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ON CRACK PROPAGATIONS IN ELASTIC BODIES WITH THIN INCLUSIONS

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Abstract. The paper concerns an analysis of a crack propagation phenomena for an elastic body with thin inclusions and cracks. In the frame of free boundary approach, we investigate a dependence of the solutions on a rigidity parameter of the inclusion. A passage to the limit is justified as the parameter goes to infinity. Derivatives of the energy functionals are found with respect to the crack length for the models considered with different rigidity parameters. The Griffith criterion is used to describe a crack propagation. In so doing, an optimal control problem is investigated with a rigidity parameter being a control function. A cost functional coincides with a derivative of the energy functional with respect to the crack length. A solution existence is proved.

Keywords: thin elastic inclusion, Timoshenko beam, semirigid inclusion, crack, delamination, nonpenetration boundary condition, optimal control.

1. Introduction

Crack propagations in elastic bodies depend on properties of elastic bodies, their shapes, possible inclusions as well as on external forces. There are different approaches to describe inclusions in elastic bodies [15, 21, 25]. Last years, a lot of papers are published describing equilibrium states of elastic structures with thin delaminated inclusions [3, 10, 11, 19, 17, 22, 24]. The delamination means that we have a crack between the inclusion and the elastic material. In the papers mentioned, thin inclusions can be elastic, rigid and semirigid. It is known that starting with the thin elastic inclusions (Euler-Bernoulli and Timoshenko models of inclusions) with

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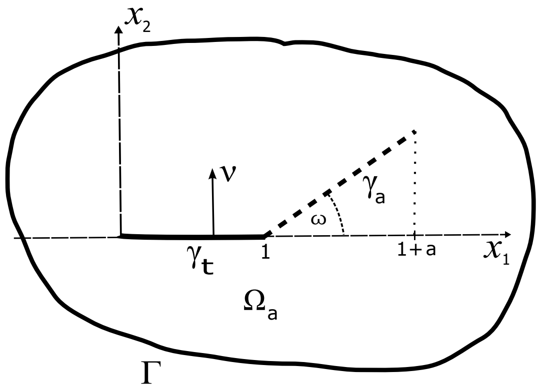

Fig. 1. Domain Ω_a with inclusion γ_t and crack γ_a

given rigidity parameters, it is possible to obtain thin semirigid and rigid inclusions by passing to the limit with respect to rigidity parameters [3, 10].

In the present paper, we consider an elastic body with a thin elastic delaminated inclusion and a crack. Free boundary approach is used to describe an equilibrium state of the body. This means that inequality type boundary conditions are considered at the crack faces which do not allow the opposite crack faces to penetrate each other, see [4, 5, 12, 13, 14, 18, 23]. In this case, contact points between the crack faces are unknown. It is well known that a classical linear approach to the crack modeling is characterized by linear boundary conditions at the crack faces what can imply a mutual penetration between the crack faces[20, 30]. We can mention results concerning optimal control problems for models with nonlinear boundary conditions at the crack faces [6, 7, 8, 9, 27, 28, 29]. In these papers, different cost functionals were considered with control functions being geometrical as well as physical parameters. Suitable results for linear models can be found in [1, 2, 26].

The thin elastic inclusion considered in the present paper is characterized by a rigidity parameter. We justify a passage to infinity of this parameter. In the limit, a thin semirigid inclusion is obtained. To describe a crack propagation, the Griffith rapture criterion is used. To this end, derivatives of the energy functional with respect to the crack length are found for the models considered, i.e for the models with the elastic and semirigid inclusions. The aim of the paper is to maximize the derivative of the energy functional over a set of rigidity parameters corresponding to elastic inclusions and the semirigid inclusion. A solution existence of this problem is established.

2. Elastic body with thin inclusion and crack

2.1. **Elastic inclusion.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ; $\gamma_t = (0,1) \times \{0\}$, γ_a is an interval with tips $(1,0)$ and $(1+a, a\,tg\,\omega)$, $a > 0$, $-\pi/2 < \omega < \pi/2$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to γ_t and to γ_a; $\tau = (\nu_2, -\nu_1)$ is a tangential vector, and set $\Omega_a = \Omega \setminus (\bar{\gamma}_t \cup \bar{\gamma}_a)$, see Fig. 1.

In what follows, the domain Ω_a represents a region filled with an elastic material, γ_a is a crack, and γ_t is an elastic inclusion with specified properties. In particular, we consider γ_t as a Timoshenko beam incorporated in the elastic body.

Assume that the delamination of the inclusion takes place at γ_t^+, thus we have a crack between the elastic body and the thin inclusion. Displacements of the inclusion should coincide with the displacements of the elastic body at γ_t^-. In our model, inequality type boundary conditions will be considered on γ_t and γ_a to prevent a mutual penetration between the crack faces. An equilibrium problem for the body Ω_a with the inclusion γ_t and the crack γ_a is formulated as follows. For given external forces $f = (f_1, f_2) \in L^2(\Omega)^2$ acting on the body, we want to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_a, and thin inclusion displacements v, w and a rotation angle φ defined on γ_t such that

$$-\mathrm{div}\ \sigma = f,\ \sigma - A\varepsilon(u) = 0 \text{ in } \Omega_a, \tag{1}$$

$$-\alpha w_{,11} = [\sigma_\tau],\ -v_{,11} - \varphi_{,1} = [\sigma_\nu] \text{ on } \gamma_t, \tag{2}$$

$$-\alpha\varphi_{,11} + v_{,1} + \varphi = 0 \text{ on } \gamma_t, \tag{3}$$

$$v = u_\nu^-,\ w = u_\tau^- \text{ on } \gamma_t, \tag{4}$$

$$u = 0 \text{ on } \Gamma;\ \varphi + v_{,1} = w_{,1} = \varphi_{,1} = 0 \text{ as } x_1 = 0, 1, \tag{5}$$

$$[u_\nu] \ge 0,\ \sigma_\nu^+ \le 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+[u_\nu] = 0 \text{ on } \gamma_t, \tag{6}$$

$$[u_\nu] \ge 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \le 0,\ \sigma_\tau = 0,\ \sigma_\nu[u_\nu] = 0 \text{ on } \gamma_a. \tag{7}$$

Here, $[\phi] = \phi^+ - \phi^-$ is a jump of a function ϕ on γ_t, γ_a, where $\phi^\pm$ are the traces of ϕ on the crack faces $\gamma_t^\pm, \gamma_a^\pm$. The signs $\pm$ fit to positive and negative directions of ν; $h_{,1} = \frac{dh}{dx_1}$, $(x_1, x_2) \in \Omega$; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $i, j = 1, 2$; $\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j)$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_\tau = \sigma\nu \cdot \tau$, $u_\nu = u\nu$, $u_\tau = u\tau$. The parameter $\alpha > 0$ characterizes a rigidity of the inclusion γ_t.

Functions defined on γ_t, we identify with functions of the variable x_1. Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices. By $A = \{a_{ijkl}\}, i, j, k, l = 1, 2$, we denote a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij},\ \ i, j, k, l = 1, 2,\ a_{ijkl} \in \mathbb{R},$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2\ \ \forall \xi_{ij},\ c_0 = const > 0\ .$$

Relations (1) are the equilibrium equations for the elastic body and Hooke's law, (2)-(3) are the Timoshenko equilibrium equations for the inclusion γ_t. According to (4) the vertical (along the axis x_2) and tangential (along the axis x_1) displacements of the elastic body coincide at γ_t with the inclusion displacements. The right-hand sides $[\sigma_\tau], [\sigma_\nu]$ in (2) describe forces acting on γ_t from the surrounding elastic media. The second group of boundary conditions (5) fit to zero moments and forces at the beam tips. The first inequalities in (6), (7) provide a mutual nonpenetration between the crack faces. All the rest relations of (6)-(7) are typical for crack problems with unknown set of a contact, see [5, 10].

We can provide a variational formulation of the problem (1)-(7). Introduce a space $W_a = H^1_\Gamma(\Omega_a)^2 \times H^1(\gamma_t)^3$ and a set of admissible displacements, with $u = (u_1, u_2)$,

$$K_a = \{(u, v, w, \varphi) \in W_a \,|\, [u_\nu] \ge 0 \text{ on } \gamma_t \cup \gamma_a;\ v = u_\nu^-,\ w = u_\tau^- \text{ on } \gamma_t\}$$

and the energy functional

$$\Pi_\alpha(u,v,w,\varphi)=\frac{1}{2}\int\limits_{\Omega_a}\sigma(u)\varepsilon(u)-\int\limits_{\Omega_a}fu+\frac{1}{2}\int\limits_{\gamma_t}\{\alpha w_{,1}^2+\alpha\varphi_{,1}^2+(v_{,1}+\varphi)^2\},$$

where the Sobolev space $H^1_\Gamma(\Omega_a)$ is defined as

$$H^1_\Gamma(\Omega_a)=\{\psi\in H^1(\Omega_a)\mid\psi=0\text{ on }\Gamma\}.$$

Here and below we write $\sigma(u)\varepsilon(u)$ instead of $\sigma_{ij}(u)\varepsilon_{ij}(u)$. There exists a unique solution of the problem:

$$\text{Find }(u,v,w,\varphi)\in K_a\text{ such that }\Pi_\alpha(u,v,w,\varphi)=\inf_{K_a}\Pi_\alpha.$$

This solution satisfies the variational inequality

$$(u,v,w,\varphi)\in K_a,\tag{8}$$

$$\int\limits_{\Omega_a}\sigma(u)\varepsilon(\bar u-u)-\int\limits_{\Omega_a}f(\bar u-u)+\int\limits_{\gamma_t}\{\alpha w_{,1}(\bar w_{,1}-w_{,1})+\alpha\varphi_{,1}(\bar\varphi_{,1}-\varphi_{,1})\}\tag{9}$$
$$+\int\limits_{\gamma_t}(v_{,1}+\varphi)(\bar v_{,1}+\bar\varphi-v_{,1}-\varphi)\ge 0\quad\forall\,(\bar u,\bar v,\bar w,\bar\varphi)\in K_a.$$

To check a solvability of the problem (8)–(9) for any fixed α it suffices to establish a coercivity of the functional Π_α on the set K_a since its weak lower semicontinuity is clear. The coercivity can be proved as that in [3], and we omit the details.

Problem formulations (1)–(7) and (8)–(9) are equivalent provided that the solutions are smooth. Suitable arguments can be found in [3].

2.2. **Semirigid inclusion.** It is turned out that it is possible to justify a passage to the limit in the model (8)-(9) as $\alpha\to\infty$. This subsection concerns a justification of the passage. The limit problem will describe an equilibrium state of the elastic body with a thin semirigid inclusion γ_t and the crack γ_a.

For a fixed value α a solution of the problem (8)-(9) satisfies the variational inequality

$$(u^\alpha,v^\alpha,w^\alpha,\varphi^\alpha)\in K_a,\tag{10}$$

$$\int\limits_{\Omega_a}\sigma(u^\alpha)\varepsilon(\bar u-u^\alpha)-\int\limits_{\Omega_a}f(\bar u-u^\alpha)+\int\limits_{\gamma_t}\{\alpha w^\alpha_{,1}(\bar w_{,1}-w^\alpha_{,1})+\alpha\varphi^\alpha_{,1}(\bar\varphi_{,1}-\varphi^\alpha_{,1})\}\tag{11}$$
$$+\int\limits_{\gamma_t}(v^\alpha_{,1}+\varphi^\alpha)(\bar v_{,1}+\bar\varphi-v^\alpha_{,1}-\varphi^\alpha)\ge 0\quad\forall\,(\bar u,\bar v,\bar w,\bar\varphi)\in K_a.$$

$>$From (11) it follows for $\beta > 0$:

$$\int_{\Omega_a} \sigma(u^\alpha)\varepsilon(u^\alpha) - \int_{\Omega_a} f u^\alpha \tag{12}$$
$$+ \int_{\gamma_t} \{\alpha (w^\alpha_{,1})^2 + \alpha(\varphi^\alpha_{,1})^2 + (v^\alpha_{,1} + \varphi^\alpha)^2\} \pm \beta \int_{\gamma_t} \{(w^\alpha)^2 + (v^\alpha)^2\} = 0.$$

By Korn's inequality, boundary conditions (4) and imbedding theorems, we have for a small β

$$\frac{1}{2}\int_{\Omega_a} \sigma(u^\alpha)\varepsilon(u^\alpha) - \beta \int_{\gamma_t} \{(w^\alpha)^2 + (v^\alpha)^2\} \geq 0.$$

Next, by Lemma proved in [3], there exists $c > 0$ such that

$$\int_{\gamma_t} \{v^2 + w^2 + w^2_{,1} + \varphi^2_{,1} + (v_{,1} + \varphi)^2\} \tag{13}$$
$$\geq c\|(v, w, \varphi)\|^2_{H^1(\gamma_t)^3} \quad \forall (v, w, \varphi) \in H^1(\gamma_t)^3.$$

Consequently, from (12) the following estimate follows being uniform in α

$$\|u^\alpha\|_{H^1_\Gamma(\Omega_a)^2} \leq c, \tag{14}$$

and, moreover, for $\alpha \geq \alpha_0 > 0$

$$\|(v^\alpha, w^\alpha, \varphi^\alpha)\|_{H^1(\gamma_t)^3} \leq c, \tag{15}$$

$$\alpha \int_{\gamma_t} \{(w^\alpha_{,1})^2 + (\varphi^\alpha_{,1})^2\} \leq c. \tag{16}$$

By (14)-(16), we can assume that as $\alpha \to \infty$

$$u^\alpha \to u \text{ weakly in } H^1_\Gamma(\Omega_a)^2, \tag{17}$$

$$(v^\alpha, w^\alpha, \varphi^\alpha) \to (v, w, \varphi) \text{ weakly in } H^1(\gamma_t)^3, \tag{18}$$
$$w_{,1} = 0,\ \varphi_{,1} = 0 \text{ on } \gamma_t.$$

Hence, it follows that constants c_1, c_2 exist such that

$$w = c_1,\ \varphi = c_2 \text{ on } \gamma_t;\ c_1, c_2 \in \mathbb{R}. \tag{19}$$

Now define a set of admissible displacements for a limit problem, with $u = (u_1, u_2)$,

$$K^a = \{(u, v, w, \varphi) \in W_a \mid [u_\nu] \geq 0 \text{ on } \gamma_t \cup \gamma_a;\ v = u^-_\nu,$$
$$w = u^-_\tau \text{ on } \gamma_t;\ w, \varphi \in \mathbb{R}\}.$$

In the definition of the set K^a, a function $\varphi \in \mathbb{R}$ is arbitrary.

Take an arbitrary element $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K^a$. Then $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K_a$. We substitute $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi})$ in (11) as a test function. In this case, by (17)-(18), a passage to the limit in (11) can be fulfilled as $\alpha \to \infty$. It gives

$$(u, v, w, \varphi) \in K^a, \tag{20}$$

$$\int_{\Omega_a} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_a} f(\bar{u} - u) \tag{21}$$
$$+ \int_{\gamma_t} (v_{,1} + \varphi)(\bar{v}_{,1} + \bar{\varphi} - v_{,1} - \varphi) \geq 0 \;\; \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K^a.$$

Hence, the following statement is proved.

Theorem 1. *As $\alpha \to \infty$, the solutions of the problem* (10)-(11) *converge in the sense* (17)–(19) *to the solution of* (20)–(21).

Along with the variational formulation (20)-(21), a differential formulation of this problem can be provided: find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_a, a function v defined on γ_t, and constants c_1, c_2 such that

$$-\text{div}\ \sigma = f, \;\; \sigma - A\varepsilon(u) = 0 \text{ in } \Omega_a, \tag{22}$$

$$-v_{,11} = [\sigma_\nu] \text{ on } \gamma_t, \tag{23}$$

$$v = u_\nu^-,\ c_1 = u_\tau^- \text{ on } \gamma_t, \tag{24}$$

$$u = 0 \text{ on } \Gamma;\; v_{,1} + c_2 = 0 \text{ as } x_1 = 0, 1, \tag{25}$$

$$[u_\nu] \geq 0,\ \sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+[u_\nu] = 0 \text{ on } \gamma_t, \tag{26}$$

$$[u_\nu] \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ \sigma_\nu[u_\nu] = 0 \text{ on } \gamma_a, \tag{27}$$

$$\int_{\gamma_t} \sigma_\tau^- = 0,\ \int_{\gamma_t} (v_{,1} + c_2) = 0. \tag{28}$$

It is possible to prove that problem formulations (20)-(21) and (22)-(28) are equivalent provided that the solutions are smooth. We omit the arguments.

The problem (22)-(28) describes an equilibrium state of the elastic body Ω_a with a crack γ_a and the semirigid inclusion γ_t. We see that in this case, the only one equilibrium equation (23) is considered for the inclusion γ_t.

We have proved a solution existence of the problem (20)-(21) by passing to the limit in the problem (8)-(9). On the other hand, a principal point for the sequel is that we can formulate the problem (20)-(21) in a variational form. Namely, define a functional $\Pi :\ K^a \to \mathbb{R}$,

$$\Pi(u, v, w, \varphi) = \frac{1}{2} \int_{\Omega_a} \sigma(u)\varepsilon(u) - \int_{\Omega_a} fu + \frac{1}{2} \int_{\gamma_t} (v_{,1} + \varphi)^2.$$

Then the problem:

$$\text{Find } (u, v, w, \varphi) \in K^a \text{ such that } \Pi(u, v, w, \varphi) = \inf_{K^a} \Pi \tag{29}$$

has a solution satisfying the variational inequality (20)-(21). Indeed, a coercivity of the functional Π on the set K^a can be proved as follows. For a small $\beta > 0$ and $(u, v, w, \varphi) \in K^a$,

$$\frac{1}{4} \int_{\Omega_a} \sigma(u)\varepsilon(u) - \beta \int_{\gamma_t} \{w^2 + v^2\} \geq 0.$$

Consequently, by Korn's inequality and (13), the following relations hold

$$\Pi(u,v,w,\varphi) \geq \frac{1}{4}\int\limits_{\Omega_a} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_a} fu$$

$$+\beta\int\limits_{\gamma_t}(v^2+w^2) + \frac{1}{2}\int\limits_{\gamma_t}\{w_{,1}^2+\varphi_{,1}^2+(v_{,1}+\varphi)^2\}$$

$$\geq c_1\|(u,v,w,\varphi)\|^2_{W_a} - c_2\|u\|_{H^1_\Gamma(\Omega_a)^2} \to \infty,\ \|(u,v,w,\varphi)\|_{W_a}\to\infty$$

with positive constants c_1, c_2 independent of functions what means the coercivity of the functional Π on the set K^a. In so doing, we take into account that $w, \varphi \in \mathbb{R}$.

By the weak lower semicontinuity of the functional Π, the problem (29) has a solution.

3. Derivatives of energy functionals

It is known that the Griffith rupture criterion is formulated in terms of derivatives of energy functionals. Namely, we have to consider a family of perturbed problems with perturbed crack lengths and find a derivative of the energy functional with respect to the crack length. In so doing, a family of problems is introduced with a small parameter δ. In particular, $\delta = 0$ fits to the unperturbed problem. We will illustrate the approach with respect to the problems (8)-(9) and (20)-(21).

3.1. **Derivatives of energy functionals for model** (8)-(9)**.** Assume that the perturbed crack $\gamma_{a+\delta}$ has tips $(1,0)$ and $(1+a+\delta,(a+\delta)\,tg\,\omega)$. Denote $\Omega_{a+\delta} = \Omega \setminus (\bar\gamma_t \cup \bar\gamma_{a+\delta})$. Setting of the problem perturbed with respect to (8)-(9) is as follows. We have to find a displacement field $u = (u_1^\delta, u_2^\delta)$, a stress tensor $\sigma^\delta = \{\sigma_{ij}^\delta\}, i,j = 1,2$, defined in $\Omega_{a+\delta}$, and thin inclusion displacements v^δ, w^δ and a rotation angle φ^δ defined on γ_t such that

$$-\mathrm{div}\,\sigma^\delta = f,\ \sigma^\delta - A\varepsilon(u^\delta) = 0 \text{ in } \Omega_{a+\delta}, \tag{30}$$

$$-\alpha w^\delta_{,11} = [\sigma^\delta_\tau],\ -v^\delta_{,11} - \varphi^\delta_{,1} = [\sigma^\delta_\nu] \text{ on } \gamma_t, \tag{31}$$

$$-\alpha\varphi^\delta_{,11} + v^\delta_{,1} + \varphi^\delta = 0 \text{ on } \gamma_t, \tag{32}$$

$$v^\delta = u^{\delta-}_\nu,\ w^\delta = u^{\delta-}_\tau \text{ on } \gamma_t, \tag{33}$$

$$u^\delta = 0 \text{ on } \Gamma;\ \varphi^\delta + v^\delta_{,1} = w^\delta_{,1} = \varphi^\delta_{,1} = 0 \text{ as } x_1 = 0,1, \tag{34}$$

$$[u^\delta_\nu] \geq 0,\ \sigma^{\delta+}_\nu \leq 0,\ \sigma^{\delta+}_\tau = 0,\ \sigma^{\delta+}_\nu[u^\delta_\nu] = 0 \text{ on } \gamma_t, \tag{35}$$

$$[u^\delta_\nu] \geq 0,\ [\sigma^\delta_\nu] = 0,\ \sigma^\delta_\nu \leq 0,\ \sigma^\delta_\tau = 0,\ \sigma^\delta_\nu[u^\delta_\nu] = 0 \text{ on } \gamma_{a+\delta}. \tag{36}$$

Like (1)-(7), the problem (30)-(36) admits a variational formulation. To this end, introduce a set of admissible displacements, with $u = (u_1, u_2)$,

$$K_{a+\delta} = \{(u,v,w,\varphi) \in W_{a+\delta} \,|\, [u_\nu] \geq 0 \text{ on } \gamma_t \cup \gamma_{a+\delta};$$
$$v = u^-_\nu,\ w = u^-_\tau \text{ on } \gamma_t\}$$

and the energy functional

$$\Pi^\delta_\alpha(u,v,w,\varphi) = \frac{1}{2}\int\limits_{\Omega_{a+\delta}}\sigma(u)\varepsilon(u) - \int\limits_{\Omega_{a+\delta}} fu + \frac{1}{2}\int\limits_{\gamma_t}\{\alpha w^2_{,1} + \alpha\varphi^2_{,1} + (v_{,1}+\varphi)^2\}.$$

Then we can solve the problem:

$$\text{Find } (u,v,w,\varphi)\in K_{a+\delta} \text{ such that } \Pi_\alpha^\delta(u,v,w,\varphi)=\inf_{K_{a+\delta}}\Pi_\alpha^\delta.$$

This solution satisfies the variational inequality

$$(u^\delta,v^\delta,w^\delta,\varphi^\delta)\in K_{a+\delta}, \tag{37}$$

$$\int\limits_{\Omega_{a+\delta}}\sigma(u^\delta)\varepsilon(\bar u-u^\delta)-\int\limits_{\Omega_{a+\delta}}f(\bar u-u^\delta)$$
$$+\int\limits_{\gamma_t}\{\alpha w^\delta_{,1}(\bar w_{,1}-w^\delta_{,1})+\alpha\varphi^\delta_{,1}(\bar\varphi_{,1}-\varphi^\delta_{,1})\} \tag{38}$$
$$+\int\limits_{\gamma_t}(v^\delta_{,1}+\varphi^\delta)(\bar v_{,1}+\bar\varphi-v^\delta_{,1}-\varphi^\delta)\ge 0\ \ \forall(\bar u,\bar v,\bar w,\bar\varphi)\in K_{a+\delta}.$$

Next consider a function $\xi\in C_0^\infty(\Omega)$ with a support in a neighborhood of the point $(1+a, a\,tg\,\omega)$, assuming that $\xi=1$ near the point $(1+a,a\,tg\,\omega)$. We can consider a transformation of the independent variables

$$y=x+\delta V(x);\ x=(x_1,x_2)\in\Omega_a,\ y=(y_1,y_2)\in\Omega_{a+\delta}, \tag{39}$$
$$V(x)=(V_1(x),V_2(x))=(\xi(x),\xi(x)\,tg\,\omega).$$

Transformation (39) provides a one-to-one correspondence between the domains Ω_a and $\Omega_{a+\delta}$, as well as between the sets K_a and $K_{a+\delta}$. The derivative of the energy functional with respect to δ is calculated as

$$J_a(\alpha)=\lim_{\delta\to 0}\frac{\Pi_\alpha^\delta(u^\delta,v^\delta,w^\delta,\varphi^\delta)-\Pi_\alpha(u,v,w,\varphi)}{\delta}, \tag{40}$$

where (u,v,w,φ) and $(u^\delta,v^\delta,w^\delta,\varphi^\delta)$ are solutions of the problems (8)-(9) and (37)-(38), respectively.

A general scheme of finding the derivative (40) is described in [5], and we do not go in details. The result is as follows

$$J_a(\alpha)=\frac{1}{2}\int\limits_{\Omega_a}\{\sigma_{ij}(u)\varepsilon_{ij}(u)\mathrm{div}V-2\sigma_{ij}(u)E_{ij}(V;u)\}-\int\limits_{\Omega_a}\mathrm{div}(Vf_i)u_i, \tag{41}$$

where

$$E_{ij}(V;u)=\frac{1}{2}(u_{i,k}V_{k,j}+u_{j,k}V_{k,i}),\ i,j,k=1,2, \tag{42}$$

and (u,v,w,φ) is the solution of the unperturbed problem (8)-(9).

3.2. **Derivatives of energy functionals for model** (20)-(21)**.** Similar arguments can be applied to find the derivative of the energy functional with respect to the parameter δ in the problem perturbed to (20)-(21) and corresponding to the semirigid inclusion γ_t. Indeed, consider a set of admissible displacements for the problem perturbed to (20)-(21), assuming $u=(u_1,u_2)$,

$$K^{a+\delta}=\{(u,v,w,\varphi)\in W_{a+\delta}\,|\,[u_\nu]\ge 0 \text{ on } \gamma_t\cup\gamma_{a+\delta};\ v=u_\nu^-,$$
$$w=u_\tau^- \text{ on } \gamma_t;\ w,\varphi\in\mathbb{R}\}$$

and the energy functional

$$\Pi^\delta(u,v,w,\varphi)=\frac{1}{2}\int\limits_{\Omega_{a+\delta}}\sigma(u)\varepsilon(u)-\int\limits_{\Omega_{a+\delta}}fu+\frac{1}{2}\int\limits_{\gamma_t}(v_{,1}+\varphi)^2.$$

There exists a function $(u^\delta,v^\delta,w^\delta,\varphi^\delta)$ minimizing the functional Π^δ over the set $K^{a+\delta}$. This function satisfies the variational inequality

$$(u^\delta,v^\delta,w^\delta,\varphi^\delta)\in K^{a+\delta},$$

$$\int\limits_{\Omega_{a+\delta}}\sigma(u^\delta)\varepsilon(\bar u-u^\delta)-\int\limits_{\Omega_{a+\delta}}f(\bar u-u^\delta)$$

$$+\int\limits_{\gamma_t}(v^\delta_{,1}+\varphi^\delta)(\bar v_{,1}+\bar\varphi-v^\delta_{,1}-\varphi^\delta)\ge 0\ \ \forall\,(\bar u,\bar v,\bar w,\bar\varphi)\in K^{a+\delta}.$$

A formula for the derivative of the energy functional

$$J_a(\infty)=\lim_{\delta\to 0}\frac{\Pi^\delta(u^\delta,v^\delta,w^\delta,\varphi^\delta)-\Pi(u,v,w,\varphi)}{\delta}$$

can be found by the same scheme (see [5]), and we have

$$J_a(\infty)=\frac{1}{2}\int\limits_{\Omega_a}\{\sigma_{ij}(u)\varepsilon_{ij}(u)\mathrm{div}V-2\sigma_{ij}(u)E_{ij}(V;u)\}-\int\limits_{\Omega_a}\mathrm{div}(Vf_i)u_i\ , \tag{43}$$

where $E_{ij}(V,u)$ are defined by (42), and (u,v,w,φ) is the solution of the unperturbed problem (20)-(21) corresponding to $\alpha=\infty$.

Remark The formulae (41), (43) provide the derivatives of the energy functional with respect to δ, i.e. with respect to the length of the crack projection on the axis x_1. Formulae for the derivatives $J^c_a(\alpha), J^c_a(\infty)$ with respect to the crack length would be as follows:

$$J^c_a(\alpha)=J_a(\alpha)(1+tg^2\omega)^{-1/2};\ J^c_a(\infty)=J_a(\infty)(1+tg^2\omega)^{-1/2}.$$

4. Optimal control problem

In this section, an optimal control problem of the rigidity parameter α is analyzed. We know that for any fixed $\alpha>0$ the problem (8)-(9) is solvable, thus a solution $(u^\alpha,v^\alpha,w^\alpha,\varphi^\alpha)$ exists. On the other hand, it is possible to find a solution (u,v,w,φ) of the problem (20)-(21) corresponding to $\alpha=\infty$. Let $\alpha_0>0$ be a given number. An optimal control problem considered in this section provides the best rigidity parameter of the inclusion γ_t which maximizes a derivative of the energy functional with respect to the crack length. From the state of applications, it provides the most safe situation according to the Griffith criterion. To formulate the problem, define a cost functional $J_a(\alpha)$ for $\alpha\in[\alpha_0,\infty]$. Observe that $J_a(\alpha)\le 0$ for $\alpha>0,\alpha=\infty$. Optimal control problem is formulated as follows

$$\sup_{\alpha\in[\alpha_0,\infty]}J_a(\alpha). \tag{44}$$

For any finite value α we have to find a solution of the problem (8)-(9) and calculate the derivative (41). For $\alpha=\infty$, it is necessary to solve the problem (20)-(21) and find the derivative (43). We have to underline at this step that the mathematical models (8)-(9) and (20)-(21) corresponding to finite and infinite values of the parameter α are different.

The principal result of the paper is as follows.

Theorem 2. *There exists a solution of the optimal control problem* (44).

Proof. Let $\alpha^n \in [\alpha_0, \infty]$ be a maximizing sequence. For any α^n we can find a unique solution of the problem (8)-(9) provided that α^n is finite, or of the problem (20)-(21) for $\alpha = \infty$, and next we can find derivatives of the energy functionals. We may assume that the sequence α^n is convergent. There are two possible cases:

1. $\alpha^n \to \alpha^*,\ n \to \infty,\ \alpha^n \in [\alpha_0, \infty),\ \alpha^* \in \mathbb{R}$;
2. $\alpha^n \to \infty,\ n \to \infty,\ \alpha^n \in [\alpha_0, \infty)$.

If $\alpha^n = +\infty$ for $n \geq n_0$, then a solution of the problem (44) exists. We analyze two above cases separately.

Case 1. Assume that $\alpha^n \to \alpha^*,\ n \to \infty,\ \alpha^n \in [\alpha_0, \infty),\ \alpha^* \in \mathbb{R}$. For every n we can find a solution of the problem

$$(u^n, v^n, w^n, \varphi^n) \in K_a, \tag{45}$$

$$\int\limits_{\Omega_a} \sigma(u^n)\varepsilon(\bar{u} - u^n) - \int\limits_{\Omega_a} f(\bar{u} - u^n) + \int\limits_{\gamma_t} \{\alpha^n w^n_{,1}(\bar{w}_{,1} - w^n_{,1}) + \alpha^n \varphi^n_{,1}(\bar{\varphi}_{,1} - \varphi^n_{,1})\} \tag{46}$$
$$+ \int\limits_{\gamma_t} (v^n_{,1} + \varphi^n)(\bar{v}_{,1} + \bar{\varphi} - v^n_{,1} - \varphi^n) \geq 0 \ \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K_a.$$

Like in Section 2, in deriving a priori estimates, from (45)-(46) it follows the estimate, being uniform with respect to n

$$\|(u^n, v^n, w^n, \varphi^n)\|_{W_a} \leq c. \tag{47}$$

By (47), we can assume choosing a subsequence, if necessary, such that as $n \to \infty$

$$(u^n, v^n, w^n, \varphi^n) \to (u, v, w, \varphi) \text{ weakly in } W_a. \tag{48}$$

Since $\alpha^n \to \alpha^*$, it is possible to pass to the limit in (45)-(46) as $n \to \infty$ which gives

$$(u, v, w, \varphi) \in K_a, \tag{49}$$

$$\int\limits_{\Omega_a} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_a} f(\bar{u} - u) + \int\limits_{\gamma_t} \{\alpha^* w_{,1}(\bar{w}_{,1} - w_{,1}) + \alpha^* \varphi_{,1}(\bar{\varphi}_{,1} - \varphi_{,1})\} \tag{50}$$
$$+ \int\limits_{\gamma_t} (v_{,1} + \varphi)(\bar{v}_{,1} + \bar{\varphi} - v_{,1} - \varphi) \geq 0 \ \ \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K_a.$$

The relations (49)-(50) prove that the limit solution (u, v, w, φ) from (48) corresponds to the parameter α^*.

Let us check that, in fact, we have a strong convergence in W_a of the sequence $(u^n, v^n, w^n, \varphi^n)$. Indeed, from (45)-(46), (48)-(50) it follows

$$\int\limits_{\Omega_a} \sigma(u^n)\varepsilon(u^n) + \int\limits_{\gamma_t} \{\alpha^n (w^n_{,1})^2 + \alpha^n (\varphi^n_{,1})^2 + (v^n_{,1} + \varphi^n)^2\} = \int\limits_{\Omega_a} f u^n$$
$$\to \int\limits_{\Omega_a} f u \int\limits_{\Omega_a} \sigma(u)\varepsilon(u) + \int\limits_{\gamma_t} \{\alpha^* (w_{,1})^2 + \alpha^* (\varphi_{,1})^2 + (v_{,1} + \varphi)^2\}.$$

Thus, by the estimate (13), we have a convergence of norms in W_a, and by (48),

$$(u^n, v^n, w^n, \varphi^n) \to (u, v, w, \varphi) \text{ strongly in } W_a.$$

Now we are ready to complete an existence proof of optimal value of α in the case 1:

$$\sup_{\alpha\in[\alpha_0,\infty]} J_a(\alpha) = \lim_{n\to\infty} J_a(\alpha^n)$$
$$= \lim_{n\to\infty} \Big\{ \frac{1}{2} \int\limits_{\Omega_a} \{\sigma_{ij}(u^n)\varepsilon_{ij}(u^n)\mathrm{div}V - 2\sigma_{ij}(u^n)E_{ij}(V; u^n)\}$$
$$- \int\limits_{\Omega_a} \mathrm{div}(V f_i) u^n_i \Big\}$$
$$= \frac{1}{2} \int\limits_{\Omega_a} \{\sigma_{ij}(u)\varepsilon_{ij}(u)\mathrm{div}V - 2\sigma_{ij}(u)E_{ij}(V; u)\} - \int\limits_{\Omega_a} \mathrm{div}(V f_i) u_i$$
$$= J_a(\alpha^*) \le \sup_{\alpha\in[\alpha_0,\infty]} J_a(\alpha).$$

Hence, α^* is a solution of the problem (44) in the case 1.

Case 2. Consider the situation when $\alpha^n \to \infty$, $n \to \infty$, $\alpha^n \in [\alpha_0, \infty)$. In this case, for any n, the solution $(u^n, v^n, w^n, \varphi^n)$ satisfies the relations (45)-(46). From (45)-(46) we obtain

$$\int\limits_{\Omega_a} \sigma(u^n)\varepsilon(u^n) - \int\limits_{\Omega_a} f u^n + \tag{51}$$
$$+ \int\limits_{\gamma_t} \{\alpha^n (w^n_{,1})^2 + \alpha^n (\varphi^n_{,1})^2 + (v^n_{,1} + \varphi^n)^2\} = 0.$$

Using the arguments of section 2, from (51) it follows uniformly in n

$$\|u^n\|_{H^1_\Gamma(\Omega_a)^2} \le c, \tag{52}$$

and uniformly in $n \ge n_0$

$$\alpha^n \|(w^n_{,1}, \varphi^n_{,1})\|_{L^2(\gamma_t)^2} \le c, \ \|(v^n, w^n, \varphi^n)\|_{H^1(\gamma_t)^3} \le c. \tag{53}$$

By (52)-(53), we can assume that as $n \to \infty$

$$(u^n, v^n, w^n, \varphi^n) \to (u, v, w, \varphi) \text{ weakly in } W_a\,; \tag{54}$$
$$w_{,1} = 0, \ \varphi_{,1} = 0 \text{ on } \gamma_t.$$

As we know, the limit functions from (54) satisfy a variational inequality

$$(u,v,w,\varphi)\in K^a,$$
$$\int\limits_{\Omega_a}\sigma(u)\varepsilon(\bar u-u)-\int\limits_{\Omega_a} f(\bar u-u)$$
$$+\int\limits_{\gamma_t}(v_{,1}+\varphi)(\bar v_{,1}+\bar\varphi-v_{,1}-\varphi)\ge 0\ \ \forall\,(\bar u,\bar v,\bar w,\bar\varphi)\in K^a,$$

which means that (u,v,w,φ) corresponds to $\alpha=\infty$ (compare with (20)-(21)). For the sequel, we have to prove a strong convergence in W_a of the sequence (u^n,v^n,w^n,φ^n). It is a more delicate question compared to the previous case. Since $K^a\subset K_a$, we can define the functional Π outside of K^a setting $\Pi\equiv+\infty$ on $K_a\backslash K^a$. In this case, $\Pi=\Pi_\alpha$ on K^a and $\Pi_\alpha\le\Pi$ for all finite α, consequently,

$$\Pi_{\alpha^n}(u^n,v^n,w^n,\varphi^n)\le\Pi_{\alpha^n}(u,v,w,\varphi)\le\Pi(u,v,w,\varphi),$$

thus

$$\limsup_{n\to\infty}\{\frac12\int\limits_{\Omega_a}\sigma(u^n)\varepsilon(u^n)-\int\limits_{\Omega_a} fu^n+\frac12\int\limits_{\gamma_t}(v^n_{,1}+\varphi^n)^2\}$$
$$\le\limsup_{n\to\infty}\Pi_{\alpha^n}(u^n,v^n,w^n,\varphi^n)\le\Pi(u,v,w,\varphi).$$

Since the linear terms are converging, it provides

$$\limsup_{n\to\infty}\{\frac12\int\limits_{\Omega_a}\sigma(u^n)\varepsilon(u^n)+\frac12\int\limits_{\gamma_t}(v^n_{,1}+\varphi^n)^2\}$$
$$\le\frac12\int\limits_{\Omega_a}\sigma(u)\varepsilon(u)+\frac12\int\limits_{\gamma_t}(v_{,1}+\varphi)^2.$$

By lower semi-continuity of the functional Π, and taking into account that $w_{,1}(x_1)=\varphi_{,1}(x_1)=0$ on γ_t, we obtain the convergence of norms in W_a,

$$\int\limits_{\Omega_a}\sigma(u^n)\varepsilon(u^n)+\int\limits_{\gamma_t}\{(w^n_{,1})^2+(\varphi^n_{,1})^2+(v^n_{,1}+\varphi^n)^2\}$$
$$\to\int\limits_{\Omega_a}\sigma(u)\varepsilon(u)+\int\limits_{\gamma_t}\{(w_{,1})^2+(\varphi_{,1})^2+(v_{,1}+\varphi)^2\}.$$

Consequently, by (54),

$$(u^n,v^n,w^n,\varphi^n)\to(u,v,w,\varphi)\text{ strongly in }W_a. \tag{55}$$

Taking into account (55), we get

$$\sup_{\alpha\in[\alpha_0,\infty]} J_a(\alpha) = \lim_{n\to\infty} J_a(\alpha^n)$$

$$= \lim_{n\to\infty} \Big\{ \frac{1}{2}\int\limits_{\Omega_a} \{\sigma_{ij}(u^n)\varepsilon_{ij}(u^n)\mathrm{div}V - 2\sigma_{ij}(u^n)E_{ij}(V;u^n)\}$$

$$- \int\limits_{\Omega_a} \mathrm{div}(Vf_i)u_i^n \Big\}$$

$$= \frac{1}{2}\int\limits_{\Omega_a} \{\sigma_{ij}(u)\varepsilon_{ij}(u)\mathrm{div}V - 2\sigma_{ij}(u)E_{ij}(V;u)\} - \int\limits_{\Omega_a} \mathrm{div}(Vf_i)u_i$$

$$= J_a(\infty) \le \sup_{\alpha\in[\alpha_0,\infty]} J_a(\alpha).$$

Hence, an existence of a solution in the case 2 is proved. The proof of Theorem 2 is complete. □

5. Conclusions

A mathematical analysis of models describing elastic bodies with thin inclusions and cracks is provided. Free boundary models are used, in particular, a set of a contact domain is unknown. A solution dependence on a rigidity parameter is investigated which takes into account both finite and infinite values of the parameter. In the frame of the Griffith approach, we prove a solution existence of an optimal control problem with the rigidity parameter being a control function.

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