S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 14, стр. 493–510 (2017) УДК 517.958, 532.5
DOI 10.17377/semi.2017.14.040 MSC 35A21, 35A22, 35D30, 76L05

LINEAR PROBLEM OF SHOCK WAVE DISTURBANCE ANALYSIS. PART 3: REFRACTION AND REFLECTION IN THE NEUTRAL STABILITY CASE

E. V. SEMENKO, T. I. SEMENKO

Abstract. The refraction and reflection in the neutral stability case, particularly the spontaneous emission, is investigated. The real agent (source) of spontaneous emission under the linear theory is indicated. The availability of infinite wave amplitudes at the plane waves refraction/reflection is disproved. The generalized generation and reflection coefficients for the appropriate description of refraction and reflection in the neutral stability case is proposed.

Keywords: Shock wave, shock disturbance, entropy-vorticity wave, acoustic wave, incident wave, refraction, transmitted wave, reflection, reflected wave, stability, neutral stability, spontaneous emission, Fourier transform.

1. Introduction

This part is organized as follows. In short Section 2, we mark the principal peculiarity of neutral stability case: the equation for shock disturbance provides the additional singular terms here. In Section 3, we consider an appearance of post-shock plane waves at the absence of the initial data for plane waves, i.e. the spontaneous emission. In Section 4, we discuss the peculiarity of plane waves refraction/reflection, in particular the situation, when wave amplitudes of the transmitted/reflected waves formally reduces to infinity (resonance). Subsection 4.1 is devoted to the refraction, and Subsection 4.2 is devoted to the reflection. In subsection 4.3, we introduce the generalized coefficients for more appropriate refraction/reflection description.

Semenko, E.V., Semenko, T.I., Linear problem of shock wave disturbance analysis. Part 3: refraction and reflection in the neutral stability case.
© 2017 Semenko E.V., Semenko T.I.
Received December, 13, 2016, published May, 23, 2017.

In Summary, we combine and briefly describe all obtained in three parts of paper results.

The necessary details of using the mathematical technique of all three parts are collected in Appendix.

2. Peculiarity of neutral stability

The equation for shock disturbance (Part 1, Eq. (4.3)) in terms of dimensionless variable α (Part 2, section 3) assumes the form

$$|\eta|\rho c\left(\frac{M^+}{R}-M\right)Y_1(\alpha)\widetilde{f}(\eta,\alpha)=\widetilde{Z}(\eta,\alpha)\,, \tag{1}$$

where $\widetilde{f}(\eta,\alpha) = \hat{f}(\eta,\eta u_y+\alpha|\eta|\sqrt{c^2-u_x^2})$, $\widetilde{Z}(\eta,\alpha) = Z(\eta,\eta u_y+\alpha|\eta|\sqrt{c^2-u_x^2})$, $\rho=\rho^-$, $c=c^-$, $M=M^-$, $u_x=u_x^-$. In the neutral stability case, function $Y_1(\alpha)$ has two single real roots $\alpha=\pm\alpha_1$, $\alpha_1>1$. So this equation provides additional singular terms by virtue of statement 2 in Appendix, Subsection 6.2.

3. Damped initial data. Spontaneous emission

Let initial data are damped (tend to zero) with respect to x and for simplicity let they are plane wave with respect to y, i.e. $G_0^{\pm}(x,y)=G_0^{\pm}(x)e^{i\eta_0 y}$, $G_0^{\pm}(x)\to 0$ at $x\to\infty$. In spectral variables, we have $\hat{G}_0^{\pm}=\hat{G}_0^{\pm}(\xi)\delta(\eta-\eta_0)$, where $\hat{G}_0^{\pm}(\xi)$ is regular. Then, by virtue of formulas of the solution (Part 1, Eqs. (4.1–4.7)), we sequentially conclude:

- the boundary pre-shock (incident) value has a form

  $$\hat{G}_\Gamma^+(\eta,\omega)=\sum_{j=1}^{4}\Omega_j^+(\eta,\omega)\hat{G}_0^+(\xi_j^+(\eta,\omega))\delta(\eta-\eta_0)=\hat{G}_\Gamma^+(\omega)\delta(\eta-\eta_0)\,,$$

  where $\hat{G}_\Gamma^+(\omega)$ is regular;
- the agent of incidence of post-shock initial wave has a form

  $$\Phi(\eta,\omega)=g_1^-(\eta,\omega)\hat{G}_0^-(\xi_1^-(\eta,\omega))\delta(\eta-\eta_0)=\Phi(\omega)\delta(\eta-\eta_0)\,,$$

  where $\Phi(\omega)$ is regular;
- the right side in the equation for shock disturbance (Part 1, Eq. (4.3)) has a form

$$Z(\eta,\omega)=-ig_1^-(\eta_0,\omega)(A^-)^{-1}A^+A_0^+\hat{G}_\Gamma^+(\omega)\delta(\eta-\eta_0)-i\Phi(\omega)\delta(\eta-\eta_0)=Z(\omega)\delta(\eta-\eta_0)\,,$$

  where $Z(\omega)$ is regular, consequently right side in equation (1) has a form

  $$\widetilde{Z}(\eta,\alpha)=Z(\eta u_y+\alpha|\eta|\sqrt{c^2-u_x^2})\delta(\eta-\eta_0)=\widetilde{Z}(\alpha)\delta(\eta-\eta_0)\,,$$

  where $\widetilde{Z}(\alpha)$ is regular, and so due to statement 2 (Appendix, Subsection 6.2, Eq. (17)) shock disturbance is

$$\widetilde{f}(\eta,\alpha)=-2\pi i\sum_{s=\pm 1}\frac{\widetilde{Z}(s\alpha_1)}{|\eta_0|\rho c(M^+/R-M)Y_1'(s\alpha_1)}I_1(\alpha-s\alpha_1)\delta(\eta-\eta_0)+\hat{\varphi}(\alpha)\delta(\eta-\eta_0)\,,$$

where $\hat{\varphi}(\alpha)$ is regular, or in initial spectral variables

$$\hat{f}(\eta,\omega) = -2\pi i \sum_{s=\pm 1} \frac{\widetilde{Z}(s\alpha_1)\sqrt{1-M^2}}{\rho(M^+/R-M)Y_1'(s\alpha_1)} I_1(\omega-\omega_s)\delta(\eta-\eta_0) + \hat{\varphi}(\omega)\delta(\eta-\eta_0)$$
$$= \sum_{s=\pm 1} \gamma_0(s) I_1(\omega-\omega_s)\delta(\eta-\eta_0) + \hat{\varphi}(\omega)\delta(\eta-\eta_0),$$

where $\hat{\varphi}(\omega)$ is regular, $\omega_{\pm 1} = \eta_0 u_y \pm \alpha_1|\eta_0|\sqrt{c^2-u_x^2}$, $\gamma_0(\pm 1) = \text{const}$;

- boundary value for transmitted and reflected waves has a form

$$\hat{H}^- = (A^-)^{-1}A^+A_0^+\hat{G}_\Gamma^+(\omega)\delta(\eta-\eta_0) - i(A^-)^{-1}\hat{F}_0\hat{f} =$$
$$= \sum_{s=\pm 1} \hat{H}_s^- I_1(\omega-\omega_s)\delta(\eta-\eta_0) + \hat{\varphi}(\omega)\delta(\eta-\eta_0)\,,$$

where $\hat{H}_s^- = -i\gamma_0(s)(A^-)^{-1}\hat{F}_0(\eta_0,\omega_s)$ are constant vectors, $s=\pm 1$, and $\hat{\varphi}$ is regular;

- boundary values for transmitted/reflected acoustic $\hat{G}_{2\Gamma}^-$, vorticity $\hat{G}_{3\Gamma}^-$ and entropy $\hat{G}_{4\Gamma}^-$ waves up to regular terms $\hat{\varphi}$ have a form

$$\hat{G}_{j\Gamma}^- = \sum_{s=\pm 1} \gamma_j(s) e_j^-(\eta_0,\omega_s) I_1(\omega-\omega_s)\delta(\eta-\eta_0) + \hat{\varphi}\,,$$

$$\gamma_j(s) = \frac{g_j^-(\eta_0,\omega_s)\hat{H}_s^-}{s_j^-(\eta_0,\omega_s)} = -i\gamma_0(s)\frac{Y_j(\eta_0,\omega_s)}{s_j^-(\eta_0,\omega_s)}\,, \quad j=\overline{2,4}\,, \quad s=\pm 1\,.$$

Finally, by virtue of representation of post-shock boundary wave (Part 1, Eq. (5.2)), using dimensionless variable α (and $s_0 = \operatorname{sgn}(\eta_0)$), and up to damped waves φ, we obtain in physical variables shock disturbance f and post-shock transmitted/ reflected waves G^- (decomposed into the sum of acoustic G_2^-, vorticity G_3^- and entropy G_4^-) in a form

$$f(y,t) = \sum_{s=\pm 1} \gamma_0(s) e^{i(\eta_0 y - \omega_s t)} + \varphi\,, \quad G^-(x,y,t) = \sum_{j=2}^{4} G_j^-\,,$$
$$G_j^-(x,y,t) = \sum_{s=\pm 1} \gamma_j(s) e_j^-(s\alpha_1,s_0) e^{i(\xi_j(s)x+\eta_0 y-\omega_s t)} + \varphi\,, \tag{2}$$

where

$$\omega_s = \eta_0 u_y + s\alpha_1|\eta_0|\sqrt{c^2-u_x^2}\,, \quad \xi_j(s) = |\eta_0|\xi_j^-(s\alpha_1)\,,$$
$$\gamma_0(s) = -\frac{2\pi\sqrt{1-M^2}}{\rho(M^+/R-M)Y_1'(s\alpha_1)}\left[g_1^-(s\alpha_1,s_0)(A^-)^{-1}A^+A_0^+\hat{G}_\Gamma^+(\omega_s) + \Phi(\omega_s)\right]\,,$$
$$\gamma_j(s) = -i\frac{|\eta_0|\rho(M^+/R-M)Y_j(s\alpha_1,s_0)}{s_j^-(s\alpha_1)}\gamma_0(s)\,, \quad j=\overline{2,4}\,, \quad s=\pm 1\,,$$
$$c=c^-\,, \quad u_x = u_x^-\,, \quad \rho=\rho^-\,, \quad M=M^-\,.$$

So for damped with respect to x initial data, we obtain the principal parts of shock disturbance and transmitted/reflected waves as plane waves, by two plane waves for shock disturbance and each kind of post-shock waves (acoustic, vorticity, entropy), six post-shock plane waves on the whole. It is spontaneous emission: the presence of reflected or transmitted plane waves at the absence of initial data for plane waves. Hence the linear theory explains and describes spontaneous emission absolutely correctly, its source (agent) is damped incident waves, although commonly used version maintains that agent of spontaneous emission has second order and is neglected under the linear theory, see [23, 24, 25].

Wavenumbers of these spontaneously emitted plane waves $\omega_s = \omega_s(\eta_0)$ and $\xi_j(s) = \xi_j(s, \eta_0)$, $s = \pm 1$, $j = \overline{2,4}$, we call, as usual (e.g. [26]), eigenfrequencies, and these plane waves themselves we call natural oscillations.

We compare obtained eigenfrequencies with results of [26], where they experimentally simulate the damped (in our terms) initial perturbation for the van der Waals gas in neutral stability case (D'yakov-Kontorovich instability in terms of mentioned work), and obtain undamped post-shock oscillations (natural oscillations in our terms). Functional relations for van der Waals gas in dimensionless form are

$$\left[\frac{p}{p_0} + 3\left(\frac{\rho}{\rho_0}\right)^2\right]\left(3\frac{\rho_0}{\rho} - 1\right) = 8\frac{T}{T_0}\,, \quad \frac{s}{s_0} = \ln\frac{T}{T_0} + \beta \ln\left(\frac{3\rho_0}{\rho} - 1\right)\,,$$

$$\frac{\epsilon}{\epsilon_0} = \frac{T}{\beta T_0} - \frac{9\rho}{8\rho_0}\,.$$

Whence

$$c^2\frac{\rho_0}{p_0} = -6\frac{\rho}{\rho_0} + \frac{3(1+\beta)(p/p_0 \cdot \rho_0/\rho + 3\rho/\rho_0)}{3 - \rho/\rho_0}\,,$$

$$B_0 = \frac{3c^2\rho_0/p_0(1+\beta)}{(3-\rho/\rho_0)(c^2\rho_0/p_0 + 6\rho/\rho_0) - 9\rho/\rho_0(1+\beta)}\,, \quad D_0 = -\frac{3\beta}{3-\rho/\rho_0}\,.$$

Then Rankine-Hugoniot conditions assume the form of three equations

$$\rho_1 M^+ = \rho M R\,,$$

$$-3\rho_1^2 + (3-\rho_1)\frac{(c^2/R^2 + 6\rho_1)\rho_1}{3(1+\beta)} + \frac{\rho_1 (M^+)^2 c^2}{R^2} = -3\rho^2 + (3-\rho)\frac{(c^2+6\rho)\rho}{3(1+\beta)} + \rho M^2 c^2\,,$$

$$(3-\rho_1)^2\frac{c^2/R^2 + 6\rho_1}{24\beta(1+\beta)} - \frac{9\rho_1}{4} + (3-\rho_1)\frac{c^2/R^2+6\rho_1}{8(1+\beta)} + \frac{3(M^+)^2c^2}{16R^2}$$
$$= (3-\rho)^2\frac{c^2+6\rho}{24\beta(1+\beta)} - \frac{9\rho}{4} + (3-\rho)\frac{c^2+6\rho}{8(1+\beta)} + \frac{3}{16}M^2c^2,$$

for six dimensionless parameters M^+, $M^- = M$, R, $\rho = \rho^-/\rho_0$, $\rho_1 = \rho^+/\rho_0$ and

$$c = c^-\sqrt{\frac{\rho_0}{p_0}} = \sqrt{-6\rho + \frac{3(p^-/p_0 + 3\rho^2)(1+\beta)}{(3-\rho)\rho}}\,.$$

This system allows to express three of these parameters through three others. Like in mentioned work, we take $\beta = 1/30$, $M^+ = 1.245$, $\rho = 1$, $p^- = 1.165p_0$, and obtain $M = 0.744459$, $R = 0.558025$, $\rho_1 = 0.333676$. These parameters ensure the neutral stability case.

Roots $Y_1(\alpha)$ calculation gives $\alpha_1 = 1.00508$. So the wave vector for spontaneously emitted (going away from shock) plane sound wave is $\boldsymbol{k} = (|\eta_0|\xi_1^-(\alpha_1), \eta_0)$, it forms with negative direction of x-axis angle θ and $\cos\theta = 0.696095$. These results good correspond to work by [26], where $M = 0.745$, $\cos\theta = 0.696$.

4. Refraction and reflection of plane waves. Generalized transmittion/reflection coefficients

4.1. **Refraction.** We consider pre-shock plane waves with boundary wavenumbers (η_0, ω_0). Let as in Part 2

$$\alpha_0 = \frac{\omega_0 - \eta_0 u_y}{|\eta_0|\sqrt{c^2 - u_x^2}}\,, \quad s_0 = \operatorname{sgn}(\eta_0)\,.$$

Boundary value for pre-shock (incident) plane wave is (Part 2, Subsection 5.3)

$$\hat{G}_\Gamma^+ = (\Psi^+\beta)\delta(\eta - \eta_0)I_1(\omega - \omega_0)\,,$$

and right side in equation (1) assumes the form

$$\widetilde{Z}(\eta, \alpha) = -ig_1^-(\alpha, s_0)(A^-)^{-1}A^+A_0^+(\Psi^+\beta)\delta(\eta - \eta_0)I_1(\omega(\alpha) - \omega_0)\,,$$

$\omega(\alpha) = \eta_0 u_y + \alpha|\eta_0|\sqrt{c^2 - u_x^2}$. The function

$$Z_0 = -2\pi i(\alpha - \alpha_0)\widetilde{Z}(\alpha, \eta) = -ig_1^-(\alpha, s_0)(A^-)^{-1}A^+A_0^+(\Psi^+\beta)\frac{\delta(\eta - \eta_0)}{|\eta_0|\sqrt{c^2 - u_x^2}}$$

is regular with respect to α, i.e. condition (15) (Appendix, Subsection 6.2) is fulfilled and so, due to statement 2 and up to regular term $\hat{\varphi}$, we have

$$\widetilde{f}(\eta, \alpha) = -iS(g_1^-, Y_1 \mid \alpha, \alpha_0)(A^-)^{-1}A^+A_0^+(\Psi^+\beta)\delta(\eta - \eta_0) + \hat{\varphi}\,, \tag{3}$$

where the operator S has a form

$$S(Z, Y \mid \alpha, \alpha_0) = \frac{Z(\alpha_0)}{Y(\alpha_0)}I_1(\alpha - \alpha_0) - \sum_{s=\pm 1}\frac{Z(s\alpha_1)}{(\alpha_0 - s\alpha_1)Y'(s\alpha_1)}I_1(\alpha - s\alpha_1)\,, \tag{4}$$

see Appendix, Subsection 6.2, Eq. (18). So the shock disturbance in physical variables up to damped wave φ is

$$f(y, t) = \gamma_0(0)e^{i(\eta_0 y - \omega_0 t)} - \sum_{s=\pm 1}\gamma_0(s)e^{i(\eta_0 y - \omega_s t)} + \varphi\,, \tag{5}$$

where

$$\gamma_0(0) = -\frac{i}{Y_1(\alpha_0)}g_1^-(\alpha_0, s_0)(A^-)^{-1}A^+A_0^+(\Psi^+\beta)\,,$$

$$\gamma_0(s) = -\frac{i}{(\alpha_0 - s\alpha_1)Y_1'(s\alpha_1)}g_1^-(s\alpha_1, s_0)(A^-)^{-1}A^+A_0^+(\Psi^+\beta)\,, \quad s = \pm 1\,.$$

Thus the principal part of shock disturbance consists of three summands: plane wave with incident frequency ω_0 (incident oscillation), and natural oscillations, i.e. two plane waves with eigenfrequencies ω_s, $s = \pm 1$. When incident frequency ω_0 coincides with some eigenfrequency, i.e. we come to limit in principal part in (5) (Appendix, Subsection 6.2, Statement 2), then resonance occurs, e.g. at $\omega_0 = \omega_1$ we get

$$f(y, t) = (C_1 t + C_0)e^{i(\eta_0 y - \omega_1 t)} + \gamma_0(-1)e^{i(\eta_0 y - \omega_{-1} t)}\,,$$

$$C_1 = -\frac{g_1^-(\alpha_1, s_0)}{Y_1'(\alpha_1)}(A^-)^{-1}A^+A_0^+(\Psi^+\beta)\,,$$

$$C_0 = -i\left[\frac{(g_1^-)_\alpha'(\alpha_1, s_0)}{Y_1'(\alpha_1)} - \frac{g_1^-(\alpha_1, s_0)Y_1''(\alpha_1)}{2(Y_1'(\alpha_1))^2}\right](A^-)^{-1}A^+A_0^+(\Psi^+\beta)\,.$$

Absolutely similar we obtain (up to damped wave φ) expressions for the transmitted acoustic G_2^-, vorticity G_3^- and entropy G_4^- waves:

$$G_j^- = \left(\gamma_j(0)e_j^-(\eta_0,\omega_0)e^{i(\xi_j x-\omega_0 t)} - \sum_{s=\pm1}\gamma_j(s)e_j^-(\eta_0,\omega_s)e^{i(\xi_j(s)x-\omega_s t)}\right)e^{i\eta_0 y} + \varphi\,,$$

where $\xi_j = \xi_j^-(\eta_0,\omega_0)$, $\xi_j(s) = \xi_j^-(\eta_0,\omega_s)$, $j=\overline{2,4}$;

$$\gamma_2(0) = \frac{1}{1-M^2}\left[\frac{\widetilde{Y}_2(\alpha_0)g_1^-(\alpha_0,s_0)}{Y_1(\alpha_0)} - \widetilde{g}_2(\alpha_0,s_0)\right](A_2^-)^{-1}A_R A_2^+ A_{01}^+(\Psi^+\beta)\,,$$

$$\gamma_2(s) = \frac{\widetilde{Y}_2(s\alpha_1)g_1^-(s\alpha_1,s_0)}{(1-M^2)(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)}(A_2^-)^{-1}A_R A_2^+ A_{01}^+(\Psi^+\beta)\,,\quad s=\pm1\,;$$

$$\gamma_j(0) = \frac{1}{M}\left[g_j^-(\alpha_0,s_0) - \frac{Y_j(\alpha_0,s_0)}{Y_1(\alpha_0)}g_1^-(\alpha_0,s_0)\right](A_2^-)^{-1}A_R A_2^+ A_{01}^+(\Psi^+\beta)\,,$$

$$\gamma_j(s) = -\frac{Y_j(s\alpha_1,s_0)}{M(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)}g_1^-(s\alpha_1,s_0)(A_2^-)^{-1}A_R A_2^+ A_{01}^+(\Psi^+\beta),\quad j=\overline{3,4},$$

$s=\pm1$. When incident frequency ω_0 coincides with some eigenfrequency ω_s, $s=\pm1$, the resonance occurs, e.g. at $\omega_0=\omega_1$

$$G_j^- = (U_{1,j}t + U_{2,j}x + U_{3,j})e^{i(\xi_j(1)x+\eta_0 y-\omega_1 t)} +$$
$$+\gamma_j(-1)e_j^-(\eta_0,\omega_{-1})e^{i(\xi_j(-1)x+\eta_0 y-\omega_{-1}t)} + \varphi\,,$$

where $U_{k,j}$, $k=\overline{1,3}$, $j=\overline{2,4}$ are definite vectors.

General conclusion reads: the principal part of each kind of transmitted wave G_j^-, $j=\overline{2,4}$ is a sum of three plane waves. One summand has wave amplitude $\gamma_j(0)$ and wavenumbers $(\xi_j^-(\eta_0,\omega_0),\eta_0,\omega_0)$, i.e. the same, as for transmission in the stability case, we call it incident oscillation. Two other summands have wave amplitudes $\gamma_j(s)$ and eigenfrequencies $(\xi_j^-(\eta_0,\omega_s),\eta_0,\omega_s)$, $s=\pm1$, i.e. they are natural oscillations. The wave amplitudes of separate summands $\gamma_j(k)$, $k=\overline{-1,1}$ in general tend to infinity, when incident frequency ω_0 tends to some eigenfrequency ω_s. That is the reason of commonly used conclusion about the infinite wave amplitudes at resonance, e.g. [11, 12, 13, 14, 25]. In fact, as we see, there are no any infinite wave amplitudes in the resonant case, there are growing wave amplitudes and the growth coefficients (the coefficients before t or x) are finite and may be compared with the amplitudes of incident wave. Anyway, the comparison of incident waves' amplitudes and separate transmitted waves' amplitudes $\gamma_j(k)$ is not appropriate in the neutral stability case, and the need arises to introduce more appropriate quantity for refraction's description.

4.2. **Reflection.** As usual, here we mean superscript minus on default.

As initial data we consider the initial data for plane wave directed towards the shock:

$$G_0 = \beta e_1(\eta_0,\omega_0)e^{i(\xi_0 x+\eta_0 y)}\,,\quad \xi_0 = \xi_1(\eta_0,\omega_0)\,,$$

which generates initial post-shock acoustic wave with principal part

$$G^- = \beta e_1(\eta_0,\omega_0)e^{i(\xi_0 x+\eta_0 y-\omega_0 t)}$$

(see Part 2, Subsection 2.3). In the spectral and dimensionless variables we have $\hat{G}_0 = \beta e_1(\alpha_0, s_0)\delta(\eta-\eta_0)I_1(\xi_0 - |\eta_0|\xi_1(\alpha))$, and agent of incidence has a form

$$\Phi = \frac{\beta}{|\eta_0|} g_1(\alpha, s_0)e_1(\alpha_0, s_0)\delta(\eta-\eta_0)I_1(\xi_1(\alpha_0)-\xi_1(\alpha))\,.$$

Here for reflected waves computation, we cannot use principal part of Φ only, because wave amplitudes of reflected waves depend also on its damped part.

The equation for the shock disturbance (1) has a form

$$|\eta|\rho c\left(\frac{M^+}{R} - M\right) Y_1(\alpha)\widetilde{f}(\eta,\alpha) = \widetilde{Z} = -i\Phi =$$

$$= -i\frac{\beta}{|\eta_0|} g_1(\alpha, s_0)e_1(\alpha_0, s_0)\delta(\eta-\eta_0)I_1(\xi_1(\alpha_0)-\xi_1(\alpha))\,.$$

The function

$$Z_0 = -2\pi i(\alpha-\alpha_0)\widetilde{Z} = -i\frac{\beta}{|\eta_0|} g_1(\alpha, s_0)e_1(\alpha_0, s_0)\delta(\eta-\eta_0)\frac{\alpha-\alpha_0}{\xi_1(\alpha_0)-\xi_1(\alpha)}$$

is regular with respect to α, and so due to statement 2 (Appendix, Subsection 6.2) and up to regular with respect to α term $\hat{\varphi}$, we obtain
$\widetilde{f}(\eta,\alpha) = \frac{1}{|\eta_0|\rho c\left(\frac{M^+}{R}-M\right)}\left(\frac{Z_0(\alpha_0)}{Y_1(\alpha_0)}I_1(\alpha-\alpha_0) - \sum_{s=\pm1}\frac{Z_0(s\alpha_1)}{(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)}I_1(\alpha-s\alpha_1)\right) +$
$+\hat{\varphi}$. But $Z_0(\alpha_0) = i\frac{\beta}{|\eta_0|\sqrt{1-M^2}}s_1(\alpha_0)\delta(\eta-\eta_0)$ and so

$$\hat{f}(\eta,\omega) = \left[\gamma_0(0)I_1(\omega-\omega_0) - \sum_{s=\pm1}\gamma_0(s)I_1(\omega-\omega_s)\right]\frac{i\beta\delta(\eta-\eta_0)\sqrt{1-M^2}}{|\eta_0|\rho\left(\frac{M^+}{R}-M\right)} + \hat{\varphi}\,,$$

$$\gamma_0(0) = \frac{s_1(\alpha_0)}{Y_1(\alpha_0)\sqrt{1-M^2}}\,,\quad \gamma_0(s) = \frac{g_1(s\alpha_1, s_0)e_1(\alpha_0, s_0)}{(\xi_1(\alpha_0)-\xi_1(s\alpha_1))Y_1'(s\alpha_1)}\,,\quad s=\pm1\,;$$

or in the physical variables

$$f(y,t) = \frac{i\beta\sqrt{1-M^2}}{|\eta_0|\rho\left(\frac{M^+}{R}-M\right)}\left[\gamma_0(0)e^{i(\eta_0 y-\omega_0 t)} - \sum_{s=\pm1}\gamma_0(s)e^{i(\eta_0 y-\omega_s t)}\right] + \varphi\,.$$

At $\alpha_0 = \pm\alpha_1$ the resonance occurs, i.e. principal part assumes a form

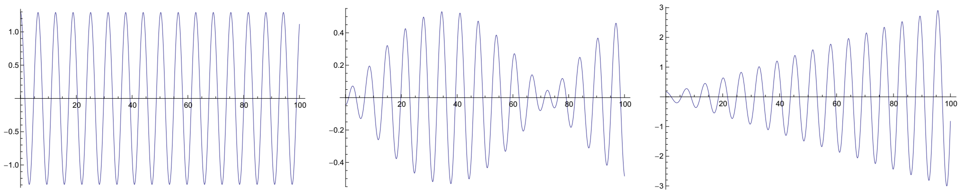
$$f = (C_0 t + C_1)e^{i(\eta_0 y-\omega_0 t)}.$$

For correct comparison with incident wave amplitude, we reduce shock deformation characteristic to density dimension, taking quantity

$$f_1 = \frac{\rho^-}{c^-}\left(\frac{\partial f}{\partial t} + u_y\frac{\partial f}{\partial y}\right) = \gamma_1(0)e^{i(\eta_0 y-\omega_0 t)} - \sum_{s=\pm1}\gamma_1(s)e^{i(\eta_0 y-\omega_s t)} + \varphi\,,$$

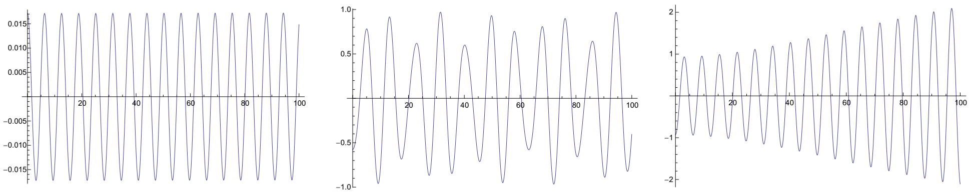
where

$$\begin{gathered}\gamma_1(0) = \frac{\sqrt{1-M^2}\,\alpha_0 s_1(\alpha_0)}{(M^+/R-M)Y_1(\alpha_0)}\,,\\ \gamma_1(s) = \frac{(1-M^2)s\alpha_1 g_1(s\alpha_1, s_0)e_1(\alpha_0, s_0)}{(\xi_1(\alpha_0)-\xi_1(s\alpha_1))Y_1'(s\alpha_1)}\,,\quad s=\pm1\,.\end{gathered}\tag{6}$$



normal incidence | oblique non-resonant incidence | resonant incidence

Fig. 1: Generalized transmission coefficient.



normal incidence | oblique non-resonant incidence | resonant incidence

Fig. 2: Generalized reflection coefficient.

Absolutely similar previous considerations, we obtain principal parts of reflected waves

$$G_j = \beta\left[\gamma_j(0)e^{i(\xi_j(\eta_0,\omega_0)x+\eta_0 y-\omega_0 t)} - \sum_{s=\pm 1}\gamma_j(s)e^{i(\xi_j(\eta_0,\omega_s)x+\eta_0 y-\omega_s t)}\right] + \varphi\,,$$

where

(7)
$$\gamma_j(0) = \frac{s_1(\alpha_0)Y_j(\alpha_0,s_0)}{s_j(\alpha_0)Y_1(\alpha_0)}\,,$$
$$\gamma_j(s) = \frac{Y_j(s\alpha_1,s_0)g_1(s\alpha_1,s_0)e_1(\alpha_0,s_0)\sqrt{1-M^2}}{s_j(s\alpha_1)(\xi_1(\alpha_0)-\xi_1(s\alpha_1))Y_1'(s\alpha_1)}\,,\quad s=\pm 1\,,\quad j=\overline{2,4}\,,$$

including the resonant case as a limit at $\alpha_0 \to \pm\alpha_1$. Taking the ratio of reflected wave amplitudes to incident amplitude β, we get reflection coefficients $\gamma_j(0)$, $\gamma_j(s)$. At resonance $\alpha_0 = \pm\alpha_1$, these coefficients (separately) reduce to infinity, like for refraction.

4.3. **Generalized coefficients.** To avoid infinite wave amplitudes and describe refraction/reflection more correctly, we suppose to take into account principal parts of each kind of transmitted waves on the whole. Principal parts of shock disturbance and transmitted/reflected waves have common form

$$g = \gamma_j(0)e^{i(\eta_0 y-\omega_0 t)} - \sum_{s=\pm 1}\gamma_j(s)e^{i(\eta_0 y-\omega_s t)}$$

or

$$g = e^{i\eta_0(y-u_y t)}\left[\gamma_j(0)e^{-i\alpha_0|\eta_0|c\sqrt{1-M^2}\,t} - \sum_{s=\pm1}\gamma_j(s)e^{-is\alpha_1|\eta_0|c\sqrt{1-M^2}\,t}\right],$$

$j=\overline{1,4}$, $c=c^-$, $M=M^-$. So for appropriate refraction/reflection description, it is reasonable to take for comparison with incident wave quantity

$$\gamma_j(\alpha_0,\tau) = \mathrm{Re}\left[\gamma_j(0)e^{-i\tau} - \sum_{s=\pm1}\gamma_j(s)e^{-is\alpha_1\tau/\alpha_0}\right] =$$

$$= \gamma_j(0)\cos\tau - [\gamma_j(1)+\gamma_j(-1)]\cos\frac{\alpha_1\tau}{\alpha_0},$$

$j=\overline{2,4}$, where $\tau = t|\alpha_0||\eta_0|c\sqrt{1-M^2}$ is dimensionless time. The choice of dimensionless time τ means, we reduce incident frequency ω_0 to unit, and then eigenfrequency is α_1/α_0.

The ratio of $\gamma_j(\alpha_0,\tau)$ and the wave amplitude of incident wave we call generalized transmission/reflection coefficient.

In this way, for refraction we obtain the matrix of generalized transmission coefficients in a form

$$\Psi_0(\alpha_0,\tau) = \Psi_0^-(A_2^-)^{-1}A_R A_2^+ A_{01}^+\Psi^+,$$

where the matrixes A_2^-, A_R, A_2^+, A_{01}^+, Ψ^+ are the same, as for refraction in stability case (Part 2, Subsection 5.3), and

$$\Psi_0^-(\alpha_0,\tau) = \Psi^-\cos\tau - \left[\Psi_1^- + \Psi_{-1}^-\right]\cos\frac{\alpha_1\tau}{\alpha_0},$$

where Ψ^- is again the same, as for refraction in stability case, and at last

$$\Psi_s^- = \begin{pmatrix} -\dfrac{s\alpha_1\sqrt{1-M^2}}{(M^+/R-M)(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)} & 0 & 0 & 0 \\ \dfrac{\widetilde{Y}_2(s\alpha_1)}{(1-M^2)(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)} & 0 & 0 & 0 \\ -\dfrac{Y_3(s\alpha_1)}{M(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)} & 0 & 0 & 0 \\ -\dfrac{Y_4}{M(\alpha_0-s\alpha_1)Y_1'(s\alpha_1)} & 0 & 0 & 0 \end{pmatrix}$$

$$\times\begin{pmatrix} \dfrac{1}{2} & -\dfrac{\xi_1^-(s\alpha_1)}{2Q_1^-(s\alpha_1)} & -\dfrac{s_0}{2Q_1^-(s\alpha_1)} & -\dfrac{1}{2} \\ 0 & \dfrac{s\alpha_1\sqrt{1-M^2}}{2Q_1^-(s\alpha_1)} & \dfrac{Ms_0}{2Q_1^-(s\alpha_1)} & 0 \\ 0 & \dfrac{s_0}{\sqrt{(\xi_0^-(s\alpha_1))^2+1}} & -\dfrac{\xi_0^-(s\alpha_1)}{\sqrt{(\xi_0^-(s\alpha_1))^2+1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We may calculate this matrix, using some mathematical programmes. Naturally, final expressions are quite complicated and unwieldy.

Accordingly, we introduce generalized reflection coefficients in a form

$$\gamma_j(\alpha_0,\tau) = \gamma_j(0)\cos\tau - [\gamma_j(1) + \gamma_j(-1)]\cos\frac{\alpha_1\tau}{\alpha_0}\,, \quad j = \overline{1,4}\,,$$

where $\gamma_j(k)$, $j = \overline{1,4}$, $k = \overline{-1,1}$ have a form (6), (7).

We carry out numerical calculations for van der Waals gas with the same parameters, as in mentioned work by [26] (see section 3). In fig.1, we depict the plots of generalized generation coefficient for transmitted acoustic wave caused by incident fast acoustic wave at two non-resonant cases – for normal ($\alpha_0 = \infty$) and for oblique incidence, and at resonance ($\alpha_0 = \alpha_1$). In fig.2, we depict the analogous plots of generalized reflection coefficient for reflected sound wave. We clearly see the difference between resonant and non-resonant cases (and, by the way, the absence of infinite wave amplitudes).

5. Summary

In three parts of article we've considered a number of questions related to the classical linear problem of shock wave disturbance.

In first part, the new representation of initial value problem's solution is obtained. This representation has strictly algebraic form in plane of Fourier transform, which gives the additional opportunities for solution's analysis. In this way, we obtain some new results or specify some known ones.

Thus, the decomposition of any solution into the sum of the acoustic, vorticity and entropy waves occurs naturally. First we use this decomposition for the refraction and reflection inspection in general form (not only for plane waves, as before).

For the refraction, we establish the correspondence between incident waves on one hand and results of refraction: shock deformation and transmitted waves on the other hand. In this way, we find the peculiarity of refraction, which haven't been noted before: the decomposition of incident pre-shock wave into the sum of waves with physically different interaction with shock. Namely, one summand interacts with shock, i.e. generates shock disturbance, but doesn't generate any transmitted waves, or somehow "sticks" to the shock and spends all its energy on the shock deformation. Three other summands don't interact with shock, i.e. don't generate shock disturbance or don't spend energy on shock deformation, but generate the transmitted waves, and each of them generates only one kind of transmitted wave: sound, vorticity or entropy.

What about reflection, then it is shown: the post-shock entropy-vorticity waves aren't incident upon the shock (known result, though for plane waves only, not in general situation), but any post-shock acoustic wave is incident upon the shock and creates shock disturbance and four-wave configuration at reflection, i.e. one incident (acoustic) wave and three reflected: acoustic, entropy and vorticity. It contradicts and, so, corrects (specifies) some previous works, where it was tacitly adopted plane acoustic post-shock wave being directed away from the shock isn't incident upon the shock, and furthermore the three-wave configurations were detected for some cases.

The obtained algebraic form of solution allows us easy separate singular (e.g. plane waves) and regular, i.e. damped waves, terms of solution, and use this separation as for analytical considerations as for numerical calculations. This approach turns out rather useful for plane waves' refraction and reflection analysis. In this way, we establish: any post-shock plane wave is accompanied by the damped wave.

These damped waves were neglected in the linear theory before and that led to some mistakes and misunderstandings, in particular it is damped waves' neglect that is the reason of mentioned (corrected) statements about reflection: non-incidence of acoustic waves and three-wave configuration.

Then we carry out the analysis of only singular terms, i.e. plane waves, as parts of explicitly expressed solution.

In second part, we consider the stability case. Here we summarize the new or specified results.

(1) The form of transmitted plane acoustic waves between critical angles of incidence is specified. They are the evanescent waves with respect to x, what strongly reminds the case of total internal reflection in optics. Before it was stated the non-existence of transmitted acoustic wave for this situation, see e.g. [14].
(2) The correspondence between angles of incidence and angles of refraction or reflection (i.e. Snell's laws) is established. The results show: known formulas (e.g. [5, 14]) need to be corrected.
(3) All sixteen generation coefficients for refraction are calculated, more accurately the representation of matrix of generation coefficients is obtained. Similarly all four reflection coefficients are calculated. The results show principal distinction with known (in [5]) ones: in fact and in contradiction of [5], the generation/reflection coefficients don't depend on angle of obliquity of basic solution's velocity with respect to shock, in particular on the cases of subsonic/supersonic downstream basic flow.
(4) The numerical calculations of generation/reflection coefficients is carried out. The obtained plots are more or less like the analogous plots from [5] up to mentioned principal distinctions.
(5) The degree of amplification of generation coefficients, when the upstream Mach number tends to infinity, is found out. This degree is two in common case and one for some kinds of incident waves, e.g. for incident vorticity wave, and doesn't depend on angle of incidence, particularly on closeness of angle of incidence to critical values. These results contradict to known ones by [5, 14], in particular the so-called abnormal amplification doesn't exist.
(6) The numerical calculation of amplification is carried out, the results illustrate and fully confirm analytical conclusions.
(7) The reflection at critical angles of incidence is inspected. The results show: principal parts of shock deformation and reflected entropy-vorticity waves vanish and principal parts of incident and reflected acoustic waves suppress (eliminate) each other and so total principal part of solution vanish, what ensures the continuous transition to the case between critical angles, when only damped parts of incident and reflected waves remain.
(8) The analytical investigations and numerical calculations show: the reflection coefficient for acoustic wave has zeroes, and it is this vanishing of reflection coefficient, that make impression of three-wave configuration (because of damped waves' neglecting).

At last the neutral stability case, particularly the so-called spontaneous emission, i.e. presence of the transmitted or refracted plane waves at the absence of incident plane waves, is inspected in third part, again on the basis of singular and regular terms separation. The principal conclusions are:

(1) When incident waves are damped (with respect to x), then there are six (by two acoustic, entropy and vorticity) transmitted or reflected plane waves. It is just spontaneous emission. Wavenumbers of transmitted and reflected plane waves and shock deformation depend only on η – wavenumber of incident wave with respect to y. We call these wavenumbers eigenfrequencies, and plane waves with these wavenumbers we call natural oscillations.

(2) When incident waves are plane, then in addition to transmitted/reflected natural oscillations there appear transmitted/reflected waves with the same wavenumbers, as in stability case. At that the wave amplitudes of these waves tend to infinity when wavenumbers of incident waves tends to eigenfrequencies. It is the reason for made before conclusion about the existence of infinite generation or reflection coefficients. In fact, when wavenumbers of incident waves coincide with eigenfrequencies the resonance occurs: transmitted/reflected waves are plane waves with growing (as first degree with respect to x and t) wave amplitudes.

(3) In order to describe refraction and reflection (including resonant case) more clear and accurately, the generalized generation/reflection coefficients are proposed. Actually they are principal parts of transmitted/reflected waves on the whole. These coefficients, naturally, describe relations between each kind of incident plane wave and each kind of transmitted or reflected plane wave. They are continuous with respect to angle of incidence, and at resonance give plane waves with growth amplitudes, i.e. exactly resonant waves. The numerical calculations of these principal parts clearly show the difference between resonant and non-resonant cases and absence of infinite wave amplitudes at resonance.

Customary explanation of spontaneous emission is that its real agent (source) has second order of smallness and so is neglected under the linear theory. As we see, the linear theory explains and describes spontaneous emission absolutely correctly, its source (agent) is damped incident wave. Moreover, similar to characteristic of plane waves refraction or reflection with the help of generalized generation or reflection coefficients, it is possible to introduce quantity characteristic of spontaneous emission as ratio of transmitted/reflected plane waves and "strength" of incident damped wave. The only problem is to introduce mentioned "strength" as meaningful parameter.

Finally it should be accentuated: the principal difference between stability and neutral stability cases isn't an amplification of reflected sound wave or infinite wave amplitudes, furthermore there are no infinite amplitudes. It is absence/presence of resonance (eigenfrequencies), that is the principal distinction between stability and neutral stability cases.

6. Appendix

6.1. **One-side Fourier transformation.** Everywhere we mean on default

$$\int \cdots = \int_{-\infty}^{+\infty} \cdots$$

Actually one-side Fourier transform

$$\hat{g}^{+}(\xi) = \frac{1}{2\pi}\int\limits_{-\infty}^{0} g(x)e^{-i\xi x}\,dx\,, \quad \hat{g}^{-}(\xi) = \frac{1}{2\pi}\int\limits_{0}^{+\infty} g(x)e^{-i\xi x}\,dx$$

is a partial case of general Fourier transform for functions $g(x)$, vanishing at $x > 0$ (g^{+}) or $x < 0$ (g^{-}). So the inverse transform has usual form [28, p.302]:

$$g(x) = \begin{cases} \displaystyle\int \hat{g}^{+}(\xi)e^{i\xi x}\,d\xi, & x < 0, \\ \displaystyle\int \hat{g}^{-}(\xi)e^{i\xi x}\,d\xi, & x > 0, \end{cases}$$

and inverse transforms for desired and given quantities are

$$\begin{aligned} G^{\pm}(x,y,t) &= \iiint \hat{G}^{\pm}(\xi,\eta,\omega)e^{i(x\xi+y\eta-t\omega)}\,d\xi d\eta d\omega\,, \\ f(y,t) &= \iint \hat{f}(\eta,\omega)e^{i(y\eta-t\omega)}\,d\eta d\omega\,, \\ G_0^{\pm}(x,y) &= \iint \hat{G}_0^{\pm}(\xi,\eta)e^{i(x\xi+y\eta)}\,d\xi d\eta\,, \\ G_\Gamma^{\pm}(y,t) &= \iint \hat{G}_\Gamma^{\pm}(\eta,\omega)e^{i(y\eta-t\omega)}\,d\eta d\omega\,. \end{aligned} \tag{8}$$

On the other hand one-side Fourier transform is in fact Laplace transform in other notations:

$$\hat{g}^{+}(\xi) = \frac{1}{2\pi}\int\limits_{0}^{\infty} g(-x)e^{-\alpha x}\,dx\,\Big|_{\alpha=-i\xi}\,,$$

$$\hat{g}^{-}(\xi) = \frac{1}{2\pi}\int\limits_{0}^{\infty} g(x)e^{-\alpha x}\,dx\,\Big|_{\alpha=i\xi}\,,$$

therefore its properties are, up to notations, the same as for Laplace transform [28, p.347-348]:

- $\hat{g}^{+}(\xi)$ is analytic in the upper half plane $\operatorname{Im}\xi > 0$, $\hat{g}^{-}(\xi)$ is analytic in the lower half plane $\operatorname{Im}\xi < 0$;
- the transform of derivative is

$$\widehat{g'(x)}^{\pm} = i\xi\hat{g}^{\pm}(\xi) \pm \frac{g(0)}{2\pi}\,.$$

It should be accentuated the analyticity here means in weak or generalized sense (e.g. [29]), i.e. $g^{+}(\omega)$ is analytic in the upper half plane $\operatorname{Im}\omega > 0$ if for any analytic (in common sense) function $h^{+}(\omega)$ we have

$$\int g^{+}(\omega)h^{+}(\omega)\,d\omega = 0\,.$$

We bring several examples. Let $\operatorname{Im}\omega_0 > 0$ and

$$I(\omega) = \delta(\omega-\omega_0) - \frac{1}{2\pi i(\omega-\omega_0)}\,.$$

The analyticity of function $I(\omega)$ in upper half plane $\operatorname{Im}\omega > 0$ immediately follows from Cauchy's integral formula [28, p.37]. It's easy to see

$$\int I(\omega)e^{-i\omega t}\,d\omega = \begin{cases} e^{-i\omega_0 t}, & t>0, \\ 0, & t<0, \end{cases} \Longrightarrow \widehat{e^{-i\omega_0 t}} = I(\omega-\omega_0)\,, \quad \operatorname{Im}\omega_0 > 0\,. \tag{9}$$

Now let $\operatorname{Im}\omega_0 = 0$ and

$$I_1(\omega-\omega_0) = \begin{cases} -\dfrac{1}{2\pi i(\omega-\omega_0)}, & \operatorname{Im}\omega>0, \\ \dfrac{\delta(\omega-\omega_0)}{2} - \dfrac{1}{2\pi i(\omega-\omega_0)}, & \operatorname{Im}\omega = 0, \end{cases} \tag{10}$$

$$I_2(\omega-\omega_0) = \frac{\partial I_1(\omega-\omega_0)}{\partial\omega_0} = \begin{cases} -\dfrac{1}{2\pi i(\omega-\omega_0)^2}, & \operatorname{Im}\omega>0, \\ -\dfrac{1}{2\pi i(\omega-\omega_0)^2} - \dfrac{\delta'(\omega-\omega_0)}{2}, & \operatorname{Im}\omega = 0. \end{cases} \tag{11}$$

Analyticity of functions I_1, I_2 with respect to ω in upper half plane follows from Plemelj formulas [28, p.414]. Also it follows via Plemelj formulas

$$\begin{gathered} \int I_1(\omega-\omega_0)e^{-i\omega t}\,d\omega = \begin{cases} e^{-i\omega_0 t}, & t>0, \\ 0, & t<0, \end{cases} \\ \int I_2(\omega-\omega_0)e^{-i\omega t}\,dt = \begin{cases} -ite^{-i\omega_0 t}, & t>0, \\ 0, & t<0. \end{cases} \end{gathered} \tag{12}$$

Consequently

$$\widehat{e^{-i\omega_0 t}} = I_1(\omega-\omega_0)\,, \quad (\widehat{e^{i\xi_0 x}})^+ = I_1(\xi-\xi_0)\,, \quad (\widehat{e^{i\xi_0 x}})^- = I_1(\xi_0-\xi)\,. \tag{13}$$

6.2. **The linear equations' solution.** We consider the equation

$$Y(\alpha)\hat{f}(\alpha) = Z(\alpha) \tag{14}$$

in class of analytic (in weak sense) functions in upper half plane $\operatorname{Im}\alpha > 0$, where $Y(\alpha)$, $Z(\alpha)$ are given and $Y(\alpha)$ has real roots.

Let α_1 is real, $Y(\alpha) = \alpha - \alpha_1$ and $Z(\alpha)$ is regular. Then evidently

$$\hat{f}(\alpha) = \frac{Z(\alpha)}{\alpha-\alpha_1} + C\delta(\alpha-\alpha_1)\,, \quad C = \text{const}\,.$$

But for any analytic function $h^+(\alpha)$ due to Plemelj formulas we get

$$\int \hat{f}(\alpha)h^+(\alpha)\,d\alpha = \pi i\,Z(\alpha_1)h^+(\alpha_1) + Ch^+(\alpha_1)\,,$$

and analyticity condition gives $C = -\pi i\,Z(\alpha_1)$,

$$\hat{f} = \frac{Z(\alpha)}{\alpha-\alpha_1} - \pi i\,Z(\alpha_1)\delta(\alpha-\alpha_1) = -2\pi i\,Z(\alpha_1)I_1(\alpha-\alpha_1) + \frac{Z(\alpha)-Z(\alpha_1)}{\alpha-\alpha_1}\,.$$

Absolutely similar if $Y(\alpha)$ has several single roots $\alpha_1,\dots,\alpha_n$, then

$$\hat{f}(\alpha) = \frac{Z(\alpha)}{Y(\alpha)} - \pi i\sum_{j=1}^{n}\frac{Z(\alpha_j)}{Y'(\alpha_j)}\delta(\alpha-\alpha_j) = -2\pi i\sum_{j=1}^{n}\frac{Z(\alpha_j)}{Y'(\alpha_j)}I_1(\alpha-\alpha_j) + \hat{\varphi}\,,$$

in particular for two roots $Y = (\alpha - \alpha_1)(\alpha - \alpha_2)$ we have

$$\hat{f}(\alpha) = \frac{Z(\alpha)}{(\alpha - \alpha_1)(\alpha - \alpha_2)} - \pi i \frac{Z(\alpha_1)}{\alpha_1 - \alpha_2}\delta(\alpha - \alpha_1) - \pi i \frac{Z(\alpha_2)}{\alpha_2 - \alpha_1}\delta(\alpha - \alpha_2) =$$

$$= -2\pi i \frac{Z(\alpha_1)}{\alpha_1 - \alpha_2} I_1(\alpha - \alpha_1) - 2\pi i \frac{Z(\alpha_2)}{\alpha_2 - \alpha_1} I_1(\alpha - \alpha_2) + \hat{\varphi}\,,$$

where $\hat{\varphi}$ is regular.

At last if $Y(\alpha) = (\alpha - \alpha_1)^2$, then

$$\hat{f} = \frac{Z(\alpha)}{(\alpha - \alpha_1)^2} + \pi i Z(\alpha_1)\delta'(\alpha - \alpha_1) - \pi i Z'(\alpha_1)\delta(\alpha - \alpha_1) =$$

$$= -2\pi i Z(\alpha_1) I_2(\alpha - \alpha_1) - 2\pi i Z'(\alpha_1) I_1(\alpha - \alpha_1) + \hat{\varphi}\,.$$

But if $Y = (\alpha - \alpha_1)(\alpha - \alpha_2)$ and $\alpha_2 \to \alpha_1$, then

$$\frac{Z(\alpha_1)}{\alpha_1 - \alpha_2}\delta(\alpha - \alpha_1) + \frac{Z(\alpha_2)}{\alpha_2 - \alpha_1}\delta(\alpha - \alpha_2) =$$

$$= \frac{Z(\alpha_2) - Z(\alpha_1)}{\alpha_2 - \alpha_1}\delta(\alpha - \alpha_1) + Z(\alpha_2)\frac{\delta(\alpha - \alpha_2) - \delta(\alpha - \alpha_1)}{\alpha_2 - \alpha_1} \to$$

$$\to Z'(\alpha_1)\delta(\alpha - \alpha_1) - Z(\alpha_1)\delta'(\alpha - \alpha_1)\,,$$

$$\frac{Z(\alpha_1)}{\alpha_1 - \alpha_2} I_1(\alpha - \alpha_1) + \frac{Z(\alpha_2)}{\alpha_2 - \alpha_1} I_1(\alpha - \alpha_2) \to Z'(\alpha_1) I_1(\alpha - \alpha_1) + Z(\alpha_1) I_2(\alpha - \alpha_1)\,,$$

and so the solution for double root and its singular term are limits of solution (its singular term respectively) for two single roots when roots coalesce.

Finally if $Z(\alpha)$ is singular, e.g. $(\alpha - \alpha_0)Z(\alpha) = Z_0(\alpha)$, where $Z_0(\alpha)$ is regular, then equation $Y(\alpha)\hat{f}(\alpha) = Z(\alpha)$ reduces to $(\alpha - \alpha_0)Y(\alpha)\hat{f}(\alpha) = Z_0(\alpha)$. Combining all previous considerations and taking into account singular parts of solutions only, we come to following results (in form we need).

Statement 1. Let $Y(\alpha)$ has no roots, then singular (up to regular term $\hat{\varphi}$) part of solution of the equation (14) under the condition

$$-2\pi i(\alpha - \alpha_0)Z(\alpha) = Z_0(\alpha) - \text{regular} \tag{15}$$

is

$$\hat{f} = \frac{Z_0(\alpha_0)}{Y(\alpha_0)} I_1(\alpha - \alpha_0) + \hat{\varphi}\,. \tag{16}$$

Statement 2. Let $Y(\alpha)$ has two single real roots $\alpha = \pm\alpha_1$, then singular part of solution of the equation (14) has a form:

- if $Z(\alpha)$ is regular, then

$$\hat{f} = -2\pi i \sum_{s=\pm 1} \frac{Z(s\alpha_1)}{Y'(s\alpha_1)} I_1(\alpha - s\alpha_1) + \hat{\varphi}\,, \tag{17}$$

- if under the condition (15) $\alpha_0 \neq \pm\alpha_1$ (non-resonant singular term), then

$$\hat{f} = S(Z_0, Y \mid \alpha, \alpha_0) + \hat{\varphi}\,,$$

$$S(Z_0, Y \mid \alpha, \alpha_0) = \frac{Z_0(\alpha_0)}{Y(\alpha_0)} I_1(\alpha - \alpha_0) - \sum_{s=\pm 1} \frac{Z_0(s\alpha_1)}{(\alpha_0 - s\alpha_1)Y'(s\alpha_1)} I_1(\alpha - s\alpha_1)\,; \tag{18}$$

- if under the condition (15) $\alpha_0 = \alpha_1$ or $\alpha_0 = -\alpha_1$ (resonant singular term), then solution has the same form $\hat{f} = S(Z_0, Y \mid \alpha, \pm\alpha_1) + \hat{\varphi}$, where

$$S(Z_0, Y \mid \alpha, \pm\alpha_1) = \lim_{\alpha_0 \to \pm\alpha_1} S(Z_0, Y \mid \alpha, \alpha_0)\,.$$

In particular if $\alpha_0 = \alpha_1$, then

$$S(Z_0, Y \mid \alpha, \alpha_1) = \frac{Z_0(\alpha_1)}{Y'(\alpha_1)} I_2(\alpha - \alpha_1) + \left[\frac{Z_0'(\alpha_1)}{Y'(\alpha_1)} - \frac{Z_0(\alpha_1) Y''(\alpha_1)}{2(Y'(\alpha_1))^2}\right] I_1(\alpha - \alpha_1) +$$

$$+ \frac{Z_0(-\alpha_1)}{(\alpha_0 + \alpha_1) Y'(-\alpha_1)} I_1(\alpha + \alpha_1)\,.$$

The situation with solvability of linear equations quite differs when we consider functions of two variables. Let $R(\xi,\omega) g(\xi,\omega) = h(\xi,\omega)$, where R and h are given analytic at $\operatorname{Im}\omega > 0$, $\operatorname{Im}\xi > 0$ functions, and let R has in analyticity region one root $\omega = \omega_0(\xi)$. Then similar to previous considerations $g = h/R + C\delta(\omega - \omega_0(\xi))$ and analyticity with respect to ω condition gives

$$C = -2\pi i \frac{h(\xi, \omega_0(\xi))}{R'_\omega(\xi, \omega_0(\xi))} \Longrightarrow g = g_0 + \hat{\varphi}\,, \quad g_0 = -2\pi i \frac{h(\xi, \omega_0(\xi))}{R'_\omega(\xi, \omega_0(\xi))} I(\omega - \omega_0(\xi))\,,$$

$$\hat{\varphi} = \frac{h(\xi,\omega)}{R(\xi,\omega)} - \frac{h(\xi, \omega_0(\xi))}{(\omega - \omega_0(\xi)) R'_\omega(\xi, \omega_0(\xi))}\,,$$

i.e. $\hat{\varphi}$ is analytic. But first summand g_0 isn't analytic with respect to ξ: for analytic function $f^+(\xi)$ we get

$$\int f^+(\xi) g_0 \, d\xi = -2\pi i \int \frac{f^+(\xi) h(\xi, \omega_0(\xi))}{R'_\omega(\xi, \omega_0(\xi))} I(\omega - \omega_0(\xi))\, d\xi$$

$$= -4\pi i \frac{f^+(\xi_0(\omega)) h(\xi_0(\omega), \omega) \xi_0'(\omega)}{R'_\omega(\xi_0(\omega), \omega)} \neq 0,$$

where $\xi_0(\omega)$ is inverse function to $\omega_0(\xi)$. So analytic solution exists if and only if $h(\xi_0(\omega), \omega) \equiv 0$, and this solution is $g = h/R$.

Statement 3. Linear equation $R(\xi,\omega) g(\xi,\omega) = h(\xi,\omega)$ in class of analytic functions of two variables has solvability conditions $h(\xi_j(\omega), \omega) \equiv 0$, where $\xi = \xi_j(\omega)$ are roots of $R(\xi,\omega)$ in analyticity region. If solvability conditions are fulfilled, then $g = h/R$.

6.3. **Change of variable.** Let $\hat{f} = I_1(\xi(\omega) - \xi(\omega_0))$, where $\xi'(\omega) \neq 0$. Then obviously function

$$Z_0(\omega) = -2\pi i (\omega - \omega_0) \hat{f} = \frac{\omega - \omega_0}{\xi(\omega) - \xi(\omega_0)}$$

is regular and due to statement 1, Eq. (16) we have $\hat{f} = Z_0(\omega_0) I_1(\omega - \omega_0)$. But $Z_0(\omega_0) = 1/\xi'(\omega_0)$ and we obtain formula for singular part at change of variable

$$I_1(\xi(\omega) - \xi(\omega_0)) = \frac{1}{\xi'(\omega_0)} I_1(\omega - \omega_0) + \hat{\varphi}\,, \tag{19}$$

where $\hat{\varphi}$ is regular.

References

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Vol. 6, Fluid Mechanics*, Elsiver, 2nd edition, 1987. MR0961259

[2] S.P. D'yakov, *On the stability of shock waves*, Sov. Phys. JETP, **27** (1954), 288–295. MR0066173

[3] S.P. D'yakov, *The interaction of shock waves with small perturbations*, Sov. Phys. JETP, **33** (1957), 948–974.

[4] V. M. Kontorovitch, *Reflection and refraction of sound by shock waves*, Sov. Phys. JETP, **33** (1957), 1527–1528.

[5] J.F. McKenzie, K.O. Westphal, *Interaction of linear waves with oblique shock waves*, Phys. Fluids, **11** (1968), 2350–2362. Zbl 0172.53203

[6] W.R. Johnson, O. Laporte, *Interaction of Cylindrical Sound Waves with a Stationary Shock Wave*, Phys. Fluids, **1** (1958), 82–94. MR0112501

[7] G.R. Fowles, G.W. Swan, *Stability of plane shock waves*, Phys.Rev.Lett., **30** (1973), 1023–1025.

[8] W.K. Van Moorhem, A.R.George, *On the stability of plane shocks*, Journal of Fluid Mechanics, **68** (1975), 97–108. Zbl 0303.76038

[9] G. W. Swan, G. R. Fowles, *Shock wave stability*, Phys. Fluids, **18** (1975), 28–35. Zbl 0321.76022

[10] G.R. Fowles, *Conditional stability of shock waves – a criterion for detonation*, Phys. Fluids, **19** (1976), 227–238. Zbl 0336.76027

[11] G.R. Fowles, *Stimulated and spontaneous emission of acoustic waves from shock fronts*, Phys. Fluids, **24** (1981), 220–227. Zbl 0464.76049

[12] M. Mond, I.M. Rutkevich, *Spontaneous acoustic emission from strong ionizing shocks*, Journal of Fluid Mechanics, **275** (1994), 121–146. MR1297865

[13] J.-C. Robinet, G. Casalis, *Critical interaction of a shock wave with an acoustic wave*, Phys. Fluids, **13** (2001), 1047–1059. MR1830406

[14] I. Men'shov, Y. Nakamura, *Abnormal amplification of sound waves refracted by an oblique shock wave*, JAXA Special Publication, **SP-03-002** (2004), 23–28.

[15] S.V. Iordanski, *On stability of a plane shock wave*, Journal of Applied Mathematics and Mechanics, **21** (1957), 465–472. MR0092523

[16] J.J. Erpenbeck, *Stability of Steady-State Equilibrium Detonations*, Phys.Fluids, **5** (1962), 604–614.

[17] J.J. Erpenbeck, *Stability of Step Shocks*, Phys.Fluids, **5** (1962), 1181–1187. MR0155515

[18] R.M. Zaidel, *Shock wave from a slightly curved piston*, Journal of Applied Mathematics and Mechanics, **24** (1960), 316–320. MR0129707

[19] R.M. Zaidel, *The perturbations propagation in plane shock waves*, Journal of Applied Mechanics and Technical Physics, **4** (1967), 30–39.

[20] J.W.Bates, *Initial-value-problem solution for isolated rippled shock fronts in arbitrary fluid media*, Phys. Rev. E, **69** (2004), 056313-1–056313-16. MR2096556

[21] J.W.Bates, *Instability of isolated planar shock waves*, Phys. Fluids, **19** (2007), 1–15. Zbl 1182.76053

[22] A.Tumin, *Initial-value problem for small disturbances in an idealized one-dimensional detonation*, Phys. Fluids, **19** (2007), 106105-1–106105-12.

[23] G.R. Fowles, A.F.P. Houwing, *Instabilities of shock and detonation waves*, Phys. Fluids, **27** (1984), 1982–1990. MR0758730

[24] N. M. Kuznetsov, *Contribution to shock-wave stability theory*, Zh. Eksp. Teor. Fiz., **88** (1985), 470–486.

[25] N. M. Kuznetsov, *Stability of shock waves*, Usp. Fiz. Nauk, **159** (1989), 493–527.

[26] J.W. Bates, D.C. Montgomery, *The D'yakov-Kontorovich Instability of Shock Waves in Real Gases*, Phys. Rev. Lett., **84** (2000), 1180–1183.

[27] A.V. Konyukhov, A.P. Likhachev, V.E. Fortov, S.I. Anisimov, A.M. Oparin, *Stability and ambiguous representation of shock wave discontinuity in thermodynamically nonideal media*, Pis'ma Zh. Eksp. Teor. Fiz., **90** (2009), 28–34.

[28] G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a complex variable*, Hod Books, Ithaka, N. Y., 1983. Zbl 0548.30001

[29] E. Zauderer, *Partial Differential Equations*, Wiley, 3rd edition, 2006. MR2244913

Evgeny Veniaminovich Semenko, Tatiana Ivanovna Semenko
Novosibirsk State Technical University,
Prospekt K. Marksa 20,
630073, Novosibirsk, Russia
E-mail address: semenko54@gmail.com