

In works [5, 14], the amplification of third degree \( O((M^+)^3) \) (abnormal amplification) for close to critical angles of incidence is stated. The seeming singularity of transmission coefficients at critical values misleads authors. In fact, as we see, there is no any singularity in transmission coefficients and so abnormal amplification doesn’t exist.

We illustrate last conclusion by the numerical calculations. In fig.5, we depict in logarithmic scale for both axes the generation coefficient for transmitted acoustic wave, caused by incident fast acoustic wave, as a function of \( M^+ \) at critical angle of incidence, and zero angle of incidence (normal incidence). As we see, asymptotically both plots become parallel lines, i.e. they have the same degree of amplification (two). In fig.6, we depict plots of generation coefficient for transmitted acoustic wave caused by incident sound wave and by vorticity wave.
wave, caused by incident fast acoustic wave, and caused by incident vorticity wave. We clearly see the difference between degrees of amplification: second for incident sound wave and first for incident vorticity wave.

6. POST-SHOCK PLANE WAVES AND REFLECTION

Here we mean superscript minus on default. As we see above (Subsection 2.3), initial post-shock wave cannot be a plane wave, so as incident plane wave we may consider only the principal part of initial post-shock wave, generated by the initial plane data, it is wave \( G_1 \) in form (3). Since \( G_1 \equiv 0 \) at wavenumbers \((\eta_0, \omega_0)\), located between critical values (i.e. at \(|\alpha_0| < 1\) in dimensionless variables), then we consider only wavenumbers out of critical values (\(|\alpha_0| > 1\)).

Using the principal part \( \Phi_1 \) in form (4) of the agent of incidence \( \Phi \), we separate singular and regular parts in the formulas (9) for the shock disturbance \( \hat{f} \) and for the boundary values of reflected waves \( G_j \), \( j = \overline{2,4} \), and get expressions for their principal parts (under the same notations \( \hat{f}, G_j \)):

\[
\hat{f} = i\beta s_1(\eta_0, \omega_0) Y_1(\eta_0, \omega_0) \delta(\eta - \eta_0) I_1(\omega - \omega_0),
\]

\[
G_j = \beta_j e_j(\eta_0, \omega_0), \quad \gamma_j = s_1(\eta_0, \omega_0) Y_j(\eta_0, \omega_0) Y_1(\eta_0, \omega_0) \delta(\eta - \eta_0) I_1(\omega - \omega_0), \quad j = \overline{2,4}.
\]

Hence the principal parts of reflected waves themselves have a form (see Subsection 2.3)

\[
G_j = \beta_j e_j(\eta_0, \omega_0) e^{i(\xi_j x + \eta_j y - \omega_j t)}, \quad \xi_j = \xi_j(\eta_0, \omega_0), \quad j = \overline{2,4}.
\]

First we compare angles of incidence and reflection. The wave vector for incident wave \( G_1 \) in form (3) is \( k_1 = (\xi_1(\eta_0, \omega_0), \eta_0) \), and the wave vectors for reflected waves are \( k_j = (\xi_j(\eta_0, \omega_0), \eta_0), \quad j = \overline{2,4} \), or in dimensionless variables \( k_j = |\eta_0|(\xi_j(\alpha_0), s_0), \quad j = \overline{1,3} \). Like above, we introduce the angle of incidence \( \theta_1 \), and the angles of reflection \( \theta_j, \quad j = \overline{2,3} \) in form \( \theta_j = s_0 \arccotan(\xi_j(\alpha_0)), \quad j = \overline{1,3} \). Then we express \( \alpha_0 \) through angle of incidence \( \theta_1 \); condition \(|\alpha_0| > 1\) means \( |\theta_1| \notin (\theta_-, \theta_+) \), where \( \theta_\pm = \arccotan(\mp M/\sqrt{1 - M^2}) \) are critical post-shock angles of incidence, and

\[
\alpha_0 = \begin{cases} M \cos \theta_1 + 1, & |\theta_1| > \theta_+, \\ M \cos \theta_1 - 1, & |\theta_1| < \theta_- \end{cases}
\]

We substitute \( \alpha_0 \) into the formulas for angles of reflection and obtain the correspondence between angles of incidence and reflection (laws of reflection): for the reflected sound wave

\[
\cos \theta_2 = \begin{cases} \frac{2M - \cos \theta_1(1 + M^2)}{1 + M^2 - 2M \cos \theta_1}, & |\theta_1| < \theta_- \\ \frac{2M + \cos \theta_1(1 + M^2)}{1 + M^2 + 2M \cos \theta_1}, & |\theta_1| > \theta_+ \end{cases}
\]
for the reflected entropy-vorticity wave

\[
\cos \theta_3 = \begin{cases} 
\frac{M \cos \theta_1 - 1}{\sqrt{2(1 + M^2 - 2M \cos \theta_1)}}, & |\theta_1| < \theta_- , \\
\frac{M \cos \theta_1 + 1}{\sqrt{2(1 + M^2 + 2M \cos \theta_1)}}, & |\theta_1| > \theta_+ .
\end{cases}
\]

These formulas strongly differ from those of [5], but it is evident misunderstanding: in mentioned work they simply make inaccurate conclusion from absolutely correct equation.

Now let’s turn to reflection coefficients. We naturally take the quantity \( \beta \) (dimension of density) as wave amplitude for incident wave \( G_1 \) in form (3). In order to reduce shock disturbance \( f \) to the same dimension we, like for refraction, take principal part of \( f_1 = \rho(f'_1 + u_y f''_1)/c \) or in spectral variables due to (14)

\[
\hat{f}_1 = -\frac{\rho(\omega_0 - \eta_0 u_y)}{c} \hat{f} = \frac{\rho(\omega_0 - \eta_0 u_y) s_1(\eta_0, \omega_0)}{c Y_1(\eta_0, \omega_0)} \delta(\eta - \eta_0) I_1(\omega - \omega_0),
\]

so reflection coefficient for shock disturbance is

\[
\gamma_1 = \frac{\rho(\omega_0 - \eta_0 u_y) s_1(\eta_0, \omega_0)}{c Y_1(\eta_0, \omega_0)} = \frac{\sqrt{1 - M^2} \alpha_0 s_1(\alpha_0)}{(M^+ / R - M) Y_1(\alpha_0)},
\]

The formulas for other reflection coefficients follow from (15):

\[
\gamma_j = \frac{s_1(\alpha_0) Y_j(\alpha_0, s_0)}{s_j(\alpha_0) Y_1(\alpha_0)}, \quad j = 2, 4.
\]

Finally, by means of direct calculations and keeping in mind the branch of \( \sqrt{\alpha^2 - 1} \) (see Section 3), we obtain

\[
\gamma_1 = \frac{2(1 - M^2)|\alpha_0|\sqrt{\alpha_0^2 - 1}}{(M^+ / R - M)(Y_{11}^0 \alpha_0^2 + Y_{12}^0 |\alpha_0| \sqrt{\alpha_0^2 - 1} + Y_{13}^0)} = \gamma_1(\alpha_0),
\]

\[
\gamma_2 = -\frac{Y_{11}^0 \alpha_0^2 - Y_{12}^0 |\alpha_0| \sqrt{\alpha_0^2 - 1} + Y_{13}^0}{Y_{11}^0 \alpha_0^2 + Y_{12}^0 |\alpha_0| \sqrt{\alpha_0^2 - 1} + Y_{13}^0} = \gamma_2(\alpha_0),
\]

\[
\gamma_3 = \frac{2(1 - M^2)(1 - MR/M^+)|\alpha_0| \sqrt{\alpha_0^2 - 1} s_0}{(Y_{11}^0 \alpha_0^2 + Y_{12}^0 |\alpha_0| \sqrt{\alpha_0^2 - 1} + Y_{13}^0) \sqrt{M^2 + \alpha_0^2(1 - M^2)}} = s_0 \gamma_3(\alpha_0),
\]

\[
\gamma_4 = \frac{2D_0(1 - MR/M^+)(1 - M^2)|\alpha_0| \sqrt{\alpha_0^2 - 1}}{(Y_{11}^0 \alpha_0^2 + Y_{12}^0 |\alpha_0| \sqrt{\alpha_0^2 - 1} + Y_{13}^0)} = \gamma_4(\alpha_0),
\]

where \( Y_{11}^0, Y_{12}^0, Y_{13}^0 \) are defined in section 3, Eq. (6).

In fig.7, we depict the result of numerical calculations of reflection coefficient for acoustic wave \( \gamma_2 \) as a function of angle of incidence \( \theta_1 \) for an ideal gas.

The comparison with corresponding plot of [5] shows the same principal distinction as for refraction: in work [5] they give different plots at different values of \( u_y \), although in fact reflection coefficients don’t depend on \( u_y \). There is one strange difference more: critical angles of [5] are located near zero \( \theta_1 = 0 \), while in fact they are located near right angle \( \theta_1 = 90^\circ \), and plot in mentioned work looks like plot in fig.7 shifted along \( \theta \)-axis.

Now we establish some properties of reflection. First let’s consider critical values \( \alpha_0 = \pm 1 \). Here wave vectors for incident and reflected sound waves coincide \( k_1 = k_2 \), and group velocities of these waves are parallel to the shock. At that \( \gamma_1(\pm 1) = \)
Fig. 7: The angular dependence of the reflection coefficient for acoustic wave at $M^+ = 8$.

$\gamma_3(\pm 1) = \gamma_4(\pm 1) = 0$, i.e. the shock disturbance and reflected entropy-vorticity waves vanish, and $\gamma_2(\pm 1) = -1$, i.e. $G_1 + G_2 = 0$, and reflected sound wave suppresses (eliminates) the incident wave, so that the total principal part of the wave behind the shock vanishes. So the picture of reflection at critical values ensures the continuous transition to the case between critical values $|\alpha_0| < 1$, when incident acoustic wave, shock deformation and all reflected waves are damped and their principal parts are zero.

The equality of the reflection coefficient to unity at the critical angles was noted before (see e.g. [13]), but mutual suppression of the incident and reflected acoustic waves was not discussed before.

Dynamics (behaviours) of reflection coefficients may be investigated comprehensively. In particular, in is easy to show $|\gamma_2| \leq 1$, i.e. the wave amplitude of reflected sound wave is less than the wave amplitude of incident wave. Further, it can be shown for an ideal gas, that the coefficient $\gamma_2$ always has eight zero independently on $M^+$, and just this situation we see in fig.7. But $\gamma_2 = 0$ means principal part of reflected sound wave vanish, and it is this vanishing that creates an impression of a three-wave configuration at reflection, see [11, 25]. In fact, $\gamma_2 = 0$ only means reflected sound wave is damped, but four-wave configuration is maintained. The obvious reason for this misunderstanding is the neglect of the damped waves.

REFERENCES


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