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ON THE LATTICE OF F-SUBGROUPS

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Abstract. In this paper we introduce the notions of f-subgroups and normal f-subgroups as natural generalizations of subgroups, normal subgroups and group topologies. It will be proved that the set of all f-subgroups on a group is a lattice containing the lattice of all normal f-subgroups as a sublattice. Some open questions are also presented.

Keywords: group topology, f-subgroup, normal f-subgroup.

1. Introduction

The close connection between the lattice of normal subgroups and the lattice of group topologies, led us to ask whether subgroup lattice can be generalized in a similar way. In this paper, we define the notion of f-subgroup and investigate a number of its basic properties. We shall prove that the dual of the subgroup lattice of a group is a sublattice of the f-subgroup lattice on it. On the other hand, the lattice of group topologies on a group can be lattice-embedded in the f-subgroup lattice. So, f-subgroups are also a generalization of group topologies.

f-subgroups are structures similar to group topologies. Mathematicians interested in topological groups, also investigate semitopological groups, paratopological groups and other partially topological groups to have the most general version of the propositions they prove. It seems they miss right topological groups as defined in this paper, which is an equivalent structure to f-subgroups. A right topological group, as defined in this paper, is another partially topological group, which can be called be most close structure to topological groups. Because f-subgroups are, in fact, the same as right group topologies, studying f-subgroups is a natural strategy for generalizing some propositions in topological groups theory. Also, we believe, as complex analysis provides powerful tools for studying real numbers, f-subgroups

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can provide some tools for studying topological groups and they may provide good connections to propositions proved in group theory; so, the existing notions and propositions in group theory can be inspiring for mathematicians interested in topological groups. Also, some existing notions and propositions in topological groups theory, may be inspiring for mathematicians interested in infinite groups.

Before defining an f-subgroup, we describe some notations and definitions that will be used throughout this paper. Let $f : X \to Y$ be any function, $A \subseteq X$, $B \subseteq Y$, $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$. Define:

$$\begin{aligned} f[A] &= \{f(a) \mid a \in A\}, \quad f^{-1}[B] = \{x \in X \mid f(x) \in B\}, \\ \dot{f}[\mathcal{A}] &= \{f[A] \mid A \in \mathcal{A}\}, \quad \dot{f^{-1}}[\mathcal{B}] = \big\{f^{-1}[B] \bigm| B \in \mathcal{B}\big\}. \end{aligned}$$

Let $(X, \leq)$ and $(Y, \leq)$ be posets. A function $f : X \to Y$ is said to be *increasing* if for every $a, b \in X$, $a \leq b \to f(a) \leq f(b)$. If f is an injection and $f^{-1} : f[X] \to X$ is also increasing then, f is said to be an *order-embedding*. An onto order-embedding is called an *order-isomorphism*. Let $(X, \leq)$ and $(Y, \leq)$ be lattices. An order-embedding $f : X \to Y$ which preserves finite supremum and finite infimum, will be called a *lattice-embedding*, or just an *embedding*. For complete lattices $(X, \leq)$ and $(Y, \leq)$, an order-embedding $f : X \to Y$ which preserves supremum and infimum is called a *complete lattice-embedding*, or just a *complete embedding*. M_n will denote the modular lattice with a greatest, a least and n pairwise incomparable elements.

The set of all subsets of a set X is denoted by $\mathcal{P}(X)$. One can easily see that $(\mathcal{P}(X), \subseteq)$ is a complete lattice. An interval in this lattice will be denoted by $[\![A, B]\!]$. By a *filter* on X we mean a filter in the lattice $\mathcal{P}(X)$. A subbase for a filter on X is a subset $\mathcal{A}$ of $\mathcal{P}(X)$ satisfying the finite intersection property. The smallest filter containing $\mathcal{A}$ will be denoted by $\langle \mathcal{A} \rangle$. A base for a filter on X is a nonempty subset $\mathcal{B}$ of $\mathcal{P}(X)$ such that $\emptyset \notin \mathcal{B}$ and

$$(\forall A, B \in \mathcal{B})(\exists C \in \mathcal{B})(C \subseteq A \cap B).$$

A *principal* filter on X is a filter which is an interval of $\mathcal{P}(X)$.

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. The set of all neighbourhoods of x will be denoted by $\mathcal{N}_x(\mathcal{T})$. Throughout this paper the notations $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_n$ will stand for the set of all integers, the set of all positive integers and the set $\{1, 2, \ldots, n\}$, respectively. A subset A of a group G is said to be *symmetric* if $A = A^{-1}$. The center of a group G will be denoted by $Z(G)$. Other notations and terms are standard and can be taken from [13].

Let G be a group and A be a nonempty subset of G. It is easy to see that A is a subgroup of G if and only if $AA^{-1} \subseteq A$, or equivalently, if and only if:

$$(\forall F \in [\![A, G]\!])(\exists E \in [\![A, G]\!])(EE^{-1} \subseteq F).$$

Clearly, $[\![A, G]\!]$ is a filter on the set G. So, we can generalize the definition of a subgroup by allowing this filter to be non-principal:

Definition 1. *An f-subgroup on a group G is a filter $\mathcal{F}$ on G satisfying:*

$$(\forall F \in \mathcal{F})(\exists E \in \mathcal{F})(EE^{-1} \subseteq F).$$

Let $\mathcal{F}$ be an f-subgroup on a group G. Clearly,

- Each $F \in \mathcal{F}$ contains the identity element 1.

- If $F \in \mathcal{F}$, then, $F^{-1} \in \mathcal{F}$.
- For each $F \in \mathcal{F}$ there is some symmetric $E \in \mathcal{F}$ with $EE \subseteq F$.
- If $\mathcal{F}$ contains a finite element, it is principal.

The following lemma will be very useful in the sequel:

Lemma 1. *Let G be a group and $\mathcal{A}$ be a subbase for a filter on G. $\langle \mathcal{A} \rangle$ is an f-subgroup on G, if and only if:*

$$(\forall A \in \mathcal{A})(\exists E \in \langle \mathcal{A} \rangle)(EE^{-1} \subseteq A).$$

Proof. Let $F \in \langle \mathcal{A} \rangle$. Then, there are $A_1, \ldots, A_n \in \mathcal{A}$ and $E_1, \ldots, E_n \in \langle \mathcal{A} \rangle$ with $A_1 \cap \cdots \cap A_n \subseteq F$ and $E_1 E_1^{-1} \subseteq A_1, \ldots, E_n E_n^{-1} \subseteq A_n$. So, we have:

$$(E_1 \cap \cdots \cap E_n)(E_1 \cap \cdots \cap E_n)^{-1} \subseteq E_1 E_1^{-1} \cap \cdots \cap E_n E_n^{-1} \subseteq A_1 \cap \cdots \cap A_n \subseteq F$$

Since $E_1 \cap \cdots \cap E_n \in \langle \mathcal{A} \rangle$,

$$(\forall F \in \langle \mathcal{A} \rangle)(\exists E \in \langle \mathcal{A} \rangle)(EE^{-1} \subseteq F),$$

that is, $\langle \mathcal{A} \rangle$ is an f-subgroup on G. □

2. Lattice structure of f-subgroups

In this section we prove that the set of all f-subgroups on a group is a complete lattice.

Notation 1. *Let G be a group. The set of all subgroups and f-subgroups of G will be denoted by $\mathrm{Sub}(G)$ and $\mathfrak{S}\mathrm{ub}(G)$, respectively.*

Theorem 1. *Let G be a group. $(\mathfrak{S}\mathrm{ub}(G), \subseteq)$ is a complete lattice in which for any $\mathfrak{U} \subseteq \mathfrak{S}\mathrm{ub}(G)$,*

$$\sup \mathfrak{U} = \left\langle \bigcup_{\mathcal{F} \in \mathcal{U}} \mathcal{F} \right\rangle.$$

Proof. Clearly, $\bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F}$ is a subbase for a filter on G and we have:

$$(\forall F \in \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F})(\exists E \in \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F}) \left(EE^{-1} \subseteq F\right).$$

Thus, by Lemma 1, $\bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F}$ is a subbase for an f-subgroup on G. On the other hand, it is evident that $\left\langle \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F} \right\rangle$ is an upper bound for $\mathfrak{U}$. Let $\mathcal{U}$ be another upper bound. Then, $\bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F} \subseteq \mathcal{U}$, which implies $\left\langle \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F} \right\rangle \subseteq \mathcal{U}$, and so, $\left\langle \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathcal{F} \right\rangle$ is the least upper bound. □

Notation 2. *An interval in the lattice $\mathfrak{S}\mathrm{ub}(G)$ will be denoted $[\mathcal{A}, \mathcal{B}]_{\mathfrak{S}}$ and the minimum and maximum elements of $\mathfrak{S}\mathrm{ub}(G)$ will be denoted, respectively, by $0_{\mathfrak{S}}$ and $1_{\mathfrak{S}}$.*

Definition 2. *Let G be a group. The f-subgroups $\mathcal{F}$ and $\mathcal{E}$ on G are said to be permutable if for every $A \in \mathcal{F}$ and $B \in \mathcal{E}$ there are $A' \in \mathcal{F}$ and $B' \in \mathcal{F}$ with $B'A' \subseteq AB$.*

In this definition, $B'A' \subseteq AB$ can be replaced by $A'B' \subseteq BA$. As a result, $\mathcal{F}$ and $\mathcal{E}$ are permutable, if and only if, $\mathcal{E}$ and $\mathcal{F}$ are permutable.

Theorem 2. *Let $\mathcal{F}$ and $\mathcal{E}$ be f-subgroups on a group G. $\mathcal{F}$ and $\mathcal{E}$ are permutable if and only if:*

$$\mathcal{F}\mathcal{E} = \{FE \mid F \in \mathcal{F}, E \in \mathcal{E}\}$$

is a subbase for an f-subgroup on G.

Proof. $\rightarrow$) Clearly $\mathcal{F}\mathcal{E}$ is a subbase for a filter on G. For every $F \in \mathcal{F}$ and $E \in \mathcal{E}$ there are symmetric $F_1 \in \mathcal{F}$ and $E_1 \in \mathcal{E}$ with:

$$F_1F_1 \subseteq F, \quad E_1E_1 \subseteq E$$

and symmetric $F_2 \in \mathcal{F}$ and $E_2 \in \mathcal{E}$ with:

$$E_2F_2 \subseteq F_1E_1$$

and symmetric $F_3 \in \mathcal{F}$ and $E_3 \in \mathcal{E}$ with:

$$E_3F_3 \subseteq F_2E_1.$$

Let $D = F_1E_2 \cap F_3E_3$. Clearly $D \in \langle \mathcal{F}\mathcal{E} \rangle$ and we have:

$$DD^{-1} \subseteq F_1E_2(F_3E_3)^{-1} = F_1E_2E_3F_3 \subseteq F_1E_2F_2E_1 \subseteq F_1F_1E_1E_1 \subseteq FE.$$

Thus, we have proved:

$$(\forall A \in \mathcal{F}\mathcal{E})(\exists D \in \langle \mathcal{F}\mathcal{E} \rangle)(DD^{-1} \subseteq A).$$

Which, by Lemma 1, means that $\mathcal{F}\mathcal{E}$ is a subbase for an f-subgroup on G.
$\leftarrow$) Let $A \in \mathcal{F}$ and $B \in \mathcal{E}$. There are symmetric $A_1, \ldots, A_n \in \mathcal{F}$ and symmetric $B_1, \ldots, B_n \in \mathcal{E}$ with $(A_1B_1 \cap \cdots \cap A_nB_n)^{-1} \subseteq AB$. We have:

$$(B_1 \cap \cdots \cap B_n)(A_1 \cap \cdots \cap A_n) \subseteq B_1A_1 \cap \cdots \cap B_nA_n = (A_1B_1 \cap \cdots \cap A_nB_n)^{-1} \subseteq AB$$

while $A_1 \cap \cdots \cap A_n \in \mathcal{F}$ and $B_1 \cap \cdots \cap B_n \in \mathcal{E}$. □

In Theorem 2, one can replace *subbase* with *base*, because if $E_1, \ldots, E_n \in \mathcal{E}$ and $F_1, \ldots, F_n \in \mathcal{F}$, then:

$$(E_1 \cap \cdots \cap E_n)(F_1 \cap \cdots \cap F_n) \subseteq E_1F_1 \cap \cdots \cap E_nF_n.$$

Theorem 3. *Let $\mathcal{F}$ and $\mathcal{E}$ be permutable f-subgroups on a group G. Then, in the lattice $\mathfrak{Sub}(G)$, $\mathcal{F} \wedge \mathcal{E} = \langle \mathcal{F}\mathcal{E} \rangle$.*

Proof. By Theorem 2, $\mathcal{F}\mathcal{E}$ is a base for filter on G and so $\langle \mathcal{F}\mathcal{E} \rangle$ is well-defined.
$\subseteq$) Let $A \in \mathcal{F} \wedge \mathcal{E}$. There is some symmetric $B \in \mathcal{F} \wedge \mathcal{E}$ with $BB \subseteq A$. Since $B \in \mathcal{F} \cap \mathcal{E}$, we have $BB \in \mathcal{F}\mathcal{E} \subseteq \langle \mathcal{F}\mathcal{E} \rangle$; which implies that $A \in \langle \mathcal{F}\mathcal{E} \rangle$. Therefore, $\mathcal{F} \wedge \mathcal{E} \subseteq \langle \mathcal{F}\mathcal{E} \rangle$.
$\supseteq$) Let $F \in \mathcal{F}$ and $E \in \mathcal{E}$. Then, $F \subseteq FE$ and $E \subseteq FE$ which implies that $FE \in \mathcal{F}$ and $FE \in \mathcal{E}$. Therefore, $\mathcal{F}\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{F}\mathcal{E} \subseteq \mathcal{E}$. This shows that $\langle \mathcal{F}\mathcal{E} \rangle \subseteq \mathcal{F}$ and $\langle \mathcal{F}\mathcal{E} \rangle \subseteq \mathcal{E}$. Now, since, by Theorem 2, $\mathcal{F}\mathcal{E}$ is a base for f-subgroup on G, we have $\langle \mathcal{F}\mathcal{E} \rangle \subseteq \mathcal{F} \wedge \mathcal{E}$. □

We have the following theorem, inspired by [7]:

Theorem 4. *Let $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{T}$ be f-subgroups on a group G and suppose*

- $\mathcal{S} \subseteq \mathcal{T}$,
- $\mathcal{A} \vee \mathcal{S}$ *and* $\mathcal{T}$ *are permutable,*
- $\mathcal{A}$ *and* $\mathcal{T}$ *are permutable.*

Then, $(\mathcal{A} \vee \mathcal{S}) \wedge \mathcal{T} = (\mathcal{A} \wedge \mathcal{T}) \vee \mathcal{S}$.

Proof. Clearly:

$$(\mathcal{A} \vee \mathcal{S}) \wedge \mathcal{T} \supseteq (\mathcal{A} \wedge \mathcal{T}) \vee \mathcal{S}.$$

Let $F \in (\mathcal{A} \vee \mathcal{S}) \wedge \mathcal{T}$. By Theorem 3, there are $E \in \mathcal{A} \vee \mathcal{S}$ and $T \in \mathcal{T}$ with $ET \subseteq F$. By Theorem 1, there are some $A \in \mathcal{A}$ and $S \in \mathcal{S}$ with $A \cap S \subseteq E$. There is a symmetric $Z \in \mathcal{S}$ with $ZZ \subseteq S$. Let $x \in A(Z \cap T) \cap Z$. There are $a \in A$ and $z \in Z \cap T$ with $x = az$. We have $a = xz^{-1} \in ZZ \subseteq S$. So, $a \in A \cap S$ and hence $x = az \in (A \cap S)T$. It follows that:

$$A(Z \cap T) \cap Z \subseteq (A \cap S)T \subseteq ET \subseteq F.$$

According to Theorem 3 and Theorem 1, we have $A(Z \cap T) \cap Z \in (\mathcal{A} \wedge \mathcal{T}) \vee \mathcal{S}$, and so, we have $F \in (\mathcal{A} \wedge \mathcal{T}) \vee \mathcal{S}$. Therefore:

$$(\mathcal{A} \vee \mathcal{S}) \wedge \mathcal{T} \subseteq (\mathcal{A} \wedge \mathcal{T}) \vee \mathcal{S}.$$

This completes the proof. □

So, if all f-subgroups on a group G are permutable then, $\mathfrak{Sub}(G)$ is modular. In particular, when G is abelian, $\mathfrak{Sub}(G)$ is modular.

Definition 3. *An f-subgroup $\mathcal{F}$ on a group G is called central if for each f-subgroup $\mathcal{E}$ on G, $\mathcal{F}$ and $\mathcal{E}$ are permutable.*

Let $\mathcal{F}$ be an f-subgroup on a group G. Clearly, $\bigcap_{F \in \mathcal{F}} F$ is a subgroup of G.

Definition 4. *Let $\mathcal{F}$ be an f-subgroup on a group G. The subgroup $\bigcap_{F \in \mathcal{F}} F$ of G is called the core of $\mathcal{F}$ and is denoted by* $\mathrm{core}(\mathcal{F})$.

An f-subgroup $\mathcal{F}$ on a group G is principal if and only if it contains its core. A finite f-subgroup contains its core and so is principal. Therefore, all f-subgroups on a finite group are principal.

Proposition 1. *Let G be an infinite group. Then, $\{G\}$ is the only finite f-subgroup on G.*

Proof. Let $\mathcal{F}$ be a finite f-subgroup on G and $\mathcal{F} \neq \{G\}$. Obviously, $\mathcal{F}$ contains its core H and $G \setminus H \neq \emptyset$ and $\mathcal{F} = [\![H, G]\!]$. Thus, $G \setminus H = \bigcup_{a \in G \setminus H} Ha$ is finite. So, H must be finite. Since H and $G \setminus H$ are finite, G cannot be infinite. □

Theorem 5. *Let G be a group. The function*

$$f : (\mathrm{Sub}(G), \subseteq^{-1}) \to (\mathfrak{Sub}(G), \subseteq)$$

given by $f(H) = [\![H, G]\!]$ is an order-embedding. If G is finite, then, f is an order-isomorphism.

Proof. Let $K, H \leq G$. We have:

$$H \subseteq^{-1} K \leftrightarrow K \subseteq H \leftrightarrow H \in f(K) \leftrightarrow f(H) \subseteq f(K).$$

Therefore, f is an order-embedding. If G is finite, f is onto and must be an order-isomorphism. □

Thus, when G is finite, $\mathfrak{Sub}(G)$ is order-isomorphic to the dual of $\mathrm{Sub}(G)$. This means that for a finite group, all we have to investigate is the lattice $\mathrm{Sub}(G)$.

Example 1. *Let G be a finite group. By Ore's theorem $\mathfrak{Sub}(G)$ is distributive (equivalently, order-isomorphic to $(D_n, |)$, the lattice of positive divisors of $n = |G|$), if and only if, it is a cyclic group* [10].

What about infinite groups? In what follows, we try to use the structure of $\mathrm{Sub}(G)$ to find some information on $\mathfrak{Sub}(G)$ for an infinite group G.

Notation 3. *Let G be a group. The set of all principal f-subgroups on G will be denoted by* $\mathrm{Prin}(G)$.

Lemma 2. *Let G be a group and $\mathfrak{U} \subseteq \mathrm{Prin}(G)$. Then,* $\inf \mathfrak{U} \in \mathrm{Prin}(G)$.

Proof. Let $\mathcal{I}$ be the infimum of $\mathfrak{U}$ in the complete lattice $\mathrm{Prin}(G)$. It is obvious that $\mathcal{I} \subseteq \inf \mathfrak{U}$. Let $F \in \inf \mathfrak{U}$. For each $n \in \mathbb{N}$, there is a symmetric $E_n \in \inf \mathfrak{U}$ such that $E_n^n = E_n E_n ... E_n \subseteq F$. Hence, $E_n \in \inf \mathfrak{U} \subseteq \bigcap_{\mathcal{F} \in \mathfrak{U}} \mathcal{F}$. This implies that for all $\mathcal{F} \in \mathfrak{U}$, $\mathrm{core}(\mathcal{F}) \subseteq E_n$, and so, $\bigcup_{\mathcal{F} \in \mathfrak{U}} \mathrm{core}(\mathcal{F}) \subseteq E_n$. Thus:

$$\left(\bigcup_{\mathcal{F} \in \mathfrak{U}} \mathrm{core}(\mathcal{F}) \right)^n \subseteq E_n^n \subseteq F.$$

Therefore:

$$\left\langle \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathrm{core}(\mathcal{F}) \right\rangle = \bigcup_{n \in \mathbb{N}} \left(\bigcup_{\mathcal{F} \in \mathfrak{U}} \mathrm{core}(\mathcal{F}) \right)^n \subseteq F.$$

So, $F \in [\![\langle \bigcup_{\mathcal{F} \in \mathfrak{U}} \mathrm{core}(\mathcal{F}) \rangle , G]\!] = \mathcal{I}$, which proves that $\inf \mathfrak{U} \subseteq \mathcal{I}$. □

Theorem 6. $\mathrm{Prin}(G)$ *is a sublattice of* $\mathfrak{Sub}(G)$.

Proof. Let $A, B \leq G$.
↑) We have:

$$[\![A, G]\!] \vee [\![B, G]\!] = \langle \{A\} \rangle \vee \langle \{B\} \rangle = \langle \{A \cap B\} \rangle = [\![A \cap B, G]\!] \in \mathrm{Prin}(G).$$

↓) By Lemma 2, $[\![A, G]\!] \wedge [\![B, G]\!] = [\![\langle A \cup B \rangle , G]\!] \in \mathrm{Prin}(G)$. □

Notice that $\mathrm{Prin}(G)$ is not a complete sublattice of $\mathfrak{Sub}(G)$:

Example 2. *Let $G = \mathbb{Z}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let:*

$$F_n = \{f : \mathbb{N} \to \mathbb{Z} \mid f[\mathbb{N}_n] = \{0\}\}.$$

Then, $\mathcal{B} = \{F_n \mid n \in \mathbb{N}\}$ is a base for an f-subgroup on G, and:

$$\langle \mathcal{B} \rangle = \bigcup_{n \in \mathbb{N}} [\![F_n, G]\!].$$

Since $\sup_{n\in\mathbb{N}} [\![F_n, G]\!] = \langle \mathcal{B} \rangle$, $\{0\} \notin \sup_{n\in\mathbb{N}} [\![F_n, G]\!]$. But, the supremum of $[\![F_n, G]\!]$ in the lattice $\mathrm{Prin}(G)$ is $[\![\{0\}, G]\!]$ which contains $\{0\}$.

One can apply some known results about subgroup lattices to f-subgroups lattices. For instance:

- Let G be a group and $\mathfrak{Sub}(G)$ be distributive. Then, G is locally cyclic [10].
- Every lattice can be lattice-embedded in the f-subgroup lattice of some group [10].
- Let G be an abelian group and $H, K \leq G$ be distinct. If $\mathfrak{Sub}(G) = [0_{\mathfrak{S}}, [\![H, G]\!]]_{\mathfrak{S}} \cup [[\![K, G]\!], 1_{\mathfrak{S}}]_{\mathfrak{S}}$ then, G is torsion and there are a group B, an $n \in \mathbb{N} \cup \{\infty\}$ and a prime number p such that $G_p \cong \mathbb{Z}_{p^n} \times B$ and $(\exists k \in \{0, 1, 2, \ldots, n-1\})(\forall b \in B)(b^k = 1)$ [2].

Also, there are other results about embedding lattices in an f-subgroup lattice which can be applied to $\mathfrak{Sub}(G)$. References [10] and [12] contain more results about subgroup lattices which can also be applied to f-subgroup lattices.

Let G be a group and $A \leq G$. Then, A is a normal subgroup of G if and only if for each $x \in G$, $A \subseteq xAx^{-1}$, equivalently, if and only if:

$$(\forall x \in G)(\forall F \in [\![A, G]\!])(xFx^{-1} \in [\![A, G]\!])).$$

In what follows, we extend this result to f-subgroups.

Definition 5. *An f-subgroup $\mathcal{F}$ on a group G is said to be normal if:*

$$(\forall F \in \mathcal{F})(\forall F \in \mathcal{F})(xFx^{-1} \in \mathcal{F}).$$

It is easy to check the following properties of normal f-subgroups:

- A filter $\mathcal{F}$ on a group G is a normal f-subgroup on G, if and only if:
$$(\forall x \in G)(\forall F \in \mathcal{F})(\exists E \in \mathcal{F})(xEE^{-1}x^{-1} \subseteq F).$$
- Let $\mathcal{A}$ be a subbase for an f-subgroup on G. $\langle \mathcal{A} \rangle$ is normal, if and only if:
$$(\forall x \in G)(\forall A \in \mathcal{A})(xAx^{-1} \in \langle \mathcal{A} \rangle).$$
- A principal f-subgroup $[\![N, G]\!]$ is normal if and only if N is a normal subgroup of G.
- Let $\mathcal{F}$ be an f-subgroup on a group G. Define:
$$\mathcal{F}^\star = \left\langle \left\{ xFx^{-1} \;\middle|\; x \in G,\ F \in \mathcal{F} \right\} \right\rangle.$$

Then, $\mathcal{F}^\star$ is a normal f-subgroup on G.

Notation 4. *Let G be a group. The set of all normal subgroups and normal f-subgroups of G will be denoted by* $\mathrm{Nor}(G)$ *and* $\mathfrak{Nor}(G)$*, respectively.*

Theorem 7. *If G is a group, then,* $\mathfrak{Nor}(G)$ *is a complete sublattice of* $\mathfrak{Sub}(G)$*.*

Proof. Let $\mathfrak{U} \subseteq \mathfrak{Sub}(G)$.

$\uparrow$) Let $x \in G$ and $F \in \sup \mathfrak{U}$. There are $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathfrak{U}$ and:

$$F_1 \in \mathcal{F}_1, \ldots, F_n \in \mathcal{F}_n$$

with $F_1 \cap \cdots \cap F_n \subseteq F$. Hence:

$$xF_1x^{-1} \cap \cdots \cap xF_nx^{-1} \subseteq xFx^{-1},$$

and so, $xFx^{-1} \in \sup \mathfrak{U}$, as desired.

$\downarrow$) Let $\mathcal{F} \in \mathfrak{U}$. For every $x \in G$ and $F \in \inf \mathfrak{U}$, we have $F \in \mathcal{F}$, and so, $xFx^{-1} \in \mathcal{F}$. Thus:

$$(\inf \mathfrak{U})^\star = \left\langle \left\{ xFx^{-1} \,\middle|\, x \in G,\ F \in \inf \mathfrak{U} \right\} \right\rangle \subseteq \mathcal{F}.$$

It follows that $(\inf \mathfrak{U})^\star$ is a lower bound for $\mathfrak{U}$. Therefore, $(\inf \mathfrak{U})^\star \subseteq \inf \mathfrak{U}$ and clearly $\inf \mathfrak{U} \subseteq (\inf \mathfrak{U})^\star$. ☐

Definition 6. *An f-subgroup $\mathcal{F}$ on a group G is called uniformly normal if*

$$(\forall F \in \mathcal{F})(\exists E \in \mathcal{F})(\forall x \in G)(xEx^{-1} \subseteq F).$$

Some easily checked propositions about uniformly normal f-subgroups are as follows:

- A uniformly normal f-subgroup is normal and central.
- A principal normal f-subgroup is uniformly normal.
- An f-subgroup $\mathcal{F}$ on a group G is uniformly normal if and only if for each $F \in \mathcal{F}$, $\bigcap_{x \in G} xFx^{-1} \in \mathcal{F}$.

Let $\mathcal{F}$ be an f-subgroup on a group G and $H \leq G$. Then, the set

$$\mathcal{F}_H = \{F \cap H \mid F \in \mathcal{F}\}$$

is an f-subgroup on H. Let G and H be groups, $\mathcal{F}$ be a (normal) (principal) f-subgroup on G and $\mathcal{E}$ be a (normal) (principal) f-subgroup on H. Then, the set

$$\{F \times E \mid F \in \mathcal{F}, E \in \mathcal{E}\}$$

is a base for a (normal) (principal) f-subgroup on $G \times H$.

The following propositions are trivial:

Proposition 2. *Let G and H be groups, $f : G \to H$ be an onto homomorphism and $\mathcal{B}$ be a base for a (normal) (principal) f-subgroup on G. The set*

$$\dot{f}[\mathcal{B}] = \{f[B] \mid B \in \mathcal{B}\}$$

is a base for a (normal) (principal) f-subgroup on H.

Proposition 3. *Let G and H be groups, $f : G \to H$ be a homomorphism and $\mathcal{B}$ be a base for a (normal) (principal) f-subgroup on H. The set*

$$\dot{f^{-1}}[\mathcal{B}] = \{f^{-1}[B] \mid B \in \mathcal{B}\}$$

is a base for a (normal) (principal) f-subgroup on G.

Definition 7. *Let $\mathcal{F}$ be an f-subgroup on a group G and $N \trianglelefteq G$. We define $\frac{\mathcal{F}}{N} = \langle \dot{\pi}(\mathcal{F}) \rangle$, where $\pi : G \to \frac{G}{N}$ is given by $\pi(x) = Nx$.*

Obviously, $\frac{\mathcal{F}}{N}$ is an f-subgroup on $\frac{G}{N}$. If N is a normal subgroup of a group G, then, the interval $[N, G]$ in $\mathrm{Sub}(G)$ is order-isomorphic to $\mathrm{Sub}(\frac{G}{N})$. We try to prove a similar result for $\mathfrak{Sub}(G)$:

Theorem 8. *Let G be a group and $N \trianglelefteq G$. The function*

$$f : [0_{\mathfrak{S}}, [\![N, G]\!]]_{\mathfrak{S}} \to \mathfrak{Sub}(\frac{G}{N})$$
$$f(\mathcal{F}) = \frac{\mathcal{F}}{N}$$

is an order-isomorphism and for each $\mathfrak{F} \in \mathfrak{Sub}(\frac{G}{N})$, $f^{-1}(\mathfrak{F}) = \left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle$.

Proof. 1) Suppose $\mathcal{F}, \mathcal{E} \in [0_{\mathfrak{S}}, [\![N, G]\!]]_{\mathfrak{S}}$ and $f(\mathcal{F}) = f(\mathcal{E})$. We have:

$$\langle \{Na \mid a \in F\} \mid F \in \mathcal{F} \rangle = \langle \{Nb \mid b \in E\} \mid E \in \mathcal{E} \rangle .$$

For each $F \in \mathcal{F}$, there is some symmetric $F_0 \in \mathcal{F}$ with $F_0F_0 \subseteq F$. There is some $E \in \mathcal{E}$ with $\{Nb \mid b \in E\} \subseteq \{Na \mid a \in F_0\}$, which implies that $NE \subseteq NF_0$. Hence

$$E \subseteq NE \subseteq NF_0 \subseteq F_0F_0 \subseteq F$$

and so $F \in \mathcal{E}$. Thus, $\mathcal{F} \subseteq \mathcal{E}$. Similarly $\mathcal{E} \subseteq \mathcal{F}$ and therefore, f is one-to-one.

2) Let $\mathfrak{F} \in \mathfrak{Sub}(\frac{G}{N})$ and $\mathcal{B}_1 \in \dot{\pi}[\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle]$. Then, there are $A_1 \in \left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle$, $A_2 \in \dot{\pi^{-1}}[\mathfrak{F}]$ and $\mathcal{B}_2 \in \mathfrak{F}$ such that $\mathcal{B}_1 = \pi[A_1]$, $A_2 \subseteq A_1$ and $A_2 = \pi^{-1}[\mathcal{B}_2]$. Hence, $\pi[A_2] = \mathcal{B}_2$ which shows that $\mathcal{B}_2 = \pi[A_2] \subseteq \pi[A_1] = \mathcal{B}_1$ and $\mathcal{B}_1 \in \mathfrak{F}$. Thus, $\dot{\pi}[\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle] \subseteq \mathfrak{F}$, and so, $\left\langle \dot{\pi}[\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle] \right\rangle \subseteq \mathfrak{F}$. It is clear that:

$$\mathfrak{F} \subseteq \left\langle \dot{\pi}[\dot{\pi^{-1}}[\mathfrak{F}]] \right\rangle \subseteq \left\langle \dot{\pi}[\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle] \right\rangle .$$

Therefore:

$$\mathfrak{F} = \left\langle \dot{\pi}[\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle] \right\rangle = \frac{\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle}{N} = f(\left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle)$$

and f is onto. Thus, f a bijection with:

$$f^{-1}(\mathfrak{F}) = \left\langle \dot{\pi^{-1}}[\mathfrak{F}] \right\rangle$$

3) Obviously, f and f^{-1} are increasing. □

This theorem says $\mathfrak{Sub}(\frac{G}{N})$ can be embedded in $\mathfrak{Sub}(G)$.

3. Concluding remarks

The concepts of f-subgroup and normal subgroups together with some of their main properties were presented. One of our motivations for introducing these concepts is their relationship with topological groups. In this section, it is shown that our concepts have natural topological counterparts. We start with a simple proposition:

Proposition 4. *$\mathcal{B}$ is a base for an f-subgroup on a group G if and only if:*

0) $\mathcal{B} \neq \emptyset$.
1) $(\forall U \in \mathcal{B})(1 \in U)$.
2) $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B})(W \subseteq U \cap V)$.
3) $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(VV \subseteq U)$.
4) $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(V^{-1} \subseteq U)$.

Let $\mathcal{F}$ be an f-subgroup on G. For each $F \in \mathcal{F}$ we define:

$$R_A = \{(x, y) \in G \times G \mid y \in Ax\}$$

and

$$L_A = \{(x, y) \in G \times G \mid y \in xA\}.$$

Using Proposition 4, we can prove:

1) The $\{R_F \mid F \in \mathcal{F}\}$ is a base for a uniformity $\mathcal{U}_r(\mathcal{F})$ on G.
2) The $\{L_F \mid F \in \mathcal{F}\}$ is a base for a uniformity $\mathcal{U}_l(\mathcal{F})$ on G.

To see the details, one can refer to [4].

The topology derived from $\mathcal{U}_r(\mathcal{F})$ and $\mathcal{U}_l(\mathcal{F}))$ will be denoted by $\mathcal{T}_r(\mathcal{F})$ and $\mathcal{T}_l(\mathcal{F})$, respectively. The uniform spaces $(G,\mathcal{U}_r(\mathcal{F}))$ and $(G,\mathcal{U}_l(\mathcal{F}))$ are uniformly isomorphic (see [4]). Thus, $(G, \mathcal{T}_r(\mathcal{F}))$ and $(G, \mathcal{T}_l(\mathcal{F}))$ are homeomorphic. Also, if $\mathcal{F}$ has a countable base, these uniform spaces are pseudo-metrizable, and so, $\mathcal{F}$ can have a base consisting of open balls in a pseudo-metric space (G, d). It is easy to see that $\mathcal{F}$ is normal if and only if $\mathcal{T}_r(\mathcal{F}) = \mathcal{T}_l(\mathcal{F})$. Using these uniform spaces, we also derive some new definitions:

Definition 8. *An f-subgroup $\mathcal{F}$ on a group G is said to be precompact (or of finite index) if for each $F \in \mathcal{F}$, there are $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathcal{F}$ with $a_1F \cup \cdots \cup a_nF = G$ (or equivalently there are $n \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathcal{F}$ with $Fb_1 \cup \cdots \cup Fb_n = G$).*

The definition above is similar to the definition of a subgroup with finite index.

Definition 9. *An f-subgroup $\mathcal{F}$ on a group G is said to be Hausdorff (or core-free) if* $\operatorname{core}(\mathcal{F}) = \{1\}$.

Note that $\mathcal{F}$, in the definition above, is Hausdorff if and only if $(G, \mathcal{T}_r(\mathcal{F}))$ is Hausdorff. The idea of the following example, originates from [6].

Example 3. *According to* [9]*, there exists an infinite group G of exponent p^2, where p is a prime number, with $|Z(G)| = p$ such that:*

$$(\forall x \in G \setminus Z(G))(x^p \neq 1).$$

Let $\mathcal{F}$ be a core-free f-subgroup on G. Define $f : G \to G$ by $f(x) = x^p$. Clearly $f[G]$ is finite and because $\mathcal{F}$ is core-free, there is some $F \in \mathcal{F}$ with $F \cap f[G] = \{1\}$. There *is some symmetric $E \in \mathcal{F}$ with $E^p = EE \dots E \subseteq F$. We have $f[E] \subseteq E^p \subseteq F$, and so, $f[E] \subseteq F \cap f[G] = \{1\}$. This implies that $E \subseteq f^{-1}(\{1\}) \subseteq Z(G)$. Therefore, E is finite, and so, $\mathcal{F}$ is principal. Thus, the only core-free principal f-subgroup on G is $1_{\mathfrak{G}} = [\![\{1\}, G]\!]$.*

In the example above, we have proved that there exists an infinite group which does not allow any core-free f-subgroups except $1_{\mathfrak{G}}$. This has been an attempt to answer the following more general question:

Question 1. *Is there an infinite group G which allows no non-principal f-subgroups?*

The following definition provides an equivalent structure for f-subgroups.

Definition 10. *We say that $\mathcal{T}$ is a right group topology on G, if G is a group, $(G, \mathcal{T})$ is a topological space, the function $G \times G \to G$ defined by $(x, y) \mapsto xy^{-1}$ is continuous at $(1, 1)$ and for every $a \in G$, the function $G \to G$ defined by $x \mapsto xa$ is continuous everywhere. $(G, \mathcal{T})$ is called a right topological group, if the topology $\mathcal{T}$ on the group G is a right group topology. A left topological group is defined similarly*

If $\mathcal{F}$ is an f-subgroup on a group G, then, $(G, \mathcal{T}_r(\mathcal{F}))$ is a right topological group. Also, if $(G, \mathcal{T})$ is any right topological group, there is a unique f-subgroup $\mathcal{F}$ on G with $\mathcal{T}_r(\mathcal{F}) = \mathcal{T}$. In [4], we have investigated topological properties of right topological groups in more details. In the current paper, we are not investigating topological properties of f-subgroups, instead, we investigate algebraic properties of f-subgroups.

Let G be a group. The set of all group topologies on G is denoted by $\mathcal{L}(G)$. Notice that a normal f-subgroup satisfies all Pontryagin conditions for a topological group:

Proposition 5. *$\mathcal{B}$ is a base for a normal f-subgroup on a group G if and only if*

0) $\mathcal{B} \neq \emptyset$.
1) $(\forall U \in \mathcal{B})(1 \in U)$.
2) $(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B})(W \subseteq U \cap V)$.
3) $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(VV \subseteq U)$.
4) $(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(V^{-1} \subseteq U)$.
5) $(\forall U \in \mathcal{B})(\forall x \in G)(\exists V \in \mathcal{B})(xVx^{-1} \subseteq U)$.

Theorem 9. *Let G be a group. The function $f : \mathcal{L}(G) \to \mathfrak{Nor}(G)$ given by $f(\mathcal{T}) = \mathcal{N}_1(\mathcal{T})$ is an order-isomorphism.*

Proof. It is clear that f is well-defined.

1) Suppose $\mathcal{T}, \mathcal{S} \in \mathcal{L}(G)$ and $f(\mathcal{T}) = f(\mathcal{S})$. Let $U \in \mathcal{T}$ and $x \in U$. There is some $V \in f(\mathcal{T}) = f(\mathcal{S})$ with $Vx \subseteq U$. So, x is in the interior of U in $(G, \mathcal{S})$. Thus, $U \in \mathcal{S}$. We have proved: $\mathcal{T} \subseteq \mathcal{S}$. Similarly $\mathcal{S} \subseteq \mathcal{T}$. Therefore, $\mathcal{S} = \mathcal{T}$ and f is one-to-one.

2) Let $\mathcal{F} \in \mathfrak{Nor}(G)$. Since $\mathcal{F}$ satisfies the Pontryagin conditions, there is a group topology $\mathcal{T}$ on G with $\mathcal{F}$ a neighbourhood base around 1 in $(G, \mathcal{T})$. Because $\mathcal{F}$ is filter, we have $\mathcal{F} = \mathcal{N}_1(\mathcal{T}) = f(\mathcal{T})$. Therefore, f is onto.

3) Clearly f is increasing.

4) Suppose $\mathcal{F}, \mathcal{E} \in \mathfrak{Nor}(G)$ and $\mathcal{F} \subseteq \mathcal{E}$. Let $U \in f^{-1}(\mathcal{F})$. For each $x \in U$ there is some $F \in \mathcal{F} \subseteq \mathcal{E}$ with $xF \subseteq U$. So, $U \in f^{-1}(\mathcal{E})$ and $f^{-1}(\mathcal{F}) \subseteq f^{-1}(\mathcal{E})$. Therefore, f^{-1} is increasing. □

Now all results about the (lattice of) group topologies can be applied to the (lattice of) normal f-subgroups. For example:

- A minimal core-free f-subgroup on an abelian group is of finite index [11].
- There is an infinite group G with $|\mathfrak{Nor}(G)| = 2$ [5].
- Let G be a group and $\{\mathcal{F} \in \mathfrak{Sub}(G) \mid \mathcal{F} \text{ is core-free}\}$ be a complete lattice. Then, $Z(G)$ is finite [3].

Example 4. *Let G be a group and $\mathfrak{Sub}(G)$ be totally ordered.* $\mathrm{Sub}(G)$ *is totally ordered, so, it must be distributive. Thus, G is locally cyclic, and so, it is abelian. We know from topological group theory that the lattice of all group topologies on an infinite abelian group cannot be linear* [1]. *So, G is finite. Clearly* $\mathrm{Sub}(G) \setminus \{\{1\}\}$ *has a minimum element M which is a group of order p, where p is a prime number. There is a nontrivial $a \in M$ which is contained in all nontrivial subgroups of G. We conclude that G is a finite abelian group of the form $\mathbb{Z}_{p^n}$. Clearly for each prime p and positive integer n, $\mathfrak{Sub}(\mathbb{Z}_{p^n})$ is totally ordered (because its dual is order-isomorphic to* $\mathrm{Sub}(\mathbb{Z}_{p^n})$*).*

Definition 11. *Let G be a group and α be a cardinality.*

- *An f-subgroup $\mathcal{F}$ on G is said to be super disciplined if:*
$$(\forall F \in \mathcal{F})(\exists H \leq G)(\{1\} \neq H \subseteq F).$$
- *An f-subgroup $\mathcal{F}$ on G is said to be super α-disciplined if:*
$$(\forall F \in \mathcal{F})(\exists H \leq G)(H \subseteq F,\ [G : H] = \alpha).$$
- *A disciplined f-subgroup on G is one with a base contained in* $\mathrm{Sub}(G)$*.*
- *An α-disciplined f-subgroup on G is one with a base $\mathcal{B}$ contained in* $\mathrm{Sub}(G)$ *such that* $(\forall H \in \mathcal{B})([G : H] = \alpha)$*.*

Evidently, every principal f-subgroup on a group, is disciplined. Not all f-subgroups on $\mathbb{Z}$ are super disciplined, let alone disciplined:

Example 5. *$\mathbb{Z}$ can be embedded in the circle group $\mathbb{T}$. $\mathbb{Z}$ endowed with the subspace topology $\mathcal{T}$ from $\mathbb{T}$, is a topological group. So, $\mathcal{F} = \mathcal{N}_1(\mathcal{T})$ is a normal f-subgroup on $\mathbb{Z}$ which is clearly not super disciplined.*

Let G be a group. By [8], there exists a core-free normal super $|\mathbb{N}|$-disciplined f-subgroup on G if and only if it is residually finite.

Theorem 10. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$ be disciplined f-subgroups on a locally cyclic group G. Then, $(\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{T} = (\mathcal{A} \wedge \mathcal{T}) \vee (\mathcal{B} \wedge \mathcal{T})$.*

Proof. Evidently:

$$(\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{T} \supseteq (\mathcal{A} \wedge \mathcal{T}) \vee (\mathcal{B} \wedge \mathcal{T}).$$

Let $F \in (\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{T}$. There are subgroups $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $T \in \mathcal{T}$ of G with $(A \cap B)T \subseteq F$. But $\mathrm{Sub}(G)$ is distributive, and so:

$$AT \cap BT = (A \cap B)T \subseteq F.$$

Thus, $F \in (\mathcal{A} \wedge \mathcal{T}) \vee (\mathcal{B} \wedge \mathcal{T})$ and:

$$(\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{T} \subseteq (\mathcal{A} \wedge \mathcal{T}) \vee (\mathcal{B} \wedge \mathcal{T}).$$

This completes the proof. □

We conclude this paper with a few questions:

Question 2. *For which $n \in \mathbb{N}$, is there a group G with $\mathfrak{Nor}(G)$ order-isomorphic to M_n?*

Question 3. *For which groups G, is $\mathfrak{Sub}(G)$ self-dual?*

Question 4. *Let p be a prime number. Is there an elementary proof which shows $\mathfrak{Sub}(\mathbb{Z}_{p^\infty})$ is not distributive?*

Question 5. *How many non-isomorphic groups with $\mathfrak{Sub}(G)$ order-isomorphic to $\mathfrak{Sub}(\mathbb{Z})$ exist?*

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