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ON THE UNIQUE DETERMINATION OF DOMAINS BY THE CONDITION OF THE LOCAL ISOMETRY OF THE BOUNDARIES IN THE RELATIVE METRICS

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Abstract. The article contains the results of the author's recent investigations of the rigidity problems of domains in Euclidean spaces undertaken for the development of a new approach to the classical problem about the unique determination of bounded closed convex surfaces.

We prove a complete characterization of a plane domain U with smooth boundary (i.e., the Euclidean boundary $\operatorname{fr} U$ of U is a one-dimensional manifold of class C^1 without boundary) that is uniquely determined in the class of domains in $\mathbb{R}^2$ with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics. In the case where U is bounded, a necessary and sufficient condition for the unique determination of the type under consideration in the class of all bounded plane domains with smooth boundary is the convexity of U. If U is unbounded then its unique determination in the class of all plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics is equivalent to its strict convexity.

Keywords: intrinsic metric, relative metric of the boundary, local isometry of the boundaries, strict convexity.

1. Introduction

Let $\mathcal{U}$ be a class of domains (i.e., open connected sets) in the real Euclidean n-dimensional space $\mathbb{R}^n$, where $n \geq 2$. We say (see, e.g., [1]) that a domain $U \in \mathcal{U}$ is uniquely determined in the class $\mathcal{U}$ by the relative metric of its (Hausdorff)

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boundary if every domain $V \in \mathcal{U}$ whose Hausdorff boundary is isometric to the Hausdorff boundary of U in the relative metrics is itself isometric to U (in the Euclidean metric).

Remark 1.1. Suppose that U is a domain in $\mathbb{R}^n$ $(n \geq 2)$ and ρ_U is its intrinsic metric, i.e., $\rho_U(x, y)$, where $x, y \in U$, is equal to the infimum of the lengths of all paths joining x and y in U[1]. Consider the Hausdorff completion of the metric space (U, ρ_U), i.e., the completion of this space in the intrinsic metric ρ_U. Identifying the points of this completion that correspond to the points of the domain U with these points themselves and removing them from the completion, we obtain a metric space $(\mathrm{fr}_H U, \rho_{\mathrm{fr}_H U,U})$; the set $\mathrm{fr}_H U$ of its elements is called the Hausdorff boundary of the domain U, and $\rho_{\mathrm{fr}_H U,U}$ is the relative metric on this Hausdorff boundary. The isometry of the Hausdorff boundaries of domains U and V in their relative metrics means the existence of a surjective isometry $f : (\mathrm{fr}_H U, \rho_{\mathrm{fr}_H U,U}) \to (\mathrm{fr}_H V, \rho_{\mathrm{fr}_H V,V})$ between these boundaries.

Results of [2], [3], [4] imply in particular that any bounded domain in $\mathbb{R}^n$ is uniquely determined by the condition of the isometry of the boundaries in the relative metrics. At the same time, according to results of [5], a bounded polygonal plane domain U is uniquely determined by the condition of the local isometry of the boundaries in the relative metrics in the class of all such domains if and only if U is convex.

Remark 1.2. Let $\mathcal{M}$ be a class of domains in $\mathbb{R}^n$, $n \geq 2$. Following [1], we say that a domain $U \in \mathcal{M}$ is uniquely determined in the class $\mathcal{M}$ by the condition of the local isometry of the (Hausdorff) boundaries of domains in the relative metrics if, for any domain V belonging to the class $\mathcal{M}$, the local isometry of its Hausdorff boundary to the Hausdorff boundary of U in the relative metrics implies the isometry of U and V (in the Euclidean metric). The local isometry in the relative metrics between the Hausdorff boundaries $\mathrm{fr}_H U$ and $\mathrm{fr}_H V$ of U and V means the existence of a bijective mapping $f : \mathrm{fr}_H U \to \mathrm{fr}_H V$ of these boundaries that is a local isometry in their relative metrics, i.e., a mapping such that, for any $y \in \mathrm{fr}_H U$, there exists a number $\varepsilon > 0$ satisfying the following condition: for any two elements a and b in the ε-neighborhood $Z(y) = \{z \in \mathrm{fr}_H U : \rho_{\mathrm{fr}_H U,U}(z, y) < \varepsilon\}$ of y, $\rho_{\mathrm{fr}_H U,U}(a, b) = \rho_{\mathrm{fr}_H V,V}(f(a), f(b))$. It is clear that f^{-1} is also a local isometry in the relative metrics of the boundaries.

In this paper, we continue the study of the unique determination of domains by the condition of the local isometry of their boundaries in the relative metrics.

The article is mainly devoted to finding a complete description of conditions that are necessary and sufficient for a plane domain with smooth boundary to be uniquely determined by the condition of the local isometry of the boundaries in the class of all domains with smooth boundary (in the case of a bounded domain, in the class of all bounded plane domains with smooth boundary).

Below $[a, b] = \{bt + (1 - t)a \in \mathbb{R}^n : 0 \leq t \leq 1\}$, $[a, b[= \{bt + (1 - t)a \in \mathbb{R}^n : 0 \leq t < 1\}$ $(]a, b] = \{bt + (1 - t)a \in \mathbb{R}^n : 0 < t \leq 1\})$, and $]a, b[= \{bt + (1 - t)a \in \mathbb{R}^n : 0 < t < 1\}$ are the segment (closed interval), the half-open interval and the interval

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[1]As a rule, the relative metric of the boundary of a domain differs from its intrinsic metric. Indeed, assume that S is a closed convex surface in $\mathbb{R}^3$, i.e., it is the boundary of a bounded convex domain $G \subset \mathbb{R}^3$. Let $U = \mathbb{R}^3 \setminus \mathrm{cl}\, G$ be the complement of the closure $\mathrm{cl}\, G$ of G. Then the intrinsic metric on the surface $S = \mathrm{fr}\, U$ coincides with the relative metric $\rho_{\mathrm{fr}\, U,U}$ on the boundary of U. But $\rho_{\mathrm{fr}\, G,G}(x, y) = |x - y|$ for all $x, y \in S$.

in $\mathbb{R}^n$ with endpoints $a, b \in \mathbb{R}^n$, $a \neq b$. $\operatorname{Int} I$ is the interior of the segment (of the half-open interval) I, $\operatorname{Int}]a, b[=]a, b[$. $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ is the open ball in $\mathbb{R}^n$ of radius r ($0 < r < \infty$) centered at $x_0 \in \mathbb{R}^n$. Id_E is the identity mapping of a set E: $\operatorname{Id}_E(x) = x$ for $x \in E$.

In what follows, paths $\gamma : [\alpha, \beta] \to \mathbb{R}^n$, where $\alpha, \beta \in \mathbb{R}$, are assumed continuous.

2. The Case of Plane Domains

The first main result of the article is the following theorem:

Theorem 2.1. *Let U be a domain in $\mathbb{R}^2$ with smooth boundary. Then*

(i) if U is bounded then it is uniquely determined in the class of all bounded plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics if and only if U is convex;

(ii) if U is unbounded then the unique determination of U in the class of all plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics is equivalent to the strict convexity of U.

Remark 2.1. Let U be a domain in $\mathbb{R}^n$. As in [1], we say that U has smooth boundary (Lipschitz boundary) if the Euclidean boundary $\operatorname{fr} U$ of U is an $(n-1)$-submanifold of class C^1 (a Lipschitz submanifold) without boundary in $\mathbb{R}^n$. In the case of a domain U with Lipschitz boundary, its Hausdorff boundary $\operatorname{fr}_H U$ is naturally identified with the Euclidean boundary and the metric $\rho_{\operatorname{fr} U, U}$ corresponding to the Hausdorff metric can be defined as follows:

$$\rho_{\operatorname{fr} U, U}(x, y) = \liminf_{x' \to x, y' \to y;\, x', y' \in U} \{\inf[l(\gamma_{x', y', U})]\},$$

where $x, y \in \operatorname{fr} U$ and $\inf[l(\gamma_{x', y', U})]$ is the infimum of the lengths $l(\gamma_{x', y', U})$ of all smooth paths $\gamma_{x', y', U} : [0, 1] \to U$ joining x' and y' in U. Recall also that a domain U is said to be strictly convex if it is convex and the interior of the segment joining any two points in its closure $\operatorname{cl} U$ is contained in U.

Lemma 2.1. *Let U and V be two plane domains with smooth boundary and $f : \operatorname{fr} U \to \operatorname{fr} V$ be a bijective mapping that is a local isometry of the boundaries of these domains in the relative metrics. Then f is a (global) isometry of the boundaries $\operatorname{fr} U$ and $\operatorname{fr} V$ in their intrinsic metrics.*

Lemma 2.2. *Suppose that domains U and V and a mapping $f : \operatorname{fr} U \to \operatorname{fr} V$ satisfy the conditions of Lemma 2.1 and $\operatorname{fr} U$ is bounded. Then the boundary $\operatorname{fr} V$ of V is also bounded and f has the following property: there exists $\varepsilon > 0$ such that $\rho_{\operatorname{fr} U, U}(a, b) = \rho_{\operatorname{fr} V, V}(f(a), f(b))$ if $a, b \in \operatorname{fr} U$ and $\rho_{\operatorname{fr} U, U}(a, b) < \varepsilon$.*

Lemma 2.3. *Under the conditions of Lemma 2.1 and the additional assumption that the boundary $\operatorname{fr} U$ of the domain U is connected, the boundary $\operatorname{fr} V$ of V is also connected.*

The proofs of these lemmas are rather easy and therefore omitted.

Proof of Theorem 2.1. **Step 1.** Prove the first part of assertion (i), i.e., show that if U is a bounded convex plane domain with smooth boundary then it is uniquely determined in the class of all bounded plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics. To this end, suppose that, for a bounded convex plane domain U with smooth boundary, there exists a bounded plane domain V with smooth boundary whose boundary $\operatorname{fr} V$ is locally isometric to the boundary $\operatorname{fr} U$ of U in the relative metrics of the boundaries (further, let $f : \operatorname{fr} U \to \operatorname{fr} V$ be a fixed mapping realizing such an isometry). Then, by Lemmas 2.2 and 2.3, the boundary of V is connected (and hence V is a Jordan domain), and f possesses the property indicated in Lemma 2.2. Performing if necessary an additional inversion with respect to a straight line, we may also assume that $f : \operatorname{fr} U \to \operatorname{fr} V$ preserves the orientation of the boundary $\operatorname{fr} U$ of U induced by the canonical orientation of this domain, i.e., f "transfers" this orientation to the orientation of the boundary $\operatorname{fr} V$ of V induced by the canonical orientation of V.

Next, let $I = [a, b]$, where $a \neq b$, be a segment such that $I \subset \operatorname{fr} U$ and the image $f(I)$ of I is no longer a segment; moreover, any other segment $I^* = [a^*, b^*]$ ($a^* \neq b^*$, $I^* \subset \operatorname{fr} U$) of $\operatorname{fr} U$ having common points with I is a subset of I (below the set of all such segments I is denoted by Λ). We assert that $f(I)$ is an arc locally convex towards the interior of V. This means that every point $P \in f(I)$ has a closed neighborhood $N = N(P)$ for which $f(I) \cap N$ is an arc convex towards the interior of V, i.e., $f(I) \cap N(P) = f(I_P)$, where $I_P = [\alpha_P, \beta_P] \subset I$, and the closed curve C_P composed of $f(I_P)$ and the segment J_P joining the endpoints $f(\alpha_P)$ and $f(\beta_P)$ of the arc $f(I_P)$ either degenerates into the segment J_P or is the boundary of a bounded convex domain with the following property: There is a segment T with $\operatorname{Int} T \subset V$ situated on the straight line τ_P that is perpendicular to J_P and passes through its midpoint; moreover, some endpoint of T belongs to the arc $f(I_P)$ and its second endpoint is on the arc $(\operatorname{fr} V) \setminus f(I_P)$; both of these endpoints are to the same side of the straight line j_P containing J_P, and the endpoint belonging to the arc $f(I_P)$ is nearer to j_P than the other endpoint. Supposing the contrary, i.e., assuming that $f(I)$ is not an arc locally convex towards the interior of V, we (reckoning with the smoothness of the boundary of V) conclude that there exists a segment $I_P = [\alpha_P, \beta_P] \subset I$ such that either (1) $f(I_P)$ is a nonconvex arc or (2) the arc $f(I_P)$ is not a segment and is an arc convex towards the interior of the complement cV of V. In both cases, for the curve $f(\operatorname{Int} I_P)$, there exist a point $Q \in f(\operatorname{Int} I_P)$ and a locally supporting segment to this curve from the side of the complement cV of V all points of which except Q belong to the interior of cV, and Q is a common point of this segment and the boundary $\operatorname{fr} V$ of V. In case (2), these point and supporting segment can be found with the use the considerations applied in the proof of the Leja–Wilkosz theorem [6] exposed in [7] with the obvious modifications concerning our case.

In case (1), the curve $f(\operatorname{Int} I_P)$ contains a point L_P such that if we draw the tangent to the curve at L_P then there exist points $R_P \in f(\operatorname{Int} I_P)$ and $S_P \in f(\operatorname{Int} I_P)$ lying to different sides of this tangent. Replace the point R_P by the point that is the nearest point to L_P on the segment $[R_P, L_P]$ if necessary (preserve the notation R_P for this point) and belongs to the arc $f(\operatorname{Int} I_P)$. Similarly, replace S_P by the point of the arc $f(\operatorname{Int} I_P)$ that is the nearest point to L_P on the segment $[L_P, S_P]$. Then consider the two Jordan domains such that the boundary of the first of them is the union of the segment $[R_P, L_P]$ and of that arc from the three arcs constituting the set $f(\operatorname{Int} I_P) \setminus \{R_P, L_P\}$ whose endpoints are R_P and L_P, and the boundary of the second domain is constructed in the same way on the basis of the points L_P and S_P and the same arc $f(\operatorname{Int} I_P)$. By construction, one of these domains is contained in V and the other domain is included in cV. Considering the first of these domains and using the above-mentioned arguments from the proof of the Leja–Wilkosz theorem in [7], it is not hard to find a desired point

Q on the part of the boundary of this domain situated on $f(\operatorname{Int} I_P)$ and a locally supporting segment j to the curve $f(\operatorname{Int} I_P)$ at the point Q from the side of the complement cV of the domain V. Hence, in both cases (1) and (2), we are in the desired situation. Translating the tangent at the point Q to $f(\operatorname{Int} I_P)$ parallelly to itself at a sufficiently short distance to the side "where V lies", we easily get the following situation: there exist three points R'_P, L_P and S'_P on the boundary $\operatorname{fr} V$ of V belonging to the arc $f(\operatorname{Int} I_P)$ and such that $]R'_P, S'_P[\subset V$; moreover, the segment $[R'_P, S'_P]$ cuts off from V a Jordan subdomain whose boundary contains L_P. Clearly, $f^{-1}(R'_P)$, $f^{-1}(L_P)$ and $f^{-1}(S'_P)$ lie consecutively on $\operatorname{Int} I_P$. Hence, the triangle inequality holds for these points in the metric $\rho_{\operatorname{fr} U,U}$, but by their choice, for the points R'_P, L_P and S'_P, we have the strict triangle inequality in the metric $\rho_{\operatorname{fr} V,V}$. Since we could initially assume that the length of I_P is less than ε, where ε is the number of Lemma 2.2 corresponding to the mapping f considered now, we get a contradiction because, by this lemma, equality in the triangle inequality must also be satisfied for R'_P, L_P and S'_P. Therefore, $f(I)$ is an arc locally convex towards the interior of V.

We assert that the set Λ is finite. Clearly, since the length $l = l(\operatorname{fr} U)$ of the boundary $\operatorname{fr} U$ of U is finite, the finiteness of Λ follows from the fact that Λ does not contain segments of length at most $\varepsilon/2$. Assuming that a segment $\Delta = [\alpha_\Delta, \beta_\Delta] \in \Lambda$ has length $l(\Delta) \le \varepsilon/2$, consider points Q and S of this segment such that $Q \neq S$, Q is situated nearer, say, to the left endpoint α_Δ of the segment, and $f(Q)$ and $f(S)$ lie to the same side (and at a positive distance) of the tangent τ to $\operatorname{fr} V$ at the point $f(\alpha_\Delta)$; finally, the (least positive) angle between the tangent rays to the arcs $(\operatorname{fr} V) \setminus f(\Delta)$ and $f([\alpha_\Delta, S])$ at the points $f(\alpha_\Delta)$ and $f(S)$, respectively, is less than $\pi/4$. Further, let a point $P \in (\operatorname{fr} U) \setminus \Delta$ be so close to α_Δ that $\rho_{\operatorname{fr} U,U}(P, \alpha_\Delta) < \varepsilon/2$ and the points $f(P)$ and $f(\alpha_\Delta)$ lie to the same side of each of the tangents to $\operatorname{fr} V$ at the points $f(Q)$ and $f(S)$. Under these assumptions, the points P, Q and S, satisfy the strict triangle inequality in the metric $\rho_{\operatorname{fr} U,U}$ and their images $f(P)$, $f(Q)$ and $f(S)$, enjoy the triangle equality (in the metric $\rho_{\operatorname{fr} V,V}$). Hence, by the choice of ε (and Lemma 2.2), we get a contradiction, which implies that $\Delta = \varnothing$ and hence Λ is finite.

Let $\omega : [0, l] \to \operatorname{fr} U$ be the natural parametrization of the boundary $\operatorname{fr} U$ of U corresponding to the orientation of $\operatorname{fr} U$ generated by the canonical orientation of the domain U, and let $[\alpha_1, \beta_1] \subset [0, l]$ and $[\alpha_2, \beta_2] \subset [0, l]$ be segments such that $\omega([\alpha_j, \beta_j]) \in \Lambda$, where $j = 1, 2$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ and the arc $\omega(]\beta_1, \alpha_2[)$ does not contain points of the segments from Λ. We assert that $f|_{\omega([\beta_1,\alpha_2])}$ is a Euclidean isometry (i.e., there exists a Euclidean isometry $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|_{\omega([\beta_1,\alpha_2])} = f|_{\omega([\beta_1,\alpha_2])}$). Indeed, if the arc $\omega([\beta_1, \alpha_2])$ does not contain segments then it is strictly convex towards the complement cU of U. Hence, considering a point $c \in \omega(]\beta_1, \alpha_2[)$ and points $a, b \in \omega(]\beta_1, \alpha_2[)$, sufficiently close to c, where $\beta_1 < \omega^{-1}(a) < \omega^{-1}(c) < \omega^{-1}(b) < \alpha_2$ (the proximity of a and b to c is such that the distance between each two of the three points a, $f^{-1}(\gamma(s_0))$, and b considered below is less than $\varepsilon/2$); such a, b, and c are easily found from the hypothesis of theorem) and assuming that $[f(a), f(b)] \cap \operatorname{Int}(cV) \neq \varnothing$, we come to the situation where, for a shortest path $\gamma : [0, s] \to \operatorname{cl} V$ joining the points $f(a)$ and $f(b)$ in the closure $\operatorname{cl} V$ of V[2], there exists a point $s_0 \in]0, s[$ for which $\gamma(s_0) \in \operatorname{fr} V$ and $f^{-1}(\gamma(s_0)) \in f^{-1}(\operatorname{Im} \gamma \cap \operatorname{fr} V) \setminus \{a, b\}$ $(\neq \varnothing)$. But then the triple of points a, $f^{-1}(\gamma(s_0))$, b satisfies

[2]The existence of such a shortest path is guaranteed, for instance, by the results of [8].

the strict triangle inequality in the metric $\rho_{\mathrm{fr}\,U,U}$ while for $f(a)$, $\gamma(s_0)$, and $f(b)$, equality is attained in the triangle inequality in the metric $\rho_{\mathrm{fr}\,V,V}$. Therefore, by Lemma 2.2, $[f(a), f(b)] \subset \mathrm{cl}\,V$, which gives the equality $|f(a) - f(b)| = |a - b|$. Hence, the restriction $f|_{U_\varepsilon \cap \omega([\beta_1,\alpha_2])}$ of f to the intersection $U_\varepsilon \cap \omega([\beta_1, \alpha_2])$ of the ε-neighborhood U_ε $(= B(P, \varepsilon))$ of every point $P \in \omega([\beta_1, \alpha_2])$ and the arc $\omega([\beta_1, \alpha_2])$ itself is an isometry in the Euclidean metric. This easily implies that $f|_{\omega([\beta_1,\alpha_2])}$ is a Euclidean isometry. In the case where $\omega([\beta_1, \alpha_2])$ contains segments (which no longer belong to the set Λ and hence their images under f are also segments), the proof of the fact that $f|_{\omega([\beta_1,\alpha_2])}$ is a Euclidean isometry is close to the proof of this fact in the previous case, i.e., in the case where $\omega([\beta_1, \alpha_2])$ is strict convex. The difference in the arguments consists of insignificant and easily reproducible details, and we omit them.

Now, we are in a position to finish the proof of the first part of item (i) of the theorem. If the boundary $\mathrm{fr}\,U$ of U is such that $\Lambda = \varnothing$ then the first part of (i) is proved on the basis of the arguments used for proving the previous item. In the case of $\Lambda \neq \varnothing$, consider a segment $\Delta \in \Lambda$ and apply appropriate translation and rotation in the plane $\mathbb{R}^2$ to get the situation when the segment Δ lies on the ordinate axis, its upper endpoint is the origin, and the domain U is situated on the left half-plane. Let $\gamma : [0, l] \to \mathrm{fr}\,U$ $(\gamma(0) = \gamma(l) = (0, 0))$ be the natural parametrization of the boundary $\mathrm{fr}\,U$ of U corresponding to the orientation of $\mathrm{fr}\,U$ generated by the canonical orientation of U. If $f|_{\gamma([0,l-l(\Delta)])}$ is a Euclidean isometry then we may assume without loss of generality that $f|_{\gamma([0,l-l(\Delta)])} = \mathrm{Id}_{\gamma([0,l-l(\Delta)])}$. Involving also the fact that $f(\gamma([l - l(\Delta), l])) = f(\Delta)$ is not a segment (because $\Delta \in \Lambda$), we see that $f(\mathrm{fr}\,U) = \mathrm{fr}\,V$ is not a closed curve i.e., $f(\gamma(l)) \neq f(\gamma(0))$. The obtained contradiction implies that $\Lambda = \varnothing$. Thus, in this case, the first part of (i) is proved. Further, assume that Λ consists of n segments $[\gamma(\alpha_1), \gamma(\beta_1)]$, $[\gamma(\alpha_2), \gamma(\beta_2)]$, $\dots$, $[\gamma(\alpha_n), \gamma(\beta_n)] = \gamma([l - l(\Delta), l]) = \Delta$, where $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n = l$. Since $f|_{\gamma([0,\alpha_1])}$ is a Euclidean isometry, we may assume with loss of generality that $f|_{\gamma([0,\alpha_1])} = \mathrm{Id}_{\gamma([0,\alpha_1])}$. Then it is easy to show using induction that the rotation of the vector ω, where $-\omega$ is the unit tangent vector to the curve $\gamma([l - l(\Delta), l])$ (i.e., to the segment Δ) at the point $\gamma(l)$ is carried out (under the action of f) by the angle

$$V = -\sum_{k=1}^{n} \left\{ \sup_{\alpha_k \le t_1 < t_2 < \cdots < t_{\varkappa+1} \le \beta_k} \sum_{\nu=1}^{\varkappa} |\theta\gamma(t_{\nu+1}) - \theta\gamma(t_\nu)| \right\} \neq 0,$$

where $\theta\gamma(t)$ is the unit tangent vector to the curve $\gamma([t, l])$ at the point $\gamma(t)$ if $0 < t < l$ and to the curve $\gamma([0, l])$ at the point $\gamma(0) = (0, 0)$ when $t = l$. If $|V| < 2\pi$ then $\omega \neq \mu e_2$, where $\mu > 0$ and e_2 is the unit basis vector of the ordinate axis. And if $|V| \geq 2\pi$ then (since f preserves the orientation of the boundary) the curve $f(\mathrm{fr}\,V)$ cannot be closed without self-intersections. In both cases, we get a contradiction to the fact that the curve $\mathrm{fr}\,V$ is closed and smooth. Hence, the first part of item (i) of the theorem is completely proved.

Step 2. Prove the second part of item (i). Assuming that U is a bounded nonconvex plane domain with smooth boundary, we will show that, by an appropriate deformation, we can obtain another domain V whose boundary $\mathrm{fr}\,V$ is smooth and locally isometric to the boundary $\mathrm{fr}\,U$ of U in the relative metrics $\rho_{\mathrm{fr}\,U,U}$ and $\rho_{\mathrm{fr}\,V,V}$ of the boundaries, and the domains U and V themselves are not isometric to each other in the Euclidean metric, i.e., there is no Euclidean isometry $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that $J(U) = V$. If the boundary of U is disconnected then the above domain V is obtained by a small motion of one of the connected components of the boundary $\operatorname{fr} U$ with remaining connected components being fixed. If the boundary of U is connected, i.e., U is a Jordan domain, then proceed as follows: By the Leja–Wilkosz theorem [6], find a segment "locally strictly supporting from inside" I lying in U except a single interior point P for I, which belongs to $\operatorname{fr} U$. Consider a closed disk K centered at P whose radius is so small r that the boundary circle of this disk intersects with I at two points and the interior of one of the half-disks K_+ and K_- such that $K_+ \cup K_- = K \setminus I$, for instance, $\operatorname{Int} K_-$, does not contain points of the boundary $\operatorname{fr} U$ of U. Let u and v be two straight lines perpendicular to I situated to different sides of the normal n to it at P and sufficiently close to n. Consider the nearest points L and S of the sets $u \cap \operatorname{fr} U$ and $v \cap \operatorname{fr} U$ to the segment I and join them by a curve μ shortest in $\operatorname{cl} U$. Moreover, let r be so small that the closure of the shorter of the two arcs appearing on the boundary $\operatorname{fr} U$ after removing L and S from it is contained in $(\operatorname{Int} K_+) \cup \{P\}$ and that (by the smoothness of $\operatorname{fr} U$) the curve μ is smooth and convex towards the interior of U. We can get one of two cases: (1) $\mu \subset \operatorname{fr} U$, and (2) μ contains segments the interior of each of which is a subset of U. Further, consider (in both cases (1) and (2)) the points L^* and S^* belonging to $\lambda \cap \mu$ and chosen as follows: L^* and L lie to the same side of both the straight line containing the segment I (moreover, the point L^* is situated nearer to this straight line than L) and the straight line ψ perpendicular to I and passing through P; moreover, L^* is situated nearer to ψ than L; finally, the point S^* is defined in the similar way with respect to the location of S. Denote by U^* the Jordan domain with the boundary $((\operatorname{fr} U) \setminus \lambda^*) \cup \mu^*$, where λ^* and μ^* are the subarcs of the arcs λ and μ with endpoints L^* and S^*.

In case (1), a desired deformation of the domain $U = U^*$ is carried out rather obviously and reduced to a deformation of the curve μ^*. The arc μ^* is replaced by a convex arc $\widetilde{\mu^*}$ of the same length lying in the disk K and also convex towards the interior of U^* (more exactly, to the interior of the new domain $\widetilde{U} = \widetilde{U^*}$), and the arc $(\operatorname{fr} U^*) \setminus \lambda^* = (\operatorname{fr} U^*) \setminus \mu^*$ remains fixed; moreover, the closed arc $\widetilde{\mu^*} \cup \{(\operatorname{fr} U^*) \setminus \mu^*\}$ forms the smooth boundary of the new domain $\widetilde{U^*}$. It is not hard to verify that the boundaries of U^* and $\widetilde{U^*}$ are locally isometric in the relative metrics $\rho_{\operatorname{fr} U^*, U^*}$ and $\rho_{\operatorname{fr} \widetilde{U^*}, \widetilde{U^*}}$ (here, as a local isometry in the relative metrics of the boundaries of U^* and $\widetilde{U^*}$, we can take a mapping f of these boundaries leaving the arc $(\operatorname{fr} U^*) \setminus \mu^*$ fixed). It is also clear that, in constructing our deformation, we can get the following situation: It is impossible to map U^* onto $\widetilde{U^*}$ by a Euclidean isometry. Consequently, $\widetilde{U^*}$ is a desired domain V.

In case (2), the new domain V is constructed as follows: If $\mu^* = \lambda^*$ then V is constructed as in case (1). If μ^* contains segments with interior in U and endpoints in $\operatorname{fr} U$ (denote the set of all such segments by $\mathcal{M}^*$) then, starting from the domain U^*, we first construct the domain $\widetilde{U^*}$ of case (1) but in addition leave invariant the length of every segment of $\mathcal{M}^*$ under the action of the boundary mapping $f^* : \operatorname{fr} U^* \to \widetilde{U^*}$ appearing in the construction. This is possible due to the large degree of freedom in the construction of the curve $\operatorname{fr} \widetilde{U^*}$ given by the condition satisfied by μ^* in case (2)[3]. In this case, the final mapping $f : \operatorname{fr} U \to \operatorname{fr} \widetilde{U}$, where $\widetilde{U}$ $(= V)$ is the desired new domain, is constructed like this: it leaves fixed the

[3]In this connection, see Lemma 3.1.

curve $(\operatorname{fr} U) \setminus \lambda$ and coincides with f^* on the set $N = \mu^* \cap \lambda^*$. If the arc χ with endpoints A and B has no common points with $(\operatorname{fr} U)\setminus\lambda^*$ and is cut off from $\operatorname{fr} U$ by a segment from $\mathcal{M}^*$ then apply to this curve an orientation-preserving Euclidean isometry $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that $J(A) = f^*(A)$ and $J(B) = f^*(B)$, and then put $f|_\chi = J|_\chi$. In this case, V is the Jordan domain with the boundary $f(\operatorname{fr} U)$ and, by construction, $f : \operatorname{fr} U \to \operatorname{fr} V$ is a local isometry of the boundaries of U and V in their relative metrics. Moreover, the large degree of freedom in the choice of the above deformation of U, which still holds, makes it possible to carry out this deformation so that the domains U and V not be isometric in the Euclidean metric. So, in both cases (1) and (2), we have the following: if U is a nonconvex bounded plane domain with smooth boundary then it is not uniquely determined in the class of all bounded plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics. Consequently, item (i) of the theorem is completely proved.

Step 3. Let us now prove (ii). The fact that an unbounded strictly convex plane domain U with smooth boundary is uniquely determined in the class of all plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics can be proved on the basis of the arguments used in the proof of the first part of item (i). Considering one more plane domain V with smooth boundary, assuming that the boundaries of U and V are locally isometric in their relative metrics, and slightly modifying the arguments of the proof of (i), we show that $\operatorname{fr} U$ and $\operatorname{fr} V$ are isometric in the Euclidean metric, which implies the isometry of the domains U and V themselves.

Step 4. Proving the second part of item (ii), we first verify that if an unbounded plane domain U with smooth boundary is not convex then, by the same method as in the proof of the second part of (i), U can be deformed to a domain V with smooth boundary such that the boundaries $\operatorname{fr} U$ and $\operatorname{fr} V$ are locally isometric in their relative metrics while for the domains themselves there is no Euclidean isometry $J : \mathbb{R}^2 \to \mathbb{R}^2$ with $V = J(U)$. In the case under consideration, not every deformation of the boundary $\operatorname{fr} U$ of U leads us to the desired result; however, the above-mentioned degree of freedom in the choice of a deformation makes it possible to easily overcome this difficulty.

Step 5. Now, let U be an unbounded plane convex domain with smooth boundary that is not strictly convex. In this case, a domain V as above is constructed by rather simple methods. Indeed, the boundary $\operatorname{fr} U$ of U contains a segment I. We may assume that any other segment having common points with I and lying on $\operatorname{fr} U$ is included in I. Without loss of generality, we will also suppose that I is a segment of the abscissa axis with endpoints $A = (-2l, 0)$ and $B = (2l, 0)$ and U lies in the lower half-plane. Transform the boundary $\operatorname{fr} U$ of U as follows: The origin divides the boundary into two curves. The curve among these two containing the segment with endpoints $(0, 0)$ and $(0, 2l)$, remains fixed under this transformation. The segment with endpoints $(-l, 0)$ and $(0, 0)$ is transformed into the quarter of the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + (y - \frac{2l}{\pi})^2 = \frac{4l^2}{\pi^2}\}$ with endpoints $(0, 0)$ and $P = (-\frac{2l}{\pi}, \frac{2l}{\pi})$. The remaining part of the boundary $\operatorname{fr} U$ is first translated parallelly by the vector $((1 - \frac{2}{\pi})l, \frac{2l}{\pi})$ and then rotated by the angle $-\frac{\pi}{2}$ around P. As a result, we obtain a curve γ dividing the plane into two unbounded domains. Of these two domains, take as V the one that locally adjoins from below to the segment with endpoints $(0, 0)$ and $(0, 2l)$. It is easy to check that the boundary $\operatorname{fr} V$ of V is locally isometric to the boundary $\operatorname{fr} U$ of U in the relative metrics $\rho_{\operatorname{fr} U,U}$ and $\rho_{\operatorname{fr} V,V}$ of the boundaries but the domains U and V themselves are not isometric to each other in Euclidean metric. Thus, item (*ii*) and Theorem 2.1 are completely proved.

In connection with Theorem 2.1, it should be noted that there exists a bounded plane domain U with smooth boundary that is not uniquely determined in the class of all plane domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics (see [9]).

Theorem 2.1 fails if the boundary of a domain $U \subset \mathbb{R}^2$ is not smooth. Namely, the following assertion holds:

Theorem 2.2. *There exists a bounded plane domain U with Lipschitz boundary that is not convex but is uniquely determined in the class of all plane domains by the condition of the local isometry of the boundaries in the relative metrics.*

Remark 2.2. Theorem 2.2 is due to M. V. Korobkov (see [10]). Its proof will be discussed in a subsequent paper.

3. Appendix

Lemma 3.1. *Suppose that $f_1 : [0, a^*] \to \mathbb{R}$ $(a^* > 0)$ is a strictly increasing smooth function convex downwards and such that $f_1(0) = f_1'(0) = 0$, the graph Γ_1 of f_1 contains straight line segments, the union of the set $\mathcal{M}$ of all such segments is dense in Γ_1, and $(0,0)$ and $(a^*, f_1(a^*))$ are limit points for the set of the left endpoints of the segments of $\mathcal{M}$ (we assume that the segments $\Delta \in \mathcal{M}$ are maximal in the sense that any segment $\widetilde{\Delta} \subset \Gamma_1$ containing Δ coincides with Δ). Then for every $\varepsilon > 0$, there exists a strictly increasing smooth function $f_2 : [0, a^*] \to \mathbb{R}$ convex downwards differing from f_1 and having the following properties: $\|f_2 - f_1\|_{C([0,a^*])} \le \varepsilon$, $f_2(0) = f_2'(0) = 0$, $f_2(a^*) = f_1(a^*)$, $f_2'(a^*) = f_1'(a^*)$, and the mapping $F : \Gamma_1 \to \Gamma_2$ of the graphs of the functions f_1 and f_2 defined by the formula*

$$F : (x, y) \mapsto (\varphi^{-1}(x), f_2(\varphi^{-1}(f_1^{-1}(y)))) \in \Gamma_2, \quad (x, y) \in \Gamma_1,$$

where $\varphi : [0, a^*] \to [0, a^*]$ is a diffeomorphism satisfying the functional equation

$$\int_0^{\varphi(x)} \{1 + [f_1'(\varphi)]^2\}^{1/2} d\varphi = \int_0^x \{1 + [f_2'(t)]^2\}^{1/2} dt, \quad 0 \le x \le a^*,$$

is an isometry in the intrinsic metrics of the curves Γ_1 and Γ_2 that transforms each straight line segment of Γ_1 into a straight line segment of Γ_2 of the same length.

Proof. Suppose that x_1, x_2 and x_3 are three points in the interval $]0, a^*[$ such that $x_1 < x_2 < x_3$ and these points are the left endpoints of segments in $\mathcal{M}$ (the choice of the points x_1, x_2, x_3 will be specified below). Let k_1, k_2, k_3 and k_4 be four real positive numbers. We will choose the function f_2 among functions having the following form:

$$f_2(x) = \begin{cases} k_1 f_1(x), & 0 \le x < x_1; \\ (k_1 - k_2)[f_1(x_1) + f_1'(x_1)(x - x_1)] + k_2 f_1(x), & x_1 \le x < x_2; \\ \sum\limits_{s=1}^{2} (k_s - k_{s+1})[f_1(x_s) + f_1'(x_s)(x - x_s)] + k_3 f_1(x), & x_2 \le x < x_3; \\ \sum\limits_{s=1}^{3} (k_s - k_{s+1})[f_1(x_s) + f_1'(x_s)(x - x_s)] + k_4 f_1(x), & x_3 \le x \le a^*. \end{cases}$$

The equalities $f_2(a^*) = f_1(a^*)$ and $f_2'(a^*) = f_1'(a^*)$ lead us to the conditions

$$\sum_{s=1}^{3}(k_s - k_{s+1})[f_1(x_s) + f_1'(x_s)(a^* - x_s)] + (k_4 - 1)f_1(a^*) = 0 \tag{3.1}$$

and

$$\sum_{s=1}^{3}(k_s - k_{s+1})f_1'(x_s) + (k_4 - 1)f_1'(a^*) = 0. \tag{3.2}$$

The last condition follows from the requirement $\varphi(a^*) = a^*$. And since this requirement is the equality

$$\int_0^{a^*} \{1 + [f_1'(t)]^2\}^{1/2} dt = \int_0^{a^*} \{1 + [f_2'(t)]^2\}^{1/2} dt$$

we have

$$\int_0^{a^*} \{1 + [f_1'(t)]^2\}^{1/2} - \sum_{j=0}^{3} \int_{x_j}^{x_{j+1}} \left\{1 + \left[\sum_{s=1}^{j}(k_s - k_{s+1})f_1'(x_s) + k_{j+1}f_1'(t)\right]^2\right\}^{1/2} dt = 0, \tag{3.3}$$

where $x_0 = 0$, $x_4 = a^*$, and $\sum\limits_{s=1}^{0} \cdots = 0$.

The element $(k_1, k_2, k_3, k_4) = (1, 1, 1, 1) \in \mathbb{R}^4$ is a solution to system (3.1)-(3.3). At the same time, by construction, each straight line segment Δ on Γ_1 is transformed into a straight line segment on Γ_2; moreover, $l(F(\Delta)) = l(\Delta)$. Now, it suffices to prove that the rank of the Jacobian matrix of the left-hand sides of the equalities (3.1)-(3.3) calculated with respect to the variables k_1, k_2, k_3, and k_4 at the point $(1, 1, 1, 1)$ is equal to 3 under an appropriate choice of x_1, x_2 and x_3.

To this end, represent the above-mentioned matrix as

$$N = (A_{js})_{\substack{j=1,2,3\\ s=1,2,3,4}}, \tag{3.4}$$

where

$$A_{11} = -u_1 - f_1'(x_1) = -\int_0^{x_1} \frac{[f_1'(t)]^2 dt}{\{1 + [f_1'(t)]^2\}^{1/2}} - f_1'(x_1)\int_{x_1}^{a^*} \frac{f_1'(t)dt}{\{1 + [f_1'(t)]^2\}^{1/2}},$$

$$A_{12} = -\int_{x_1}^{x_2} \frac{f_1'(t)[f_1'(t) - f_1'(x_1)]dt}{\{1 + [f_1'(t)]^2\}^{1/2}} - \int_{x_2}^{a^*} \frac{f_1'(t)[f_1'(x_2) - f_1'(x_1)]dt}{\{1 + [f_1'(t)]^2\}^{1/2}},$$

$$A_{13} = -\int_{x_2}^{x_3} \frac{f_1'(t)[f_1'(t) - f_1'(x_2)]dt}{\{1 + [f_1'(t)]^2\}^{1/2}} - \int_{x_3}^{a^*} \frac{f_1'(t)[f_1'(x_3) - f_1'(x_2)]dt}{\{1 + [f_1'(t)]^2\}^{1/2}},$$

$$A_{14} = -\int_{x_3}^{a^*} \frac{f_1'(t)[f_1'(t) - f_1'(x_3)]dt}{\{1 + [f_1'(t)]^2\}^{1/2}},$$

$$A_{21} = f_1(x_1) + (a^* - x_1)f_1'(x_1),$$

$$A_{22} = f_1(x_2) - f_1(x_1) + (a^* - x_2)f_1'(x_2) - (a^* - x_1)f_1'(x_1),$$

$$A_{23} = f_1(x_3) - f_1(x_2) + (a^* - x_3)f_1'(x_3) - (a^* - x_2)f_1'(x_2),$$

$$A_{24} = f_1(a^*) - f_1(x_3) - (a^* - x_3)f_1'(x_3), \quad A_{31} = f_1'(x_1),$$

$$A_{32} = f_1'(x_2) - f_1'(x_1), \ \ A_{33} = f_1'(x_3) - f_1'(x_2), \ \ A_{34} = f_1'(a^*) - f_1'(x_3).$$

The rank of matrix (3.4) coincides with the rank of the matrix

$$\widetilde{N} = \left(\sum_{\nu=1}^{s} A_{j\nu}\right)_{\substack{j=1,2,3\\ s=1,2,3,4}},$$

in which

$$\sum_{\nu=1}^{2} A_{1\nu} = -u_2 - f_1'(x_2)v_2 =$$
$$-\int_0^{x_2} \frac{[f_1'(t)]^2 dt}{\{1+[f_1'(t)]^2\}^{1/2}} - f_1'(x_2)\int_{x_2}^{a^*} \frac{f_1'(t)dt}{\{1+[f_1'(t)]^2\}^{1/2}},$$

$$\sum_{\nu=1}^{3} A_{1\nu} = -u_3 - f_1'(x_3)v_3 =$$
$$-\int_0^{x_3} \frac{[f_1'(t)]^2 dt}{\{1+[f_1'(t)]^2\}^{1/2}} - f_1'(x_3)\int_{x_3}^{a^*} \frac{f_1'(t)dt}{\{1+[f_1'(t)]^2\}^{1/2}},$$

$$\sum_{\nu=1}^{4} A_{1\nu} = -u_4 = -\int_0^{a^*} \frac{[f_1'(t)]^2 dt}{\{1+[f_1'(t)]^2\}^{1/2}},$$

$$\sum_{\nu=1}^{2} A_{2\nu} = f_1'(x_2) + (a^* - x_2)f_1'(x_2), \ \ \sum_{\nu=1}^{3} A_{2\nu} = f_1(x_3) + (a^* - x_3)f_1'(x_3),$$

$$\sum_{\nu=1}^{4} A_{2\nu} = f_1(a^*), \ \ \sum_{\nu=1}^{2} A_{3\nu} = f_1'(x_2), \ \ \sum_{\nu=1}^{3} A_{3\nu} = f_1'(x_3), \ \ \sum_{\nu=1}^{4} A_{3\nu} = f_1'(a^*).$$

Consider the determinant

$$\delta_1 = \det\left\{\left(\sum_{\nu=1}^{s} A_{j\nu}\right)_{\substack{j=2,3\\ s=3,4}}\right\} = [f_1(x_3) + (a^* - x_3)f_1'(x_3)]f_1'(a^*) - f_1(a^*)f_1'(x_3) =$$
$$f_1'(x_3)f_1'(a^*)\left\{\frac{f_1(x_3)}{f_1'(x_3)} + a^* - x_3 - \frac{f_1(a^*)}{f_1'(a^*)}\right\}$$

($0 < f_1'(x_3) < f_1'(a^*)$ by the hypothesis of the lemma and the choice of the points x_1, x_2 and x_3). Transform the second factor on the right-hand side of the last equalities as follows:

$$\text{(3.5)} \quad \frac{\delta_1}{f_1'(a^*)f_1'(x_3)} = \frac{f_1(x_3)}{f_1'(x_3)} - \frac{f_1(a^*)}{f_1'(a^*)} + a^* - x_3 =$$
$$-\frac{f_1(a^*) - f_1(x_3)}{f_1'(a^*)} - f_1(x_3)\left(\frac{1}{f_1'(a^*)} - \frac{1}{f_1'(x_3)}\right) + a^* - x_3 =$$
$$-\frac{f_1'(\theta)(a^* - x_3)}{f_1'(a^*)} - f_1(x_3)\frac{f_1'(x_3) - f_1'(a^*)}{f_1'(a^*)f_1'(x_3)} + a^* - x_3 =$$
$$-\left\{\frac{f_1'(\theta) - f_1'(a^*)}{f_1'(x_3) - f_1'(a^*)}(a^* - x_3) + \frac{f_1(x_3)}{f_1'(x_3)}\right\}\frac{f_1'(x_3) - f_1'(a^*)}{f_1'(a^*)}$$

$(x_3 < \theta < a^*)$. The convexity of f_1 implies that

$$\left|\frac{f_1'(\theta) - f_1'(a^*)}{f_1'(x_3) - f_1'(a^*)}\right| \le 1.$$

Therefore, if the condition

$$a^* - x_3 < \frac{f_1(x_3)}{f_1'(x_3)} \tag{3.6}$$

holds then $\delta \neq 0$ and hence $\operatorname{rank} \widetilde{N} = \operatorname{rank} N \geq 2$.

It is shown similarly that if the point x_1 is fixed and x_2 $(> x_1)$ is so close to x_1 that

$$x_2 - x_1 < \frac{f_1(x_1)}{f_1'(x_1)} \tag{3.7}$$

then

$$\begin{aligned}\delta_2 = \det\left\{\left(\sum_{\nu=1}^{s} A_{j\nu}\right)_{\substack{j=2,3\\ s=1,2}}\right\} &\\ = \{f_1(x_1) + (a^* - x_1)f_1'(x_1)\}f_1'(x_2) - f_1'(x_1)\{f_1(x_2) + (a^* - x_2)f_1'(x_2)\} &\\ = f_1'(x_1)f_1'(x_2)\left\{\frac{f_1(x_1)}{f_1'(x_1)} - \frac{f_1(x_2)}{f_1'(x_2)} + x_2 - x_1\right\} \neq 0.&\end{aligned}$$

If we suppose that the first row of $\widetilde{N}$ is a linear combination of the other two rows of $\widetilde{N}$ then we obtain two pairs of relations

$$-\frac{u_3}{f_1'(x_3)} - v_3 = C_1\left\{\frac{f_1(x_3)}{f_1'(x_3)} + (a^* - x_3)\right\} + C_2,$$

$$-\frac{u_4}{f_1'(a^*)} = C_1\frac{f_1(a^*)}{f_1'(a^*)} + C_2$$

and

$$-\frac{u_1}{f_1'(x_1)} - v_1 = C_1\left\{\frac{f_1(x_1)}{f_1'(x_1)} + (a^* - x_1)\right\} + C_2,$$

$$-\frac{u_2}{f_1'(x_2)} - v_2 = C_1\left\{\frac{f_1(x_2)}{f_1'(x_2)} + (a^* - x_2)\right\} + C_2,$$

which implies that, under conditions (3.6) and (3.7),

$$\begin{aligned} C_1 &= \left\{-\frac{-u_3}{f_1'(x_3)} + \frac{u_4}{f_1'(a^*)} - v_3\right\}\Big/\left\{\frac{f_1(x_3)}{f_1'(x_3)} - \frac{f_1(a^*)}{f_1'(a^*)} + (a^* - x_3)\right\} \\ &= \left\{-\frac{u_2}{f_1'(x_2)} + \frac{u_1}{f_1'(x_1)} - v_2 + v_1\right\}\Big/\left\{\frac{f_1(x_2)}{f_1'(x_2)} - \frac{f_1(x_1)}{f_1'(x_1)} - (x_2 - x_1)\right\}; \end{aligned} \tag{3.8}$$

moreover, C_1 does not depend on the location of the points x_1, x_2 and x_3.

Further, we have

$$\begin{aligned}\frac{u_4}{f_1'(a^*)} - \frac{u_3}{f_1'(x_3)} - v_3 = \frac{u_4 - u_3}{f_1'(a^*)} + u_3\left(\frac{1}{f_1'(a^*)} - \frac{1}{f_1'(x_3)}\right) - v_3 = &\\ \left\{-\frac{u_3}{f_1'(a^*)f_1'(x_3)} + O(a^* - x_3)\right\}\{f_1'(a^*) - f_1'(x_3)\}.&\end{aligned}$$

Note that here we used the estimates

$$\left|\frac{u_4-u_3}{f_1'(a^*)}-v_3\right| = \left|\frac{1}{f_1'(a^*)}\int_{x_3}^{a^*}\frac{[f_1(t)]^2dt}{\{1+[f_1'(t)]^2\}^{1/2}} - \int_{x_3}^{a^*}\frac{f_1'(t)dt}{\{1+[f_1'(t)]^2\}^{1/2}}\right| =$$
$$\frac{1}{f_1'(a^*)}\left|\int_{x_3}^{a^*}\frac{f_1'(t)[f_1'(t)-f_1'(a^*)]dt}{\{1+[f_1'(t)]^2\}^{1/2}}\right| \leq$$
$$\frac{f_1'(a^*)-f_1'(x_3)}{f_1'(a^*)}\int_{x_3}^{a^*}\frac{f_1'(t)dt}{\{1+[f_1'(t)]^2\}^{1/2}} \leq \frac{f_1'(a^*)-f_1'(x_3)}{f_1'(a^*)}\frac{f_1'(a^*)}{\{1+[f_1'(a^*)]^2\}^{1/2}}.$$

These calculations and (3.5) yield the equality

$$C_1 = \lim_{x_3\to a^*}\left\{-\frac{u_3}{f_1'(x_3)}+\frac{u_4}{f_1'(a^*)}-v_3\right\}\Big/\left\{\frac{f_1(x_3)}{f_1'(x_3)}-\frac{f_1(a^*)}{f_1'(a^*)}+(a^*-x_3)\right\} = -\frac{u_4}{f_1(a^*)} \neq 0. \tag{3.9}$$

By analogy with (3.9), using (3.8), we can also prove that

$$C_1 = -\frac{u_1}{f_1(x_1)}.$$

Since f_1 is convex downwards, we infer

$$\frac{u_1}{f_1(x_1)} = \frac{1}{f_1(x_1)}\int_0^{x_1}\frac{[f_1'(t)]^2dt}{\{1+[f_1'(t)]^2\}^{1/2}} =$$
$$\frac{1}{f_1(x_1)}\frac{f_1'(x_1)}{\{1+[f_1'(x_1)]^2\}^{1/2}}\int_\xi^{x_1}f_1'(t)dt = \frac{f_1'(x_1)}{\{1+[f_1'(x_1)]^2\}^{1/2}}\frac{f_1(x_1)-f_1(\xi)}{f_1(x_1)}$$
$$= \frac{f_1'(x_1)}{\{1+[f_1'(x_1)]^2\}^{1/2}}\frac{f_1(x_1)-f_1(\xi)}{f_1(x_1)-f_1(0)} \leq \frac{f_1'(x_1)}{\{1+[f_1'(x_1)]^2\}^{1/2}} \to_{x_1\to 0} 0$$

($0<\xi<x_1$); therefore, $C_1 = 0$. As a result, we get a contradiction to (3.9). This in turn leads to the relations $\operatorname{rank} N = \operatorname{rank} \widetilde{N} = 3$. The lemma is proved.

In conclusion, note that the main results of our article were earlier announced in [10].

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