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GENUINELY NONLINEAR FORWARD-BACKWARD ULTRA-PARABOLIC EQUATIONS

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Abstract. In this paper we have proved the existence and uniqueness of entropy solutions to the Dirichlet problem for genuinely nonlinear forward-backward ultra-parabolic equations. We have used a kinetic formulation of entropy solutions which enables also to prove the existence of their traces in the L^1 sense.

Keywords: entropy solution, forward-backward ultra-parabolic equation, kinetic formulation.

Introduction

Entropy solutions to initial-boundary value problems with homogeneous boundary conditions for nonlinear parabolic and ultra-parabolic equations were analyzed in [1], [6], [19], [20], [24], [25], [28], [29], [30], [32], [33]. It is important to pay attention to the class of ultra-parabolic equations with two time-like variables [3], [4], [9], [10], [12].

In the present paper we have generalized methods developed in [2], [13], [14], [15] while studying forward-backward parabolic equations. We have proved the existence and uniqueness of entropy solutions for genuinely nonlinear forward-backward ultra-parabolic equations.

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1. The main results

In this section we deal with entropy solutions of forward-backward ultra-parabolic equations.

1.1. Forward-backward ultra-parabolic equations.

Let scalar functions $a(z)$, $b(z)$ and vector function $\boldsymbol{\varphi}(z)$ satisfy the following conditions.

Conditions on $a, b\&\boldsymbol{\varphi}$. *Let* $a, b \in C^2(\mathbb{R})$, $a(0) = b(0) = 0$, $\boldsymbol{\varphi}(z) = (\varphi_1(z), \dots, \varphi_d(z))$, $z \in \mathbb{R}$, $\varphi_j \in C^2(\mathbb{R})$, $j = 1, \dots, d$, $\boldsymbol{\varphi}(0) = \mathbf{0}$. *Function a is non-monotonic. Moreover, a' and b' satisfy the genuine nonlinearity condition*

$$\text{mes}\,\{\lambda \in \mathbb{R} : \, a'(\lambda)\theta + b'(\lambda)\vartheta = 0\} = 0, \tag{1}$$

for every $(\theta, \vartheta) \in \mathbb{S}^1$. *Also, we assume that function $\boldsymbol{\varphi}$ satisfies* (45) *which guarantees the uniqueness of the entropy solution.*

Remark 1. *It follows from Conditions on $a, b\&\boldsymbol{\varphi}$ that the linear case $a(z) = b(z) = z$ (see, for example,* [12]*) cannot be treated here. But, for example, we can consider the case* $a(z) = \frac{z^2}{2}$, $b(z) = z$.

Under Conditions on $a, b\&\boldsymbol{\varphi}$, we formulate boundary value problem Π_0. Let $G_{T,S} = \Omega \times (0, T) \times (0, S)$, $\Gamma_0 = \overline{\Omega} \times \{t = 0\} \times [0, S]$, $\Gamma_T = \overline{\Omega} \times \{t = T\} \times [0, S]$, $\Xi_0 = \overline{\Omega} \times [0, T] \times \{s = 0\}$, $\Xi_S = \overline{\Omega} \times [0, T] \times \{s = S\}$, $\Gamma_l = \partial G_{T,S} \setminus (\Gamma_0 \cup \Gamma_T \cup \Xi_0 \cup \Xi_S)$, a bounded domain $\Omega \subset \mathbb{R}^d$ ($\text{mes}\Omega < \infty$) has smooth boundary $\partial\Omega$.

Problem Π_0. *For arbitrary initial and final data* $u_0^\Gamma, u_T^\Gamma \in L^\infty(\Omega \times (0, S)) \cap L^2(0, S; W_0^{1,2}(\Omega))$, $u_0^\Xi, u_S^\Xi \in L^\infty(\Omega \times (0, T)) \cap L^2(0, T; W_0^{1,2}(\Omega))$ *the unknown function* $u : G_{T,S} \to \mathbb{R}$ *satisfies*

$$\partial_t a(u) + \partial_s b(u) + \text{div}_x \boldsymbol{\varphi}(u) = \Delta_x u, \quad (\boldsymbol{x}, t, s) \in G_{T,S}, \tag{2a}$$

$$u|_{\Gamma_0} \approx u_0^\Gamma(\boldsymbol{x}, s), \quad (\boldsymbol{x}, s) \in \Omega \times (0, S), \tag{2b}$$

$$u|_{\Gamma_T} \approx u_T^\Gamma(\boldsymbol{x}, s), \quad (\boldsymbol{x}, s) \in \Omega \times (0, S), \tag{2c}$$

$$u|_{\Xi_0} \approx u_0^\Xi(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{2d}$$

$$u|_{\Xi_S} \approx u_S^\Xi(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{2e}$$

$$u|_{\Gamma_l} = 0, \tag{2f}$$

in the form given in Definition 3. The sign $\approx$ means the equality only on a part of the boundary.

Remark 2. *We formulate equation* (2a) *in the sense of distributions. Since function $a(z)$ is non-monotonic on $\mathbb{R}$, equation* (2a) *is a forward-backward ultra-parabolic equation. Moreover, a weak solution* $u \in L^\infty(G_{T,S}) \cap L^2((0, T) \times (0, S); W_0^{1,2}(\Omega))$ *can deviate from initial and final data* u_0^Γ, $u_T^\Gamma \in L^\infty(\Omega \times (0, S)) \cap L^2(0, S; W_0^{1,2}(\Omega))$, u_0^Ξ, $u_S^\Xi \in L^\infty(\Omega \times (0, T)) \cap L^2(0, T; W_0^{1,2}(\Omega))$. *Therefore, the difficulty of problem Π_0 is that equation* (2a) *as well as initial and final conditions* (2b)–(2e) *must be reformulated in the form of entropy inequalities given in Definition 4 or in the form of kinetic equalities given in Definition 3.*

Remark 3. *There are still open questions concerning the solvability of the forward-backward p-ultra-parabolic equation*

$$\partial_t a(u) + \partial_s b(u) + \mathrm{div}_x \boldsymbol{\varphi}(u) = \mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u), \quad (\boldsymbol{x}, t, s) \in G_{T,S}, \quad p > 1.$$

1.2. Elliptic regularization.

We are going to construct an entropy solution for problem Π_0 as a singular limit of weak solutions u_ε for the non-homogeneous Dirichlet problem Π_ε as $\varepsilon \to 0+$.

Problem Π_ε**.** *For arbitrary initial and final data* $u_0^\Gamma, u_T^\Gamma \in L^\infty(\Omega \times (0,S)) \cap L^2(0,S; W_0^{1,2}(\Omega))$, $u_0^\Xi, u_S^\Xi \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T; W_0^{1,2}(\Omega))$ *the unknown function* u_ε *satisfies the boundary value problem*

$$\text{(3a)} \quad \partial_t a(u_\varepsilon) + \partial_s b(u_\varepsilon) + \mathrm{div}_x \boldsymbol{\varphi}(u_\varepsilon) = \Delta_x u_\varepsilon + \varepsilon \partial_t^2 u_\varepsilon + \varepsilon \partial_s^2 u_\varepsilon, \quad (\boldsymbol{x}, t, s) \in G_{T,S},$$

$$\text{(3b)} \quad u_\varepsilon|_{\Gamma_0} = u_0^\Gamma, \quad u_\varepsilon|_{\Gamma_T} = u_T^\Gamma, \quad u_\varepsilon|_{\Xi_0} = u_0^\Xi, \quad u_\varepsilon|_{\Xi_S} = u_S^\Xi, \quad u_\varepsilon|_{\Gamma_l} = 0,$$

in the weak sense, see Definition 1.

We assume here that $\varepsilon \in (0,1]$. Let $V(G_{T,S}) = \{v \in W^{1,2}(G_{T,S}) : \ v|_{\Gamma_l} = 0\}$.

Definition 1. *Function* $u_\varepsilon \in L^\infty(G_{T,S}) \cap V(G_{T,S})$ *is called a weak solution for problem* Π_ε *if the following assertions hold.*

(EL.1) *Let* $\widehat{u} \in L^\infty(G_{T,S}) \cap V(G_{T,S})$ *be an arbitrary extension of functions* $u_0^\Gamma, u_T^\Gamma, u_0^\Xi$ *and* u_S^Ξ *into* $G_{T,S}$*. Therefore,* $u_\varepsilon - \widehat{u} \in L^\infty(G_{T,S}) \cap W_0^{1,2}(G_{T,S})$*.*

(EL.2) *The following equality holds*

$$\text{(4a)} \quad \int_{G_{T,S}} \big(- a(u_\varepsilon)\partial_t \phi - b(u_\varepsilon)\partial_s \phi - \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla_x \phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon \partial_t u_\varepsilon \partial_t \phi + \varepsilon \partial_s u_\varepsilon \partial_s \phi \big) \, d\boldsymbol{x}dtds = 0$$

for every $\phi \in L^\infty(G_{T,S}) \cap W_0^{1,2}(G_{T,S})$*.*

Remark 4. *We can reformulate* (4a) *in a similar way:*

$$\text{(4b)} \quad \int_{G_{T,S}} \big(\partial_t a(u_\varepsilon)\phi + \partial_s b(u_\varepsilon)\phi + \mathrm{div}_x \boldsymbol{\varphi}(u_\varepsilon)\phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon \partial_t u_\varepsilon \partial_t \phi + \varepsilon \partial_s u_\varepsilon \partial_s \phi \big) \, d\boldsymbol{x}dtds = 0.$$

Also, we assume that $\psi \in L^\infty(G_{T,S}) \cap V(G_{T,S})$*:*

$$\text{(4c)} \quad \int_{G_{T,S}} \big(\partial_t a(u_\varepsilon)\psi + \partial_s b(u_\varepsilon)\psi + \mathrm{div}_x \boldsymbol{\varphi}(u_\varepsilon)\psi + \nabla_x u_\varepsilon \cdot \nabla_x \psi + \varepsilon \partial_t u_\varepsilon \partial_t \psi + \varepsilon \partial_s u_\varepsilon \partial_s \psi \big) \, d\boldsymbol{x}dtds = \int_\Omega \int_0^S \varepsilon \partial_t u_\varepsilon|_{t=T} \psi \, d\boldsymbol{x}ds - \int_\Omega \int_0^S \varepsilon \partial_t u_\varepsilon|_{t=0} \psi \, d\boldsymbol{x}ds + \int_\Omega \int_0^T \varepsilon \partial_s u_\varepsilon|_{s=S} \psi \, d\boldsymbol{x}dt - \int_\Omega \int_0^T \varepsilon \partial_s u_\varepsilon|_{s=0} \psi \, d\boldsymbol{x}dt,$$

where $\partial_t u_\varepsilon|_{t=0}$*,* $\partial_t u_\varepsilon|_{t=T}$*,* $\partial_s u_\varepsilon|_{s=0}$ *and* $\partial_s u_\varepsilon|_{s=S}$ *are weak traces according to* [11].

Proposition 1. *Under Conditions on $a, b\&\boldsymbol{\varphi}$, problem Π_ε has at least one weak solution u_ε for all u_0^Γ, $u_T^\Gamma \in L^\infty(\Omega\times(0,S))\cap W_0^{1,2}(\Omega\times(0,S))$, u_0^Ξ, $u_S^\Xi \in L^\infty(\Omega\times(0,T))\cap W_0^{1,2}(\Omega\times(0,T))$. Moreover, the maximum principle*

$$(5)\quad \|u_\varepsilon\|_{L^\infty(G_{T,S})} \le M = \max\left(\|u_0^\Gamma\|_{L^\infty(\Omega\times(0,S))}, \|u_T^\Gamma\|_{L^\infty(\Omega\times(0,S))}, \|u_0^\Xi\|_{L^\infty(\Omega\times(0,T))}, \|u_S^\Xi\|_{L^\infty(\Omega\times(0,T))}\right),$$

and the energy estimate

$$(6)\qquad \int_{G_{T,S}} \left(|\nabla_x u_\varepsilon|^2 + \varepsilon\,|\partial_t u_\varepsilon|^2 + \varepsilon\,|\partial_s u_\varepsilon|^2\right)\, d\boldsymbol{x}dtds < C_{(14)}$$

hold. The constant $C_{(14)}$ is defined through (14) *and does not depend on $\varepsilon\in(0,1]$.*

This proposition is proved in section 2.

1.3. **Kinetic and entropy solutions.**

Consider the function χ which is defined in the following way

$$\chi(\lambda;\upsilon) = \begin{cases} +1, & \text{for } 0<\lambda<\upsilon, \\ -1, & \text{for } \upsilon<\lambda<0, \\ \ \ 0, & \text{elsewhere.} \end{cases}$$

Definition 2. *Let N be an positive integer, $\mathcal{O}$ be an open set of $\mathbb{R}^N$ and the function $h\in L^\infty(\mathcal{O}\times(-L,L))$ satisfying $0\le h(\boldsymbol{z},\lambda)\mathrm{sgn}(\lambda)\le 1$ for almost every $(\boldsymbol{z},\lambda)\in\mathbb{R}^{N+1}$. It is said that h is a χ-function if there exists a function $\upsilon\in L^\infty(\mathcal{O})$ such that*

$$h(\boldsymbol{z},\lambda) = \chi(\lambda;\upsilon(\boldsymbol{z}))$$

for a.e. $\boldsymbol{z}\in\mathcal{O}$. Note that $\upsilon(\boldsymbol{z}) = \int_{-L}^{L} h(\boldsymbol{z},\lambda)\,d\lambda = \int_{-L}^{L}\chi(\lambda;\upsilon(\boldsymbol{z}))\,d\lambda$.

The following lemma formulated and proved in [31] guarantees the link between sequences of χ-functions and its limits.

Lemma 1. *Let $\mathcal{O}$ be an open set of $\mathbb{R}^N$ and $h_n\in L^\infty(\mathcal{O}\times(-L,L))$ be a sequence of χ-functions converging weakly to $h\in L^\infty(\mathcal{O}\times(-L,L))$. We define $\upsilon_n(\cdot) = \int_{-L}^{L} h_n(\cdot,\lambda)\,d\lambda$ and $\upsilon(\cdot) = \int_{-L}^{L} h(\cdot,\lambda)\,d\lambda$. Then the three following propositions are equivalent:*

- *h_n converges strongly to h in $L^1_{\mathrm{loc}}(\mathcal{O}\times(-L,L))$,*
- *υ_n converges strongly to υ in $L^1_{\mathrm{loc}}(\mathcal{O})$,*
- *h is a χ-function.*

Definition 3. *Function $f: G_{T,S}\times(-M,M)\to\mathbb{R}$ is called a kinetic solution for problem Π_0 if it is a χ-function and satisfies the following assertions:*

FBU.1 *(Kinetic equation)*

$$(7a)\quad a'(\lambda)\partial_t f(\boldsymbol{x},t,s,\lambda) + b'(\lambda)\partial_s f(\boldsymbol{x},t,s,\lambda) + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f(\boldsymbol{x},t,s,\lambda) = \Delta_x f(\boldsymbol{x},t,s,\lambda) + \partial_\lambda(m(\boldsymbol{x},t,s,\lambda) + n(\boldsymbol{x},t,s,\lambda)),$$

FBU.2 *(Kinetic boundary conditions)*

$$(7b)\quad a'(\lambda)\big(\chi(\lambda;u_0^{\tau,\Gamma}(\boldsymbol{x},s)) - \chi(\lambda;u_0^\Gamma(\boldsymbol{x},s))\big) - \delta_{(\lambda=u_0^\Gamma(\boldsymbol{x},s))}\big(a(u_0^{\tau,\Gamma}(\boldsymbol{x},s)) - a(u_0^\Gamma(\boldsymbol{x},s))\big) = \partial_\lambda\mu_0^\Gamma(\boldsymbol{x},s,\lambda),$$

$$\text{(7c)}\quad a'(\lambda)\big(\chi(\lambda;u_T^{\tau,\Gamma}(\boldsymbol{x},s))-\chi(\lambda;u_T^{\Gamma}(\boldsymbol{x},s))\big) \\ -\delta_{(\lambda=u_T^{\Gamma}(\boldsymbol{x},s))}\big(a(u_T^{\tau,\Gamma}(\boldsymbol{x},s))-a(u_T^{\Gamma}(\boldsymbol{x},s))\big)=-\partial_\lambda\mu_T^{\Gamma}(\boldsymbol{x},s,\lambda),$$

$$\text{(7d)}\quad b'(\lambda)\big(\chi(\lambda;u_0^{\tau,\Xi}(\boldsymbol{x},t))-\chi(\lambda;u_0^{\Xi}(\boldsymbol{x},t))\big) \\ -\delta_{(\lambda=u_0^{\Xi}(\boldsymbol{x},t))}\big(b(u_0^{\tau,\Xi}(\boldsymbol{x},t))-b(u_0^{\Xi}(\boldsymbol{x},t))\big)=\partial_\lambda\mu_0^{\Xi}(\boldsymbol{x},t,\lambda),$$

$$\text{(7e)}\quad b'(\lambda)\big(\chi(\lambda;u_S^{\tau,\Xi}(\boldsymbol{x},t))-\chi(\lambda;u_S^{\Xi}(\boldsymbol{x},t))\big) \\ -\delta_{(\lambda=u_S^{\Xi}(\boldsymbol{x},t))}\big(b(u_S^{\tau,\Xi}(\boldsymbol{x},t))-b(u_S^{\Xi}(\boldsymbol{x},t))\big)=-\partial_\lambda\mu_S^{\Xi}(\boldsymbol{x},t,\lambda),$$

where $\mu_0^\Gamma,\mu_T^\Gamma\in\mathcal{M}^+(\Omega\times(0,S)\times(-M,M))$, $\mu_0^\Xi,\mu_S^\Xi\in\mathcal{M}^+(\Omega\times(0,T)\times(-M,M))$, n, $m\in\mathcal{M}^+(G_{T,S}\times(-M,M))$, $n(\boldsymbol{x},t,s,\lambda)=\delta_{(\lambda=u)}\left|\nabla_x u\right|^2$,

$$u(\boldsymbol{x},t,s)=\int_{-M}^{M} f(\boldsymbol{x},t,s,\lambda)\,d\lambda,\ (\boldsymbol{x},t,s)\in G_{T,S},$$

$$u_0^{\tau,\Gamma}(\boldsymbol{x},s)=\int_{-M}^{M} f_0^{\tau,\Gamma}(\boldsymbol{x},s,\lambda)\,d\lambda,\ u_T^{\tau,\Gamma}(\boldsymbol{x},s)=\int_{-M}^{M} f_T^{\tau,\Gamma}(\boldsymbol{x},s,\lambda)\,d\lambda,\ (\boldsymbol{x},s)\in\Omega\times(0,S),$$

$$u_0^{\tau,\Xi}(\boldsymbol{x},t)=\int_{-M}^{M} f_0^{\tau,\Xi}(\boldsymbol{x},t,\lambda)\,d\lambda,\ u_S^{\tau,\Xi}(\boldsymbol{x},t)=\int_{-M}^{M} f_S^{\tau,\Xi}(\boldsymbol{x},t,\lambda)\,d\lambda,\ (\boldsymbol{x},t)\in\Omega\times(0,T),$$

χ-functions $f_0^{\tau,\Gamma}$, $f_T^{\tau,\Gamma}$, $f_0^{\tau,\Xi}$ *and* $f_S^{\tau,\Xi}$ *are defined in subsection 3.3.*

Let $\eta\in C^2(\mathbb{R})$, $\eta''(z)\geq 0$, $\forall z\in\mathbb{R}$, $q_a\in C^2(\mathbb{R})$, $q_b\in C^2(\mathbb{R})$, $\boldsymbol{q}_\varphi\in(C^2(\mathbb{R}))^d$, where

$$\text{(8)}\qquad {q_a}'(z)=a'(z)\eta'(z),\quad q_b'(z)=b'(z)\eta'(z),\quad \boldsymbol{q}_\varphi'(z)=\boldsymbol{\varphi}'(z)\eta'(z),\quad \forall z\in\mathbb{R}.$$

Definition 4. *Function $u(\boldsymbol{x},t,s)$ is called an entropy solution for problem Π_0 if it satisfies the maximum principle*

$$\text{(9a)}\quad \|u\|_{L^\infty(G_{T,S})}\leq M= \\ \max\Big(\left\|u_0^\Gamma\right\|_{L^\infty(\Omega\times(0,S))},\left\|u_T^\Gamma\right\|_{L^\infty(\Omega\times(0,S))},\left\|u_0^\Xi\right\|_{L^\infty(\Omega\times(0,T))},\left\|u_S^\Xi\right\|_{L^\infty(\Omega\times(0,T))}\Big),$$

the entropy inequality

$$\text{(9b)}\qquad \partial_t q_a(u)+\partial_s q_b(u)+\mathrm{div}_x\boldsymbol{q}_\varphi(u)-\Delta_x\eta(u)\leq-\eta''(u)|\nabla_x u|^2$$

and the entropy boundary conditions

$$\text{(9c)}\quad q_a(u_0^{\tau,\Gamma}(\boldsymbol{x},s))-q_a(u_0^{\Gamma}(\boldsymbol{x},s))-\eta'(u_0^{\Gamma}(\boldsymbol{x},s))(a(u_0^{\tau,\Gamma}(\boldsymbol{x},s))-a(u_0^{\Gamma}(\boldsymbol{x},s)))\leq 0,$$

$$\text{(9d)}\quad q_a(u_T^{\tau,\Gamma}(\boldsymbol{x},s))-q_a(u_T^{\Gamma}(\boldsymbol{x},s))-\eta'(u_T^{\Gamma}(\boldsymbol{x},s))(a(u_T^{\tau,\Gamma}(\boldsymbol{x},s))-a(u_T^{\Gamma}(\boldsymbol{x},s)))\geq 0,$$

$$\text{(9e)}\quad q_b(u_0^{\tau,\Xi}(\boldsymbol{x},t))-q_b(u_0^{\Xi}(\boldsymbol{x},t))-\eta'(u_0^{\Xi}(\boldsymbol{x},t))(b(u_0^{\tau,\Xi}(\boldsymbol{x},t))-b(u_0^{\Xi}(\boldsymbol{x},t)))\leq 0,$$

$$\text{(9f)}\quad q_b(u_S^{\tau,\Xi}(\boldsymbol{x},t))-q_b(u_S^{\Xi}(\boldsymbol{x},t))-\eta'(u_S^{\Xi}(\boldsymbol{x},t))(b(u_S^{\tau,\Xi}(\boldsymbol{x},t))-b(u_S^{\Xi}(\boldsymbol{x},t)))\geq 0,$$

for all auxiliary functions $(\eta,\boldsymbol{q}_\varphi,q_a,q_b)$ *satisfying* (8).

Here functions $u_0^{\tau,\Gamma}$, $u_T^{\tau,\Gamma}$, $u_0^{\tau,\Xi}$ and $u_S^{\tau,\Xi}$ are traces in the L^1 sense.

Remark 5. *Under conditions of Definition 3, the function*

$$u(\boldsymbol{x},t,s)=\int_{-M}^{M} f(\boldsymbol{x},t,s,\lambda)\,d\lambda$$

is the entropy solution for problem Π_0. The equivalence between Definitions 3 and 4 can be proved in a similar way as it was done in [15].

Theorem 1. *Under Conditions on $a, b\&\boldsymbol{\varphi}$, problem Π_0 has the unique entropy solution u for all u_0^Γ, $u_T^\Gamma \in L^\infty(\Omega\times(0,S))$, u_0^Ξ, $u_S^\Xi \in L^\infty(\Omega\times(0,T))$. Moreover, L^1 stability holds:*

$$(10)\quad \|u_1-u_2\|_{L^1(G_{T,S})} \le C_{(46)}\Big(\|u_{1,0}^\Gamma-u_{2,0}^\Gamma\|_{L^1(\Omega\times(0,S))} + \|u_{1,T}^\Gamma-u_{2,T}^\Gamma\|_{L^1(\Omega\times(0,S))} + \|u_{1,0}^\Xi-u_{2,0}^\Xi\|_{L^1(\Omega\times(0,T))} + \|u_{1,S}^\Xi-u_{2,S}^\Xi\|_{L^1(\Omega\times(0,T))}\Big).$$

This theorem is proved in sections 3 and 4 which are devoted to the existence and uniqueness of the entropy solution, correspondingly. Here we use Remark 5.

It is important to note that (10) enables to decrease the smoothness of u_0^Γ, u_T^Γ, u_0^Ξ and u_S^Ξ assumed in Proposition 1.

2. Proof of Proposition 1

To establish the existence of a weak solution u_ε for problem Π_ε, we use the well-known results on elliptic equations [5], [18].

2.1. **The maximum principle** (5).

In order to prove the maximum principle (5), let us introduce the function

$$u_\varepsilon^M=\max(u_\varepsilon-M,0)=\begin{cases} u_\varepsilon-M & \text{if } u_\varepsilon>M\\ 0 & \text{if } u_\varepsilon\le M\end{cases}.$$

We see that

$$u_\varepsilon^M\big|_{\partial G_{T,S}}=0,\ \nabla_x u_\varepsilon^M=\begin{cases}\nabla_x u_\varepsilon, & u_\varepsilon>M,\\ 0, & u_\varepsilon\le M,\end{cases}\ \partial_t u_\varepsilon^M=\begin{cases}\partial_t u_\varepsilon, & u_\varepsilon>M,\\ 0, & u_\varepsilon\le M,\end{cases}$$

$$\partial_s u_\varepsilon^M=\begin{cases}\partial_s u_\varepsilon, & u_\varepsilon>M,\\ 0, & u_\varepsilon\le M.\end{cases}$$

Putting $\phi=u_\varepsilon^M$ in (4a), we obtain

$$(11)\qquad u_\varepsilon\le M$$

from

$$\int_{G_{T,S}} \Big(\big|\nabla_x u_\varepsilon^M\big|^2 + \varepsilon \big|\partial_t u_\varepsilon^M\big|^2 + \varepsilon \big|\partial_s u_\varepsilon^M\big|^2 \Big) \, \boldsymbol{dx} dt ds = \int_{G_{T,S}} a(u_\varepsilon)\partial_t u_\varepsilon^M \, \boldsymbol{dx} dt ds$$

$$+ \int_{G_{T,S}} b(u_\varepsilon)\partial_s u_\varepsilon^M \, \boldsymbol{dx} dt ds + \int_{G_{T,S}} \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla_x u_\varepsilon^M \, \boldsymbol{dx} dt ds =$$

$$\int_{G_{T,S}} a(u_\varepsilon^M + M)\partial_t u_\varepsilon^M \, \boldsymbol{dx} dt ds + \int_{G_{T,S}} b(u_\varepsilon^M + M)\partial_s u_\varepsilon^M \, \boldsymbol{dx} dt ds$$

$$+ \int_{G_{T,S}} \boldsymbol{\varphi}(u_\varepsilon^M + M) \cdot \nabla_x u_\varepsilon^M \, \boldsymbol{dx} dt ds = \int_{G_{T,S}} \partial_t \Big(\int_0^{u_\varepsilon^M} a(\lambda + M) \, d\lambda \Big) \, \boldsymbol{dx} dt ds$$

$$+ \int_{G_{T,S}} \partial_s \Big(\int_0^{u_\varepsilon^M} b(\lambda + M) \, d\lambda \Big) \, \boldsymbol{dx} dt ds + \int_{G_{T,S}} \nabla_x \Big(\int_0^{u_\varepsilon^M} \boldsymbol{\varphi}(\lambda + M) \, d\lambda \Big) \, \boldsymbol{dx} dt ds = 0.$$

The inequality

$$u_\varepsilon \geq -M \tag{12}$$

can be deduced in a similar way. Inequalities (11) and (12) lead to the maximum principle (5).

2.2. **The energy estimate** (6).

To deduce the energy estimate (6), we consider the extension denoted by $\widehat{u}$ of u_0^Γ, u_T^Γ, u_0^Ξ and u_S^Ξ into $G_{T,S}$ such that

$$\widehat{u} \in L^\infty(G_{T,S}) \cap V(G_{T,S}), \quad (u_\varepsilon - \widehat{u})|_{\partial G_{T,S}} = 0. \tag{13}$$

In equation (4a) we take $\phi = u_\varepsilon - \widehat{u}$:

$$\int_{G_{T,S}} \Big(|\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_t u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2 \Big) \, \boldsymbol{dx} dt ds =$$

$$\int_{G_{T,S}} \Big(\nabla_x u_\varepsilon \cdot \nabla_x \hat{u} + \varepsilon \partial_t u_\varepsilon \partial_t \widehat{u} + \varepsilon \partial_s u_\varepsilon \partial_s \widehat{u}$$

$$+ a(u_\varepsilon)\partial_t(u_\varepsilon - \widehat{u}) + b(u_\varepsilon)\partial_s(u_\varepsilon - \widehat{u}) + \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla_x(u_\varepsilon - \widehat{u}) \Big) \, \boldsymbol{dx} dt ds \leq$$

$$\int_{G_{T,S}} \Big(\frac{\delta^2}{2} |\nabla_x u_\varepsilon|^2 + \frac{\delta^{-2}}{2} |\nabla_x \widehat{u}|^2 + \frac{\varepsilon\delta^2}{2} |\partial_t u_\varepsilon|^2 + \frac{\varepsilon\delta^{-2}}{2} |\partial_t \widehat{u}|^2 + \frac{\varepsilon\delta^2}{2} |\partial_s u_\varepsilon|^2$$

$$+ \frac{\varepsilon\delta^{-2}}{2} |\partial_s \widehat{u}|^2 \Big) \, \boldsymbol{dx} dt ds + 2 \sup_{|z| \leq M} \left| \int_0^z a(\lambda) \, d\lambda \right| S \text{mes}\Omega$$

$$+ \int_{G_{T,S}} \Big(\frac{1}{2} |\partial_t \widehat{u}|^2 + \frac{1}{2} |a(u_\varepsilon)|^2 \Big) \, \boldsymbol{dx} dt ds + 2 \sup_{|z| \leq M} \left| \int_0^z b(\lambda) \, d\lambda \right| T \text{mes}\Omega$$

$$+ \int_{G_{T,S}} \Big(\frac{1}{2} |\partial_s \widehat{u}|^2 + \frac{1}{2} |b(u_\varepsilon)|^2 \Big) \, \boldsymbol{dx} dt ds$$

$$+ \int_{G_{T,S}} \Big(\frac{\delta^2}{2} |\nabla_x u_\varepsilon|^2 + \frac{\delta^{-2}}{2} |\boldsymbol{\varphi}(u_\varepsilon)|^2 + \frac{1}{2} |\nabla_x \widehat{u}|^2 + \frac{1}{2} |\boldsymbol{\varphi}(u_\varepsilon)|^2 \Big) \, \boldsymbol{dx} dt ds.$$

Choosing $\delta^2 = \frac{1}{2}$ so as $1 - \frac{\delta^2}{2} > 1 - \delta^2 = \frac{1}{2}$, we find that

$$
\begin{aligned}
(14)\quad & \int_{G_{T,S}} \Big(|\nabla_x u_\varepsilon|^2 + \varepsilon\, |\partial_t u_\varepsilon|^2 + \varepsilon\, |\partial_s u_\varepsilon|^2 \Big)\, d\boldsymbol{x}dtds \le \\
& \int_{G_{T,S}} \big(3\, |\nabla_x \widehat{u}|^2 + (1+2\varepsilon)\, |\partial_t \widehat{u}|^2 + (1+2\varepsilon)\, |\partial_s \widehat{u}|^2 \big)\, d\boldsymbol{x}dtds \\
& + \int_{G_{T,S}} \big(|a(u_\varepsilon)|^2 + |b(u_\varepsilon)|^2 + |\boldsymbol{\varphi}(u_\varepsilon)|^2 \big)\, d\boldsymbol{x}dtds \\
& + 4 \sup_{|z|\le M} \left| \int_0^z a(\lambda)\, d\lambda \right| S \mathrm{mes}\Omega + 4 \sup_{|z|\le M} \left| \int_0^z b(\lambda)\, d\lambda \right| T \mathrm{mes}\Omega \le \\
& 3 \int_{G_{T,S}} \big(|\nabla_x \widehat{u}|^2 + |\partial_t \widehat{u}|^2 + |\partial_s \widehat{u}|^2 \big)\, d\boldsymbol{x}dtds \\
& + TS\mathrm{mes}\Omega \big(\sup_{|z|\le M} |a(z)|^2 + \sup_{|z|\le M} |b(z)|^2 + 3 \sup_{|z|\le M} |\boldsymbol{\varphi}(z)|^2 \big) \\
& + 4 \sup_{|z|\le M} \left| \int_0^z a(\lambda)\, d\lambda \right| S \mathrm{mes}\Omega + 4 \sup_{|z|\le M} \left| \int_0^z b(\lambda)\, d\lambda \right| T \mathrm{mes}\Omega =: C_{(14)},
\end{aligned}
$$

using $1 + 2\varepsilon \le 3$.

2.3. **Existence of u_ε.**

The maximum principle (5) and the energy estimate (6) imply that the operator L defined by the formula

$$
\begin{aligned}
L(u_\varepsilon, \omega) = \int_{G_{T,S}} & \big(- a(u_\varepsilon)\partial_t(\omega - \widehat{u}) - b(u_\varepsilon)\partial_s(\omega - \widehat{u}) - \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla_x(\omega - \widehat{u}) \\
& + \nabla_x u_\varepsilon \cdot \nabla_x(\omega - \widehat{u}) + \varepsilon \partial_t u_\varepsilon \partial_t(\omega - \widehat{u}) + \varepsilon \partial_s u_\varepsilon \partial_s(\omega - \widehat{u}) \big)\, d\boldsymbol{x}dtds
\end{aligned}
$$

is coercive in $L^\infty(G_{T,S}) \cap V(G_{T,S})$, that is

$$
\begin{aligned}
L(u_\varepsilon, u_\varepsilon) = & \int_{G_{T,S}} \big(- a(u_\varepsilon)\partial_t(u_\varepsilon - \widehat{u}) - b(u_\varepsilon)\partial_s(u_\varepsilon - \widehat{u}) - \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla_x(u_\varepsilon - \widehat{u}) \\
& + \nabla_x u_\varepsilon \cdot \nabla_x(u_\varepsilon - \widehat{u}) + \varepsilon \partial_t u_\varepsilon \partial_t(u_\varepsilon - \widehat{u}) + \varepsilon \partial_s u_\varepsilon \partial_s(u_\varepsilon - \widehat{u}) \big)\, d\boldsymbol{x}dtds \ge \\
& \min\big(\frac{1}{2}, \frac{3}{4}\varepsilon\big) \int_{G_{T,S}} \Big(|\nabla_x u_\varepsilon|^2 + |\partial_t u_\varepsilon|^2 + |\partial_s u_\varepsilon|^2 \Big)\, d\boldsymbol{x}dtds - C_{(15)},
\end{aligned}
$$

$$
\begin{aligned}
(15)\quad C_{(15)} := & \frac{3}{2} \int_{G_{T,S}} |\nabla_x \widehat{u}|^2\, d\boldsymbol{x}dtds + \frac{1+2\varepsilon}{2} \int_{G_T} |\partial_t \widehat{u}|^2\, d\boldsymbol{x}dtds \\
& + \frac{1+2\varepsilon}{2} \int_{G_T} |\partial_s \widehat{u}|^2\, d\boldsymbol{x}dtds + TS\mathrm{mes}\Omega \Big(\frac{1}{2} \sup_{|z|\le M} |a(z)|^2 + \frac{1}{2} \sup_{|z|\le M} |b(z)|^2 \\
& + \frac{3}{2} \sup_{|z|\le M} |\boldsymbol{\varphi}(z)|^2 \Big) + 2\mathrm{mes}\Omega \Big(S \sup_{|z|\le M} \left| \int_0^z a(\lambda)\, d\lambda \right| + T \sup_{|z|\le M} \left| \int_0^z b(\lambda)\, d\lambda \right| \Big).
\end{aligned}
$$

According to the well-known results (see [5, Theorem 8.5], [18, Theorem 9.2, Ch. IV]) we can conclude that problem Π_ε has at least one weak solution u_ε. □

3. Existence of entropy solutions for problem Π_0

In this section we use Remark 5.

3.1. **Auxiliary kinetic equations.**

In this subsection we use results from [8], [19]. Let

$$f^{(n)} \rightharpoonup f \quad \text{in} \quad L^2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}) \quad \text{as } n \to \infty,$$

satisfying

$$\begin{aligned} (16)\quad & a'(\lambda)\partial_t f^{(n)}(\boldsymbol{x},t,s,\lambda) + b'(\lambda)\partial_s f^{(n)}(\boldsymbol{x},t,s,\lambda) \\ & + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f^{(n)}(\boldsymbol{x},t,s,\lambda) - \Delta_x f^{(n)}(\boldsymbol{x},t,s,\lambda) = \\ & \partial_\lambda k^{(n)}(\boldsymbol{x},t,s,\lambda) + \sum_{i=1}^{d}\partial_{x_i}\left(g_{0,i}^{(n)}(\boldsymbol{x},t,s,\lambda) + \partial_\lambda g_{1,i}^{(n)}(\boldsymbol{x},t,s,\lambda)\right) \\ & + \partial_t\left(g_{0,d+1}^{(n)}(\boldsymbol{x},t,s,\lambda) + \partial_\lambda g_{1,d+1}^{(n)}(\boldsymbol{x},t,s,\lambda)\right) \\ & + \partial_s\left(g_{0,d+2}^{(n)}(\boldsymbol{x},t,s,\lambda) + \partial_\lambda g_{1,d+2}^{(n)}(\boldsymbol{x},t,s,\lambda)\right), \end{aligned}$$

where $g_{0,i}^{(n)}, g_{1,i}^{(n)} \to 0$ in $L^2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $i = 1,\ldots,d+2$, while $k^{(n)}$ lies in a bounded set in the space of bounded measures $\mathcal{M}^+(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and is precompact in $W^{-1,p}_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ for $1 \le p < \frac{d+3}{d+2}$.

Here we formulate the condition [19, Equation (11)] in the following way

$$(17) \qquad \mathrm{mes}\{\lambda \in \mathbb{R} : \xi_1^2 + \ldots + \xi_d^2 + i(a'(\lambda)\theta + b'(\lambda)\vartheta) = 0\} = 0$$

for every

$$(\xi_1,\ldots,\xi_d,\theta,\vartheta) \in P = \left\{(\xi_1,\ldots,\xi_d,\theta,\vartheta) \in \mathbb{R}^{d+2} : \xi_1^4 + \ldots + \xi_d^4 + \theta^2 + \vartheta^2 = 1\right\}.$$

Condition (17) is equivalent to (1). From [19, Theorem 7] it follows that

$$\int_{\mathbb{R}} f^{(n)}(\boldsymbol{x},t,s,\lambda)\,d\lambda \to \int_{\mathbb{R}} f(\boldsymbol{x},t,s,\lambda)\,d\lambda \quad \text{strongly in} \quad L^2_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}).$$

We note that the result presented in [19] can be generalized in the case of the right hand side presented in (17).

3.2. **Precompactness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^1(G_{T,S})$.**

In this section we need to formulate kinetic representation of a weak solution for problem Π_ε. In equation (4b) we take $\phi = \eta'(u_\varepsilon)\gamma$, $\gamma \in C_0^\infty(G_{T,S})$ and integrate by parts

$$\begin{aligned} (18)\quad & \int_{G_{T,S}} \Big(- q_a(u_\varepsilon)\partial_t\gamma - q_b(u_\varepsilon)\partial_s\gamma - \boldsymbol{q}_\varphi(u_\varepsilon)\cdot\nabla_x\gamma - \eta(u_\varepsilon)\Delta_x\gamma \\ & - \varepsilon\eta(u_\varepsilon)\partial_t^2\gamma - \varepsilon\eta(u_\varepsilon)\partial_s^2\gamma + \eta''(u_\varepsilon)\big(|\nabla_x u_\varepsilon|^2 + \varepsilon|\partial_t u_\varepsilon|^2 + \varepsilon|\partial_s u_\varepsilon|^2\big)\gamma\Big)\,d\boldsymbol{x}dtds = 0. \end{aligned}$$

With the help of kinetic methods described in the monograph [27] we get

$$\begin{aligned} (19)\quad & a'(\lambda)\partial_t f_\varepsilon(\boldsymbol{x},t,s,\lambda) + b'(\lambda)\partial_s f_\varepsilon(\boldsymbol{x},t,s,\lambda) + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f_\varepsilon(\boldsymbol{x},t,s,\lambda) \\ & - \Delta_x f_\varepsilon(\boldsymbol{x},t,s,\lambda) = \partial_\lambda k_\varepsilon(\boldsymbol{x},t,s,\lambda) + \partial_t(g_{0,d+1,\varepsilon}(\boldsymbol{x},t,s,\lambda) + \partial_\lambda g_{1,d+1,\varepsilon}(\boldsymbol{x},t,s,\lambda)) \\ & + \partial_s(g_{0,d+2,\varepsilon}(\boldsymbol{x},t,s,\lambda) + \partial_\lambda g_{1,d+2,\varepsilon}(\boldsymbol{x},t,s,\lambda)), \end{aligned}$$

where

$$k_\varepsilon = \delta_{(\lambda=u_\varepsilon)}\big(|\nabla_x u_\varepsilon|^2 + \varepsilon|\partial_t u_\varepsilon|^2 + \varepsilon|\partial_s u_\varepsilon|^2\big),$$
$$g_{0,d+1,\varepsilon} + \partial_\lambda g_{1,d+1,\varepsilon} = \varepsilon\partial_t f_\varepsilon = \sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_t u_\varepsilon)\delta_{(\lambda=u_\varepsilon)},$$
$$g_{0,d+2,\varepsilon} + \partial_\lambda g_{1,d+2,\varepsilon} = \varepsilon\partial_s f_\varepsilon = \sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_s u_\varepsilon)\delta_{(\lambda=u_\varepsilon)}.$$

According to results formulated in subsection 3.1, $u_\varepsilon(\boldsymbol{x},t,s) = \int_{-M}^{M} f_\varepsilon(\boldsymbol{x},t,s,\lambda)\,d\lambda$ strongly converges in $L^1(G_{T,S})$ to $u(\boldsymbol{x},t,s) = \int_{-M}^{M} f(\boldsymbol{x},t,s,\lambda)\,d\lambda$ as $\varepsilon \to 0+$.

3.3. **Traces of an entropy solution.**

We use methods elaborated in [1], [16], [22], [23], [31] to prove the existence of strong traces on Γ_0. On Γ_T, Ξ_0 and Ξ_S the proof is the same. Let $f_0^{\tau,\Gamma}(\boldsymbol{x},s,\lambda)$, $f_T^{\tau,\Gamma}(\boldsymbol{x},s,\lambda)$, $f_0^{\tau,\Xi}(\boldsymbol{x},t,\lambda)$ and $f_S^{\tau,\Xi}(\boldsymbol{x},t,\lambda)$ be essential $*$-weak limits in L^∞ of a χ-function $f(\boldsymbol{x},t,s,\lambda)$ satisfying (7a) as $t \to 0+$, $T-0$ and, correspondingly, $s \to 0+$, $S-0$. Note that these limits exist in view of the maximum principle (9a) and the Alaoglu theorem.

The existence of the strong trace is equivalent to that $f_0^{\tau,\Gamma}$ is a χ-function in $\Omega\times(0,S)$. Up to a set of measure zero, we represent a bounded domain $\Omega\times(0,S)$ as the countable union of sets $Q_\alpha = \{(\boldsymbol{x},s) \in \mathbb{R}^{d+1} : |x_j - x_{\alpha,j}| < r_\alpha, j = 1,\dots,d, |s - s_\alpha| < r_\alpha\}$, $\alpha \in \mathbb{N}$. We fix $\alpha \in \mathbb{N}$ and consider $f_0^{\tau,\Gamma}$ only in the domain Q_α.

We are going to prove that $f_0^{\tau,\Gamma}$ is a χ-function in Q_α.

We consider (7a) in the domain $\prod_{j=1}^{d}(x_{\alpha,j} - r_\alpha, x_{\alpha,j} + r_\alpha)\times(0,r_\alpha)\times(s_\alpha - r_\alpha, s_\alpha + r_\alpha)\times(-M,M)$. Let the set $\mathcal{E} \subset Q_\alpha$ such that $\mathrm{mes}(Q_\alpha\backslash\mathcal{E}) = 0$. We fix a point $\widehat{x} \in \mathcal{E}$. Let $r = \min(r_\alpha, \mathrm{dist}(\widehat{x},\partial\Omega), \widehat{s}, S-\widehat{s})$. We apply the scaling procedure $(\boldsymbol{x},t,s,\lambda) = (\widehat{\boldsymbol{x}} + \sqrt{\epsilon}\,\boldsymbol{\omega}, \epsilon\,\theta, \widehat{s} + \epsilon\,\vartheta, \lambda)$ to (7a):

$$a'(\lambda)\partial_\theta f_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)} + b'(\lambda)\partial_\vartheta f_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)} = \Delta_\omega f_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)} + \partial_\lambda(m_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)} + n_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)}) - \sqrt{\epsilon}\,\boldsymbol{\varphi}'(\lambda)\cdot\nabla_\omega f_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)}, \tag{20a}$$

where

$$f_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)}(\boldsymbol{\omega},\theta,\vartheta,\lambda) = f(\widehat{\boldsymbol{x}} + \sqrt{\epsilon}\,\boldsymbol{\omega}, \epsilon\,\theta, \widehat{s} + \epsilon\,\vartheta, \lambda),$$
$$(\boldsymbol{\omega},\theta,\vartheta,\lambda) \in \big(-\tfrac{r}{\sqrt{\epsilon}}, \tfrac{r}{\sqrt{\epsilon}}\big)^d \times (0,\tfrac{r}{\epsilon}) \times (-\tfrac{r}{\epsilon},\tfrac{r}{\epsilon}) \times (-M,M),$$

(20b)
$$m_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)}\Big(\prod_{j=1}^{d+2}(R_{1,j},R_{2,j})\times(L_1,L_2)\Big) = \frac{1}{\epsilon^{\frac{d}{2}+1}} m\Big(\prod_{j=1}^{d}(\widehat{x}_j + \sqrt{\epsilon}R_{1,j}, \widehat{x}_j + \sqrt{\epsilon}R_{2,j}) \times (\epsilon R_{1,d+1}, \epsilon R_{2,d+1}) \times (\widehat{s} + \epsilon R_{1,d+2}, \widehat{s} + \epsilon R_{2,d+2}) \times (L_1,L_2)\Big),$$

$$n_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon)}\big(\boldsymbol{\omega},\theta,\vartheta,\lambda\big) = \delta_{(\lambda=u(\widehat{\boldsymbol{x}}+\sqrt{\epsilon}\boldsymbol{\omega},\epsilon\theta,\widehat{s}+\epsilon\vartheta))}|\nabla_\omega u(\widehat{\boldsymbol{x}} + \sqrt{\epsilon}\boldsymbol{\omega}, \epsilon\theta, \widehat{s}+\epsilon\vartheta)|^2. \tag{20c}$$

Lemma 2. *There exist a sequence* $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *and a set* $\mathcal{E} \subset Q_\alpha$ *such that* $\epsilon_n \to 0+$, $\mathrm{mes}(Q_\alpha\backslash\mathcal{E}) = 0$ *and for any* $(\widehat{\boldsymbol{x}},\widehat{s}) \in \mathcal{E}$ *and* $R > 0$

$$\lim_{n\to\infty} m_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon_n)}((-R,R)^d \times (0,R) \times (-R,R) \times (-M,M)) = 0, \tag{21a}$$

$$\lim_{n\to\infty} n_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon_n)}(\boldsymbol{\omega},\theta,\vartheta,\lambda) = 0, \tag{21b}$$

where $(\boldsymbol{\omega}, \theta, \vartheta, \lambda) \in (-R, R)^d \times (0, R) \times (-R, R) \times (-M, M)$,
(21c)

$$\lim_{n\to\infty} \int_{(-R,R)^{d+1}\times(-M,M)} \Big| f_0^{\tau,\Gamma}(\widehat{\boldsymbol{x}}, \widehat{s}, \lambda) - f_0^{\tau,\Gamma}(\widehat{\boldsymbol{x}} + \sqrt{\epsilon_n}\boldsymbol{\omega}, \widehat{s} + \epsilon_n\vartheta, \lambda)\Big| \, d\boldsymbol{\omega} d\vartheta d\lambda = 0.$$

Moreover, there exists a χ*-function* $f_{(\cdot,\cdot)} \in L^\infty(\mathcal{E} \times \mathbb{R}^d_\omega \times \mathbb{R}^+_\theta \times \mathbb{R}_\vartheta \times (-M, M))$ *such that for a fixed point* $(\widehat{\boldsymbol{x}}, \widehat{s}) \in \mathcal{E}$ *the sequence of* χ*-functions, multiplied by corresponding characteristic functions,*

$$\{f_{(\widehat{\boldsymbol{x}},\widehat{s},\epsilon_n)}\chi_{(-\frac{r}{\sqrt{\epsilon_n}},\frac{r}{\sqrt{\epsilon_n}})^d\times(0,\frac{r}{\epsilon_n})\times(-\frac{r}{\epsilon_n},\frac{r}{\epsilon_n})\times(-M,M)}\}_{n\in\mathbb{N}}$$

strongly converges to the χ*-function* $f_{(\widehat{\boldsymbol{x}},\widehat{s})}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^d_\omega \times \mathbb{R}^+_\theta \times \mathbb{R}_\vartheta \times (-M, M))$ *as* $\epsilon_n \to 0+$*. Moreover,* $f_{(\widehat{\boldsymbol{x}},\widehat{s})}$ *is a solution of the Cauchy problem*

$$a'(\lambda)\partial_\theta f_{(\widehat{\boldsymbol{x}},\widehat{s})}(\boldsymbol{\omega}, \theta, \vartheta, \lambda) + b'(\lambda)\partial_\vartheta f_{(\widehat{\boldsymbol{x}},\widehat{s})}(\boldsymbol{\omega}, \theta, \vartheta, \lambda) = \Delta_\omega f_{(\widehat{\boldsymbol{x}},\widehat{s})}(\boldsymbol{\omega}, \theta, \vartheta, \lambda), \tag{22a}$$

$$f_{(\widehat{\boldsymbol{x}},\widehat{s})}(\boldsymbol{\omega}, 0, \vartheta, \lambda) \equiv f_0^{\tau,\Gamma}(\widehat{\boldsymbol{x}}, \widehat{s}, \lambda), \quad (\boldsymbol{\omega}, \vartheta) \in \mathbb{R}^d_\omega \times \mathbb{R}_\vartheta. \tag{22b}$$

The proof of this lemma is analogous to the proof of Lemmas 2 and 3 from [14]. It follows from (22a) and (22b) that for almost all $(\boldsymbol{\omega}, \theta, \vartheta, \lambda) \in \mathbb{R}^d_\omega \times \mathbb{R}^+_\theta \times \mathbb{R}_\vartheta \times (-M, M)$

$$f_{(\widehat{\boldsymbol{x}},\widehat{s})}(\boldsymbol{\omega}, \theta, \vartheta, \lambda) \equiv f_0^{\tau,\Gamma}(\widehat{\boldsymbol{x}}, \widehat{s}, \lambda). \tag{23}$$

Therefore, (see [14, Lemma 4]) $f_0^{\tau,\Gamma}$ is a χ-function in Q_α.

3.4. **Inequalities** (9c)–(9f)**.**

Let $\rho_\delta \in C^2[0, T]$,

$$\rho_\delta(t) = 0, \quad t > \delta, \quad \rho_\delta(0) = 1, \quad |\rho'_\delta(t)| \le \frac{c}{\delta}, \quad -\lim_{\delta\to 0+} \int_0^\delta \Phi(t)\rho'_\delta(t)\, dt = \Phi(0),$$

where $\Phi \in C[0, T]$.

Here we use the assumption from [11]. We put $\psi = \eta'(u_\varepsilon)\rho_\delta(t)\vartheta(\boldsymbol{x}, s)$ in (4c) and integrate by parts:

$$\begin{aligned} -\int_\Omega \int_0^S q_a(u_0^\Gamma(\boldsymbol{x}, s))\vartheta(\boldsymbol{x}, s)\, d\boldsymbol{x}ds + \int_{\Omega\times(0,\delta)\times(0,S)} \Big(- q_a(u_\varepsilon)\rho'_\delta(t)\vartheta(\boldsymbol{x}, s) \\ - q_b(u_\varepsilon)\rho_\delta(t)\partial_s\vartheta(\boldsymbol{x}, s) + \mathrm{div}_x \boldsymbol{q}_\varphi(u_\varepsilon)\rho_\delta(t)\vartheta(\boldsymbol{x}, s) + \nabla_x\eta(u_\varepsilon) \cdot \nabla_x\vartheta\rho_\delta(t) \\ + \varepsilon\partial_t\eta(u_\varepsilon)\rho'_\delta(t)\vartheta(\boldsymbol{x}, s) + \varepsilon\partial_s\eta(u_\varepsilon)\rho_\delta(t)\partial_s\vartheta(\boldsymbol{x}, s) + \eta''(u_\varepsilon)\big(|\nabla_x u_\varepsilon|^2 + \varepsilon\, |\partial_t u_\varepsilon|^2 \\ + \varepsilon\, |\partial_s u_\varepsilon|^2 \big)\rho_\delta(t)\vartheta(\boldsymbol{x}, s)\Big)\, d\boldsymbol{x}dtds = \int_\Omega \int_0^S \varepsilon\eta'(u_0^\Gamma(\boldsymbol{x}, s))\vartheta(\boldsymbol{x}, s)(-\partial_t u_\varepsilon)\big|_{t=0}\, d\boldsymbol{x}ds. \end{aligned} \tag{24}$$

Here we need the following lemma that can be proved in a similar way as in [15].

Lemma 3. *For any test function* $\theta \in C^1_0(\Omega \times (0, S))$*,* $\kappa \in C^1_0(\Omega \times (0, T))$

$$\lim_{\varepsilon\to 0+} \int_\Omega \int_0^S \theta(\boldsymbol{x}, s)(-\varepsilon\partial_t u_\varepsilon)\big|_{t=0}\, d\boldsymbol{x}ds = \int_\Omega \int_0^S \theta(\boldsymbol{x}, s)\big(a(u_0^{\tau,\Gamma}(\boldsymbol{x}, s)) - a(u_0^\Gamma(\boldsymbol{x}, s))\big)\, d\boldsymbol{x}ds, \tag{25a}$$

$$\text{(25b)}\quad \lim_{\varepsilon\to 0+}\int_\Omega\int_0^S \theta(\boldsymbol{x},s)(-\varepsilon\partial_t u_\varepsilon)\big|_{t=T}\,d\boldsymbol{x}ds = \int_\Omega\int_0^S \theta(\boldsymbol{x},s)\big(a(u_T^{\tau,\Gamma}(\boldsymbol{x},s)) - a(u_T^{\Gamma}(\boldsymbol{x},s))\big)\,d\boldsymbol{x}ds,$$

$$\text{(25c)}\quad \lim_{\varepsilon\to 0+}\int_\Omega\int_0^T \kappa(\boldsymbol{x},t)(-\varepsilon\partial_s u_\varepsilon)\big|_{s=0}\,d\boldsymbol{x}dt = \int_\Omega\int_0^T \kappa(\boldsymbol{x},t)\big(b(u_0^{\tau,\Xi}(\boldsymbol{x},t)) - b(u_0^{\Xi}(\boldsymbol{x},t))\big)\,d\boldsymbol{x}dt,$$

$$\text{(25d)}\quad \lim_{\varepsilon\to 0+}\int_\Omega\int_0^T \kappa(\boldsymbol{x},t)(-\varepsilon\partial_s u_\varepsilon)\big|_{s=S}\,d\boldsymbol{x}dt = \int_\Omega\int_0^T \kappa(\boldsymbol{x},t)\big(b(u_S^{\tau,\Xi}(\boldsymbol{x},t)) - b(u_S^{\Xi}(\boldsymbol{x},t))\big)\,d\boldsymbol{x}dt.$$

With the help of (24) and (25a) we can deduce inequality (9c). Repeating the steps being similar to the previous ones in this subsection, we can prove inequalities (9d)–(9f). For details, see [15] where the forward-backward parabolic equation was studied.

4. Uniqueness of the entropy solution for problem Π_0

Let f_1 and f_2 be χ-functions having traces and satisfying Definition 3:

$$\text{(26a)}\quad a'(\lambda)\partial_t f_i(\boldsymbol{x},t,s,\lambda) + b'(\lambda)\partial_s f_i(\boldsymbol{x},t,s,\lambda) + \boldsymbol{\varphi}'(\lambda)\cdot\nabla_x f_i(\boldsymbol{x},t,s,\lambda) = \Delta_x f_i(\boldsymbol{x},t,s,\lambda) + \partial_\lambda(m_i(\boldsymbol{x},t,s,\lambda) + n_i(\boldsymbol{x},t,s,\lambda)),$$

$$\text{(26b)}\quad a'(\lambda)\big(\chi(\lambda;u_{i,0}^{\tau,\Gamma}(\boldsymbol{x},s)) - \chi(\lambda;u_{i,0}^{\Gamma}(\boldsymbol{x},s))\big) - \delta_{(\lambda=u_{i,0}^{\Gamma}(\boldsymbol{x},s))}\big(a(u_{i,0}^{\tau,\Gamma}(\boldsymbol{x},s)) - a(u_{i,0}^{\Gamma}(\boldsymbol{x},s))\big) = \partial_\lambda\mu_{i,0}^{\Gamma}(\boldsymbol{x},s,\lambda),$$

$$\text{(26c)}\quad a'(\lambda)\big(\chi(\lambda;u_{i,T}^{\tau,\Gamma}(\boldsymbol{x},s)) - \chi(\lambda;u_{i,T}^{\Gamma}(\boldsymbol{x},s))\big) - \delta_{(\lambda=u_{i,T}^{\Gamma}(\boldsymbol{x},s))}\big(a(u_{i,T}^{\tau,\Gamma}(\boldsymbol{x},s)) - a(u_{i,T}^{\Gamma}(\boldsymbol{x},s))\big) = -\partial_\lambda\mu_{i,T}^{\Gamma}(\boldsymbol{x},s,\lambda),$$

$$\text{(26d)}\quad b'(\lambda)\big(\chi(\lambda;u_{i,0}^{\tau,\Xi}(\boldsymbol{x},t)) - \chi(\lambda;u_{i,0}^{\Xi}(\boldsymbol{x},t))\big) - \delta_{(\lambda=u_{i,0}^{\Xi}(\boldsymbol{x},t))}\big(b(u_{i,0}^{\tau,\Xi}(\boldsymbol{x},t)) - b(u_{i,0}^{\Xi}(\boldsymbol{x},t))\big) = \partial_\lambda\mu_{i,0}^{\Xi}(\boldsymbol{x},t,\lambda),$$

$$\text{(26e)}\quad b'(\lambda)\big(\chi(\lambda;u_{i,S}^{\tau,\Xi}(\boldsymbol{x},t)) - \chi(\lambda;u_{i,S}^{\Xi}(\boldsymbol{x},t))\big) - \delta_{(\lambda=u_{i,S}^{\Xi}(\boldsymbol{x},t))}\big(b(u_{i,S}^{\tau,\Xi}(\boldsymbol{x},t)) - b(u_{i,S}^{\Xi}(\boldsymbol{x},t))\big) = -\partial_\lambda\mu_{i,S}^{\Xi}(\boldsymbol{x},t,\lambda),$$

where $\mu_{i,0}^{\Gamma},\mu_{i,T}^{\Gamma} \in \mathcal{M}^+(\Omega\times(0,S)\times(-M,M))$, $\mu_{i,0}^{\Xi},\mu_{i,S}^{\Xi} \in \mathcal{M}^+(\Omega\times(0,T)\times(-M,M))$, $n_i, m_i \in \mathcal{M}^+(G_{T,S}\times(-M,M))$, $n_i(\boldsymbol{x},t,s,\lambda) = \delta_{(\lambda=u_i(\boldsymbol{x},t,s))}\,|\nabla_x u_i(\boldsymbol{x},t,s)|^2$, $f_i(\boldsymbol{x},t,s,\lambda) = \chi(\lambda;u_i(\boldsymbol{x},t,s))$, $u_i(\boldsymbol{x},t,s) = \int_{-M}^{M} f_i(\boldsymbol{x},t,s,\lambda)\,d\lambda$, $i=1,2$.

Lemma 4. *For an arbitrary non-negative function $\xi \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ the following inequality holds*

$$
\begin{aligned}
(27)\quad & \int_{G_{T,S}} \int_{-M}^{M} \Big(-\boldsymbol{\varphi}'(\lambda) \cdot \nabla_x \xi(\boldsymbol{x}) \big|\chi(\lambda; u_1(\boldsymbol{x},t,s)) - \chi(\lambda; u_2(\boldsymbol{x},t,s))\big|^2 \\
& \quad - \Delta_x \xi(\boldsymbol{x}) \big|\chi(\lambda; u_1(\boldsymbol{x},t,s)) - \chi(\lambda; u_2(\boldsymbol{x},t,s))\big|^2 \Big)\, d\boldsymbol{x}dtdsd\lambda \leq \\
& \int_\Omega \int_0^S \int_{-M}^{M} a'(\lambda) \Big(\big|\chi(\lambda; u_{1,0}^{\tau,\Gamma}(\boldsymbol{x},s)) - \chi(\lambda; u_{2,0}^{\tau,\Gamma}(\boldsymbol{x},s))\big|^2 \\
& \quad - \big|\chi(\lambda; u_{1,T}^{\tau,\Gamma}(\boldsymbol{x},s)) - \chi(\lambda; u_{2,T}^{\tau,\Gamma}(\boldsymbol{x},s))\big|^2 \Big) \xi(\boldsymbol{x})\, d\boldsymbol{x}dsd\lambda \\
& + \int_\Omega \int_0^T \int_{-M}^{M} b'(\lambda) \Big(\big|\chi(\lambda; u_{1,0}^{\tau,\Xi}(\boldsymbol{x},t)) - \chi(\lambda; u_{2,0}^{\tau,\Xi}(\boldsymbol{x},t))\big|^2 \\
& \quad - \big|\chi(\lambda; u_{1,S}^{\tau,\Xi}(\boldsymbol{x},t)) - \chi(\lambda; u_{2,S}^{\tau,\Xi}(\boldsymbol{x},t))\big|^2 \Big) \xi(\boldsymbol{x})\ d\boldsymbol{x}dtd\lambda.
\end{aligned}
$$

Proof. Here we use methods from [7], [17], [21], [26], [27]. Set $\epsilon \in (0,1]$. We define the auxiliary function:

$$
(28a)\qquad \phi_\epsilon(\boldsymbol{x},t,s) = \frac{1}{\epsilon}\phi_1\big(\frac{t}{\epsilon}\big)\frac{1}{\epsilon}\phi_2\big(\frac{s}{\epsilon}\big)\frac{1}{\epsilon^d}\phi_3\big(\frac{x_1}{\epsilon},\dots,\frac{x_d}{\epsilon}\big),
$$

where ϕ_1, $\phi_2 \in C_c^\infty(\mathbb{R})$, $\phi_3 \in C_c^\infty(\mathbb{R}^d)$, $\phi_j \geq 0$, $\int_{\mathbb{R}} \phi_1 = \int_{\mathbb{R}} \phi_2 = \int_{\mathbb{R}^d} \phi_3 = 1$, supp$\phi_1$, supp$\phi_2 \subset (-1,1)$, supp$\phi_3 \subset (-1,1)^d$. Let us introduce the following functions and measures:

$$
(28b)\qquad f_{i,\epsilon}(\boldsymbol{x},t,s,\lambda) = \chi(\lambda; u_i(\cdot,\cdot,\cdot)) *_{(\boldsymbol{x},t,s)} \phi_\epsilon(\boldsymbol{x},t,s),
$$

$$
(28c)\qquad m_{i,\epsilon}(\boldsymbol{x},t,s,\lambda) = m_i(\cdot,\cdot,\cdot,\lambda) *_{(\boldsymbol{x},t,s)} \phi_\epsilon(\boldsymbol{x},t,s),
$$

$$
(28d)\qquad n_{i,\epsilon}(\boldsymbol{x},t,s,\lambda) = n_i(\cdot,\cdot,\cdot,\lambda) *_{(\boldsymbol{x},t,s)} \phi_\epsilon(\boldsymbol{x},t,s), \quad i = 1,2.
$$

Therefore, we multiply the following equation

$$
\begin{aligned}
& a'(\lambda)\partial_t\big(f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big) + b'(\lambda)\partial_s\big(f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big) \\
& + \boldsymbol{\varphi}'(\lambda) \cdot \nabla_x\big(f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big) - \Delta_x\big(f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big) \\
& = \partial_\lambda(m_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - m_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)) + \partial_\lambda(n_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - n_{2,\epsilon}(\boldsymbol{x},t,s,\lambda))
\end{aligned}
$$

by $2\big(f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big)$:

$$
\begin{aligned}
& a'(\lambda)\partial_t\big|f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big|^2 + b'(\lambda)\partial_s\big|f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big|^2 \\
& \quad + \boldsymbol{\varphi}'(\lambda) \cdot \nabla_x\big|f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big|^2 \\
& - \Delta_x\big|f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big|^2 + 2\big|\nabla_x\big(f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big)\big|^2 \\
& = \partial_\lambda(m_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - m_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)) + \partial_\lambda(n_{1,\epsilon}(\boldsymbol{x},t,s,\lambda) - n_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)).
\end{aligned}
$$

We multiply the latter equation by an arbitrary non-negative function $\xi \in C^2(\Omega)\cap C_0^1(\overline{\Omega})$ and integrate over $G_{T,S}\times(-M,M)$:

$$\int_{G_{T,S}}\int_{-M}^{M}\Big(a'(\lambda)\xi(\boldsymbol{x})\partial_t\big|f_{1,\epsilon}-f_{2,\epsilon}\big|^2+b'(\lambda)\xi(\boldsymbol{x})\partial_s\big|f_{1,\epsilon}-f_{2,\epsilon}\big|^2 \tag{29}$$
$$-\boldsymbol{\varphi}'(\lambda)\cdot\nabla_x\xi(\boldsymbol{x})\big|f_{1,\epsilon}-f_{2,\epsilon}\big|^2-\Delta_x\xi(\boldsymbol{x})\big|f_{1,\epsilon}-f_{2,\epsilon}\big|^2+2\xi(\boldsymbol{x})\big|\nabla_x(f_{1,\epsilon}-f_{2,\epsilon})\big|^2\Big)\,d\boldsymbol{x}dtdsd\lambda=$$
$$2\int_{G_{T,S}}\int_{-M}^{M}\xi(\boldsymbol{x})(f_{1,\epsilon}-f_{2,\epsilon})\partial_\lambda(m_{1,\epsilon}-m_{2,\epsilon})\,d\boldsymbol{x}dtdsd\lambda$$
$$+2\int_{G_{T,S}}\int_{-M}^{M}\xi(\boldsymbol{x})(f_{1,\epsilon}-f_{2,\epsilon})\partial_\lambda(n_{1,\epsilon}-n_{2,\epsilon})\,d\boldsymbol{x}dtdsd\lambda.$$

It is important to note that

$$-\int_{G_{T,S}}\int_{-M}^{M}2\xi(\boldsymbol{x})\big|\nabla_x(f_{1,\epsilon}-f_{2,\epsilon})\big|^2\,d\boldsymbol{x}dtdsd\lambda\le 0. \tag{30}$$

We construct several limits:

$$\lim_{\epsilon\to0+}\int_{-M}^{M}\big|f_{1,\epsilon}(\cdot,\cdot,\cdot,\lambda)-f_{2,\epsilon}(\cdot,\cdot,\cdot,\lambda)\big|^2\,d\lambda= \tag{31}$$
$$\int_{-M}^{M}\big|\chi(\lambda;u_1(\cdot,\cdot,\cdot))-\chi(\lambda;u_2(\cdot,\cdot,\cdot))\big|^2\,d\lambda \text{ in } L^1(G_{T,S}),$$

$$\lim_{\epsilon\to0+}\int_{-M}^{M}\varphi_i'(\lambda)\big|f_{1,\epsilon}(\cdot,\cdot,\cdot,\lambda)-f_{2,\epsilon}(\cdot,\cdot,\cdot,\lambda)\big|^2\,d\lambda= \tag{32}$$
$$\int_{-M}^{M}\varphi_i'(\lambda)\big|\chi(\lambda;u_1(\cdot,\cdot,\cdot))-\chi(\lambda;u_2(\cdot,\cdot,\cdot))\big|^2\,d\lambda \text{ in } L^1(G_{T,S}),\quad i=1,\dots,d.$$

Using of the assertion

$$\lim_{\epsilon\to0+}\int_{G_{T,S}}\int_{-M}^{M}\big(m_{1,\epsilon}(\boldsymbol{x},t,s,\lambda)\delta_{(\lambda=u_1(\cdot,\cdot,\cdot))}*\phi_\epsilon(\boldsymbol{x},t,s) \tag{33}$$
$$+m_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\delta_{(\lambda=u_2(\cdot,\cdot,\cdot))}*\phi_\epsilon(\boldsymbol{x},t,s)\big)\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda=0,$$

which is analogous to [26, Proposition 2.2], we prove that

$$2\lim_{\epsilon\to0+}\int_{G_{T,S}}\int_{-M}^{M}\partial_\lambda(m_{1,\epsilon}-m_{2,\epsilon})(f_{1,\epsilon}-f_{2,\epsilon})\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda= \tag{34}$$
$$-2\lim_{\epsilon\to0+}\int_{G_{T,S}}\int_{-M}^{M}(m_{1,\epsilon}-m_{2,\epsilon})\partial_\lambda(f_{1,\epsilon}-f_{2,\epsilon})\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda=\dots$$

We take into account that

$$\partial_\lambda(\chi(\lambda;u_1)-\chi(\lambda;u_2))=\delta_{(\lambda=0)}-\delta_{(\lambda=u_1)}-\delta_{(\lambda=0)}+\delta_{(\lambda=u_2)}=\delta_{(\lambda=u_2)}-\delta_{(\lambda=u_1)} \text{ and}$$
$$-\partial_\lambda(f_{1,\epsilon}-f_{2,\epsilon})=\big(\delta_{(\lambda=u_1)}-\delta_{(\lambda=u_2)}\big)*\phi_\epsilon.$$

Therefore, we proceed

$$\ldots = 2\lim_{\epsilon\to 0+}\int_{G_{T,S}}\int_{-M}^{M}(m_{1,\epsilon}-m_{2,\epsilon})\big(\delta_{(\lambda=u_1(\cdot,\cdot,\cdot))} - \delta_{(\lambda=u_2(\cdot,\cdot,\cdot))}\big) * \phi_\epsilon(\boldsymbol{x},t,s)\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda =$$

$$2\lim_{\epsilon\to 0+}\int_{G_{T,S}}\int_{-M}^{M}\big(m_{1,\epsilon}(\boldsymbol{x},t,s,\lambda)\delta_{(\lambda=u_1(\cdot,\cdot,\cdot))} * \phi_\epsilon(\boldsymbol{x},t,s) + m_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\delta_{(\lambda=u_2(\cdot,\cdot,\cdot))} * \phi_\epsilon(\boldsymbol{x},t,s)\big)\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda$$

$$-2\lim_{\epsilon\to 0+}\int_{G_{T,S}}\int_{-M}^{M}\big(m_{1,\epsilon}(\boldsymbol{x},t,s,\lambda)\delta_{(\lambda=u_2(\cdot,\cdot,\cdot))} * \phi_\epsilon(\boldsymbol{x},t,s) + m_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\delta_{(\lambda=u_1(\cdot,\cdot,\cdot))} * \phi_\epsilon(\boldsymbol{x},t,s)\big)\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda \le 0.$$

Analogously, we can prove that

$$\lim_{\epsilon\to 0+}\int_{G_{T,S}}\int_{-M}^{M}\partial_\lambda(n_{1,\epsilon}-n_{2,\epsilon})(f_{1,\epsilon}-f_{2,\epsilon})\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda \le 0. \tag{35}$$

The existence of traces established in subsection 3.3 implies the following results

$$-\lim_{\epsilon\to 0+}\int_{G_{T,S}}\int_{-M}^{M}a'(\lambda)\partial_t\big|f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda)-f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big|^2\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda = \int_\Omega\int_0^S\int_{-M}^{M}a'(\lambda)\Big(\big|\chi(\lambda;u^{\tau,\Gamma}_{1,0}(\boldsymbol{x},s))-\chi(\lambda;u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s))\big|^2 - \big|\chi(\lambda;u^{\tau,\Gamma}_{1,T}(\boldsymbol{x},s))-\chi(\lambda;u^{\tau,\Gamma}_{2,T}(\boldsymbol{x},s))\big|^2\Big)\xi(\boldsymbol{x})\;d\boldsymbol{x}dsd\lambda, \tag{36}$$

$$-\lim_{\epsilon\to 0+}\int_{G_{T,S}}\int_{-M}^{M}b'(\lambda)\partial_s\big|f_{1,\epsilon}(\boldsymbol{x},t,s,\lambda)-f_{2,\epsilon}(\boldsymbol{x},t,s,\lambda)\big|^2\xi(\boldsymbol{x})\,d\boldsymbol{x}dtdsd\lambda = \int_\Omega\int_0^T\int_{-M}^{M}b'(\lambda)\Big(\big|\chi(\lambda;u^{\tau,\Xi}_{1,0}(\boldsymbol{x},t))-\chi(\lambda;u^{\tau,\Xi}_{2,0}(\boldsymbol{x},t))\big|^2 - \big|\chi(\lambda;u^{\tau,\Xi}_{1,S}(\boldsymbol{x},t))-\chi(\lambda;u^{\tau,\Xi}_{2,S}(\boldsymbol{x},t))\big|^2\Big)\xi(\boldsymbol{x})\;d\boldsymbol{x}dtd\lambda. \tag{37}$$

If we apply the above-mentioned results to (29) as $\epsilon \to 0+$, we obtain (27). □

Let us estimate the second term in the right-hand side of inequality (27).

Proposition 2. *For any non-negative test function* $\xi \in C_0(\overline{\Omega})$

$$\int_\Omega\int_0^S\int_{-M}^{M}a'(\lambda)\big|\chi(\lambda;u^{\tau,\Gamma}_{1,0}(\boldsymbol{x},s))-\chi(\lambda;u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s))\big|^2\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda \le \int_\Omega\int_0^S\int_{-M}^{M}a'(\lambda)\big|\chi(\lambda;u^{\Gamma}_{1,0}(\boldsymbol{x},s))-\chi(\lambda;u^{\Gamma}_{2,0}(\boldsymbol{x},s))\big|^2\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda, \tag{38a}$$

$$-\int_\Omega\int_0^S\int_{-M}^{M}a'(\lambda)\big|\chi(\lambda;u^{\tau,\Gamma}_{1,T}(\boldsymbol{x},s))-\chi(\lambda;u^{\tau,\Gamma}_{2,T}(\boldsymbol{x},s))\big|^2\xi(\boldsymbol{x})\;d\boldsymbol{x}dsd\lambda \le -\int_\Omega\int_0^S\int_{-M}^{M}a'(\lambda)\big|\chi(\lambda;u^{\Gamma}_{1,T}(\boldsymbol{x},s))-\chi(\lambda;u^{\Gamma}_{2,T}(\boldsymbol{x},s))\big|^2\xi(\boldsymbol{x})\;d\boldsymbol{x}dsd\lambda, \tag{38b}$$

(38c)
$$\int_\Omega \int_0^T \int_{-M}^M b'(\lambda)\big|\chi(\lambda; u_{1,0}^{\tau,\Xi}(\boldsymbol{x},t)) - \chi(\lambda; u_{2,0}^{\tau,\Xi}(\boldsymbol{x},t))\big|^2 \xi(\boldsymbol{x})\, d\boldsymbol{x}dtd\lambda \leq \int_\Omega \int_0^T \int_{-M}^M b'(\lambda)\big|\chi(\lambda; u_{1,0}^{\Xi}(\boldsymbol{x},t)) - \chi(\lambda; u_{2,0}^{\Xi}(\boldsymbol{x},t))\big|^2 \xi(\boldsymbol{x})\ d\boldsymbol{x}dtd\lambda,$$

(38d)
$$-\int_\Omega \int_0^T \int_{-M}^M b'(\lambda)\big|\chi(\lambda; u_{1,S}^{\tau,\Xi}(\boldsymbol{x},t)) - \chi(\lambda; u_{2,S}^{\tau,\Xi}(\boldsymbol{x},t))\big|^2 \xi(\boldsymbol{x})\, d\boldsymbol{x}dtd\lambda \leq -\int_\Omega \int_0^T \int_{-M}^M b'(\lambda)\big|\chi(\lambda; u_{1,S}^{\Xi}(\boldsymbol{x},t)) - \chi(\lambda; u_{2,S}^{\Xi}(\boldsymbol{x},t))\big|^2 \xi(\boldsymbol{x})\, d\boldsymbol{x}dtd\lambda.$$

Proof. In order to prove Proposition 2, let us represent the part of the integrand in the left-hand side of (38a) in the following way:

$$\begin{aligned}
&\big|\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\tau,\Gamma})\big|^2 = \big|\big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big) - \big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\\
&+ \big(\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\big|^2 = \big|\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big|^2 + \big|\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big|^2\\
&+ \big|\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big|^2 - 2\big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\\
&- 2\big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\big(\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\\
&+ 2\big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\big(\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big) = \dots
\end{aligned}$$

Here we use the fact that the following formulas

$$\begin{aligned}
\big|\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big|^2 &= \big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\text{sgn}(\lambda - u_{1,0}^{\Gamma}),\\
\big|\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big|^2 &= \big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\text{sgn}(\lambda - u_{2,0}^{\Gamma})
\end{aligned}$$

hold. Therefore,

$$\begin{aligned}
\dots &= \big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\text{sgn}(\lambda - u_{1,0}^{\Gamma}) + \big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\text{sgn}(\lambda - u_{2,0}^{\Gamma})\\
&+ \big|\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big|^2 - 2\big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\\
&- 2\big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\big(\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\\
&+ 2\big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\big(\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\\
&= \big(\chi(\lambda; u_{1,0}^{\tau,\Gamma}) - \chi(\lambda; u_{1,0}^{\Gamma})\big)\alpha(\boldsymbol{x},s,\lambda) + \big(\chi(\lambda; u_{2,0}^{\tau,\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big)\beta(\boldsymbol{x},s,\lambda)\\
&+ \big|\chi(\lambda; u_{1,0}^{\Gamma}) - \chi(\lambda; u_{2,0}^{\Gamma})\big|^2,
\end{aligned}$$

where

$$\alpha(\boldsymbol{x},s,\lambda) = \text{sgn}(\lambda - u_{1,0}^{\Gamma}(\boldsymbol{x},s)) - \chi(\lambda; u_{2,0}^{\tau,\Gamma}(\boldsymbol{x},s)) + 2\chi(\lambda; u_{1,0}^{\Gamma}(\boldsymbol{x},s)) - \chi(\lambda; u_{2,0}^{\Gamma}(\boldsymbol{x},s)),$$

$$\beta(\boldsymbol{x},s,\lambda) = \text{sgn}(\lambda - u_{2,0}^{\Gamma}(\boldsymbol{x},s)) - \chi(\lambda; u_{1,0}^{\tau,\Gamma}(\boldsymbol{x},s)) - \chi(\lambda; u_{1,0}^{\Gamma}(\boldsymbol{x},s)) + 2\chi(\lambda; u_{2,0}^{\Gamma}(\boldsymbol{x},s)).$$

Lemma 5. *Functions $\alpha(\boldsymbol{x},s,\lambda)$ and $\beta(\boldsymbol{x},s,\lambda)$ are equivalent to functions $\widetilde{\alpha}(\boldsymbol{x},s,\lambda)$ and $\widetilde{\beta}(\boldsymbol{x},s,\lambda)$, respectively:*

(39)
$$\widetilde{\alpha}(\boldsymbol{x},s,\lambda) = \begin{cases} 1 & \text{if} \quad \lambda > \max\big(u_{2,0}^{\Gamma}(\boldsymbol{x},s), u_{2,0}^{\tau,\Gamma}(\boldsymbol{x},s)\big),\\ 0 & \text{if} \quad \lambda \in [\min\big(u_{2,0}^{\Gamma}(\boldsymbol{x},s), u_{2,0}^{\tau,\Gamma}(\boldsymbol{x},s)\big), \max\big(u_{2,0}^{\Gamma}(\boldsymbol{x},s), u_{2,0}^{\tau,\Gamma}(\boldsymbol{x},s)\big)],\\ -1 & \text{if} \quad \lambda < \min\big(u_{2,0}^{\Gamma}(\boldsymbol{x},s), u_{2,0}^{\tau,\Gamma}(\boldsymbol{x},s)\big), \end{cases}$$

(40) $\widetilde{\beta}(\boldsymbol{x}, s, \lambda) =$

$$\begin{cases} 1 & \text{if} \quad \lambda > \max\left(u^{\Gamma}_{1,0}(\boldsymbol{x}, s), u^{\tau,\Gamma}_{1,0}(\boldsymbol{x}, s)\right), \\ 0 & \text{if} \quad \lambda \in [\min\left(u^{\Gamma}_{1,0}(\boldsymbol{x}, s), u^{\tau,\Gamma}_{1,0}(\boldsymbol{x}, s)\right), \max\left(u^{\Gamma}_{1,0}(\boldsymbol{x}, s), u^{\tau,\Gamma}_{1,0}(\boldsymbol{x}, s)\right)], \\ -1 & \text{if} \quad \lambda < \min\left(u^{\Gamma}_{1,0}(\boldsymbol{x}, s), u^{\tau,\Gamma}_{1,0}(\boldsymbol{x}, s)\right). \end{cases}$$

Proof. (1) If we assume that $0 < u^{\Gamma}_{1,0}(\boldsymbol{x}, s) < u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s)$, function $\alpha(\boldsymbol{x}, \lambda)$ is represented in the following way:

$$\alpha(\boldsymbol{x}, s, \lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\Gamma}_{1,0}(\boldsymbol{x}, s), \\ -2 & \text{if} \quad \lambda = u^{\Gamma}_{1,0}(\boldsymbol{x}, s), \\ -1 & \text{if} \quad u^{\Gamma}_{1,0}(\boldsymbol{x}, s) < \lambda < u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 0 & \text{if} \quad u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq \lambda \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 1 & \text{if} \quad \lambda > u^{\Gamma}_{2,0}(\boldsymbol{x}, s). \end{cases}$$

In the general case $u^{\Gamma}_{1,0}(\boldsymbol{x}, s) \leq u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s)$ function $\alpha(\boldsymbol{x}, s, \lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x}, s, \lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 0 & \text{if} \quad u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq \lambda \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 1 & \text{if} \quad \lambda > u^{\Gamma}_{2,0}(\boldsymbol{x}, s). \end{cases}$$

(2) When $u^{\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{1,0}(\boldsymbol{x}, s)$, function $\alpha(\boldsymbol{x}, s, \lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x}, s, \lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 0 & \text{if} \quad u^{\Gamma}_{2,0}(\boldsymbol{x}, s) \leq \lambda \leq u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 1 & \text{if} \quad \lambda > u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s). \end{cases}$$

(3) When $u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{1,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s)$, function $\alpha(\boldsymbol{x}, s, \lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x}, s, \lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 0 & \text{if} \quad u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq \lambda \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 1 & \text{if} \quad u^{\Gamma}_{2,0}(\boldsymbol{x}, s) < \lambda. \end{cases}$$

(4) When $u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{1,0}(\boldsymbol{x}, s)$, function $\alpha(\boldsymbol{x}, s, \lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x}, s, \lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 0 & \text{if} \quad u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) \leq \lambda \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 1 & \text{if} \quad u^{\Gamma}_{2,0}(\boldsymbol{x}, s) < \lambda. \end{cases}$$

(5) When $u^{\Gamma}_{1,0}(\boldsymbol{x}, s) \leq u^{\Gamma}_{2,0}(\boldsymbol{x}, s) \leq u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s)$, function $\alpha(\boldsymbol{x}, s, \lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x}, s, \lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 0 & \text{if} \quad u^{\Gamma}_{2,0}(\boldsymbol{x}, s) \leq \lambda \leq u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s), \\ 1 & \text{if} \quad u^{\tau,\Gamma}_{2,0}(\boldsymbol{x}, s) < \lambda. \end{cases}$$

(6) When $u^{\Gamma}_{2,0}(\boldsymbol{x},s) \le u^{\Gamma}_{1,0}(\boldsymbol{x},s) \le u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s)$, function $\alpha(\boldsymbol{x},s,\lambda)$ is equivalent to function

$$\widetilde{\alpha}(\boldsymbol{x},s,\lambda) = \begin{cases} -1 & \text{if} \quad \lambda < u^{\Gamma}_{2,0}(\boldsymbol{x},s), \\ \ \ 0 & \text{if} \quad u^{\Gamma}_{2,0}(\boldsymbol{x},s) \le \lambda \le u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s), \\ \ \ 1 & \text{if} \quad u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s) < \lambda. \end{cases}$$

From these representations it follows that function $\alpha(\boldsymbol{x},s,\lambda)$ is equivalent to function $\tilde{\alpha}(\boldsymbol{x},s,\lambda)$. The same analysis can be provided for $\beta(\boldsymbol{x},s,\lambda)$. □

With the help of Lemma 5 we obtain

$$(41)\quad \int_\Omega\int_0^S\int_{-M}^{M} a'(\lambda)\big|\chi(\lambda;u^{\tau,\Gamma}_{1,0}(\boldsymbol{x},s)) - \chi(\lambda;u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s))\big|^2\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda =$$
$$\lim_{\epsilon\to 0+}\int_\Omega\int_0^S\Big(\int_{-M}^{u^{\Gamma}_{1,0}-\epsilon} + \int_{u^{\Gamma}_{1,0}+\epsilon}^{M}\Big)a'(\lambda)\big(\chi(\lambda;u^{\tau,\Gamma}_{1,0}(\boldsymbol{x},s)) - \chi(\lambda;u^{\Gamma}_{1,0}(\boldsymbol{x},s))\big)\times$$
$$\alpha(\boldsymbol{x},s,\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda$$
$$+\lim_{\epsilon\to 0+}\int_\Omega\int_0^S\Big(\int_{-M}^{u^{\Gamma}_{2,0}-\epsilon} + \int_{u^{\Gamma}_{2,0}+\epsilon}^{M}\Big)a'(\lambda)\big(\chi(\lambda;u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s)) - \chi(\lambda;u^{\Gamma}_{2,0}(\boldsymbol{x},s))\big)\times$$
$$\beta(\boldsymbol{x},s,\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda$$
$$+\int_\Omega\int_0^S\int_{-M}^{M} a'(\lambda)\big|\chi(\lambda;u^{\Gamma}_{1,0}(\boldsymbol{x},s)) - \chi(\lambda;u^{\Gamma}_{2,0}(\boldsymbol{x},s))\big|^2\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda.$$

Here the first and the second terms in the right-hand side as follows:

$$(42)\quad \lim_{\epsilon\to 0+}\int_\Omega\int_0^S\Big(\int_{-M}^{u^{\Gamma}_{1,0}-\epsilon} + \int_{u^{\Gamma}_{1,0}+\epsilon}^{M}\Big)a'(\lambda)\big(\chi(\lambda,u^{\tau,\Gamma}_{1,0}(\boldsymbol{x},s))$$
$$-\chi(\lambda;u^{\Gamma}_{1,0}(\boldsymbol{x},s))\big)\alpha(\boldsymbol{x},s,\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda =$$
$$\lim_{\epsilon\to 0+}\int_\Omega\int_0^S\Big(\int_{-M}^{u^{\Gamma}_{1,0}-\epsilon} + \int_{u^{\Gamma}_{1,0}+\epsilon}^{M}\Big)\partial_\lambda\mu^{\Gamma}_{1,0}(\boldsymbol{x},s,\lambda)\alpha(\boldsymbol{x},s,\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda =$$
$$-\int_\Omega\int_0^S(\mu^{\Gamma}_{1,0}(\boldsymbol{x},s,u^{\Gamma}_{2,0}(\boldsymbol{x},s)) + \mu^{\Gamma}_{1,0}(\boldsymbol{x},s,u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s)))\xi(\boldsymbol{x})\,d\boldsymbol{x}ds \le 0,$$

$$(43)\quad \lim_{\epsilon\to 0+}\int_\Omega\int_0^S\Big(\int_{-M}^{u^{\Gamma}_{2,0}-\epsilon} + \int_{u^{\Gamma}_{2,0}+\epsilon}^{M}\Big)a'(\lambda)\big(\chi(\lambda,u^{\tau,\Gamma}_{2,0}(\boldsymbol{x},s))$$
$$-\chi(\lambda;u^{\Gamma}_{2,0}(\boldsymbol{x},s))\big)\beta(\boldsymbol{x},s,\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda =$$
$$\lim_{\epsilon\to 0+}\int_\Omega\int_0^S\Big(\int_{-M}^{u^{\Gamma}_{2,0}-\epsilon} + \int_{u^{\Gamma}_{2,0}+\epsilon}^{M}\Big)\partial_\lambda\mu^{\Gamma}_{2,0}(\boldsymbol{x},s,\lambda)\beta(\boldsymbol{x},s,\lambda)\xi(\boldsymbol{x})\,d\boldsymbol{x}dsd\lambda =$$
$$-\int_\Omega\int_0^S(\mu^{\Gamma}_{2,0}(\boldsymbol{x},s,u^{\Gamma}_{1,0}(\boldsymbol{x},s)) + \mu^{\Gamma}_{2,0}(\boldsymbol{x},s,u^{\tau,\Gamma}_{1,0}(\boldsymbol{x},s)))\xi(\boldsymbol{x})\,d\boldsymbol{x}ds \le 0.$$

Therefore, (41), (42) and (43) complete the proof of (38a). Inequalities (38b), (38c) and (38d) can be proved in a similar way. □

Remark 6. *It is important to note that*

$$\int_{-M}^{M} \big|\chi(\lambda; u_1(\boldsymbol{x},t,s)) - \chi(\lambda; u_2(\boldsymbol{x},t,s))\big|^2 \, d\lambda$$
$$= \int_{-M}^{M} \big|\chi(\lambda; u_1(\boldsymbol{x},t,s)) - \chi(\lambda; u_2(\boldsymbol{x},t,s))\big| \, d\lambda = \big|u_1(\boldsymbol{x},t,s) - u_2(\boldsymbol{x},t,s)\big|,$$

$$\int_{-M}^{M} \varphi_i'(\lambda)\big|\chi(\lambda; u_1(\boldsymbol{x},t,s)) - \chi(\lambda; u_2(\boldsymbol{x},t,s))\big|^2 \, d\lambda =$$
$$\int_{-M}^{M} \varphi_i'(\lambda)\big|\chi(\lambda; u_1(\boldsymbol{x},t,s)) - \chi(\lambda; u_2(\boldsymbol{x},t,s))\big| \, d\lambda =$$
$$\operatorname{sgn}(\varphi_i(u_1(\boldsymbol{x},t,s)) - \varphi_i(u_2(\boldsymbol{x},t,s)))(\varphi_i(u_1(\boldsymbol{x},t,s)) - \varphi_i(u_2(\boldsymbol{x},t,s))), \quad i = 1,\dots,d,$$

for a.e. $(\boldsymbol{x},t,s) \in G_{T,S}$*;*

$$\int_{-M}^{M} a'(\lambda)\big|\chi(\lambda; u^{\Gamma}_{1,\nu}(\boldsymbol{x},s)) - \chi(\lambda; u^{\Gamma}_{2,\nu}(\boldsymbol{x},s))\big|^2 \, d\lambda =$$
$$\operatorname{sgn}(u^{\Gamma}_{1,\nu}(\boldsymbol{x},s) - u^{\Gamma}_{2,\nu}(\boldsymbol{x},s))\big(a(u^{\Gamma}_{1,\nu}(\boldsymbol{x},s)) - a(u^{\Gamma}_{2,\nu}(\boldsymbol{x},s))\big),$$

$\nu = 0, T$*, for a.e.* $(\boldsymbol{x},s) \in \Omega \times (0,S)$*;*

$$\int_{-M}^{M} b'(\lambda)\big|\chi(\lambda; u^{\Xi}_{1,\nu}(\boldsymbol{x},t)) - \chi(\lambda; u^{\Xi}_{2,\nu}(\boldsymbol{x},t))\big|^2 \, d\lambda =$$
$$\operatorname{sgn}(u^{\Xi}_{1,\nu}(\boldsymbol{x},t) - u^{\Xi}_{2,\nu}(\boldsymbol{x},t))\big(b(u^{\Xi}_{1,\nu}(\boldsymbol{x},t)) - b(u^{\Xi}_{2,\nu}(\boldsymbol{x},t))\big),$$

$\nu = 0, S$*, for a.e.* $(\boldsymbol{x},t) \in \Omega \times (0,T)$*.*

Corollary 1. *The following inequality holds*

$$(44) \quad - \int_{G_{T,S}} \sum_{i=1}^{d} \partial_{x_i}\xi(\boldsymbol{x}) \operatorname{sgn}(\varphi_i(u_1(\boldsymbol{x},t,s)) - \varphi_i(u_2(\boldsymbol{x},t,s)))\times$$
$$\big(\varphi_i(u_1(\boldsymbol{x},t,s)) - \varphi_i(u_2(\boldsymbol{x},t,s))\big)\, d\boldsymbol{x}dtds$$
$$- \int_{G_{T,S}} \Delta_x\xi(\boldsymbol{x})\big|u_1(\boldsymbol{x},t,s) - u_2(\boldsymbol{x},t,s)\big| \, d\boldsymbol{x}dtds \le$$
$$\mathcal{A}\int_{\Omega}\int_0^S \Big(\big|u^{\Gamma}_{1,0}(\boldsymbol{x},s) - u^{\Gamma}_{2,0}(\boldsymbol{x},s)\big| + \big|u^{\Gamma}_{1,T}(\boldsymbol{x},s) - u^{\Gamma}_{2,T}(\boldsymbol{x},s)\big|\Big)\xi(\boldsymbol{x})\, d\boldsymbol{x}ds$$
$$+ \mathcal{B}\int_{\Omega}\int_0^T \Big(\big|u^{\Xi}_{1,0}(\boldsymbol{x},t) - u^{\Xi}_{2,0}(\boldsymbol{x},t)\big| + \big|u^{\Xi}_{1,S}(\boldsymbol{x},t) - u^{\Xi}_{2,S}(\boldsymbol{x},t)\big|\Big)\xi(\boldsymbol{x})\, d\boldsymbol{x}dt$$

for any non-negative function $\xi \in C^2(\Omega)\cap C_0(\overline{\Omega})$, $\mathcal{A} = \sup_{|z|\le M}\big|a'(z)\big|$, $\mathcal{B} = \sup_{|z|\le M}\big|b'(z)\big|$.

Remark 7. *In Lemma 4 we assume that* $\xi \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$ $(\nabla_x\xi\big|_{\partial\Omega} = 0)$*. If we take* $\xi \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$ *in Corollary 1, we can rewrite the second term in the left-hand side in the following way:*

$$\int_{G_{T,S}} \nabla_x\xi(\boldsymbol{x}) \cdot \nabla_x\big|u_1(\boldsymbol{x},t,s) - u_2(\boldsymbol{x},t,s)\big| \, d\boldsymbol{x}dtds,$$

where u_1, $u_2 \in L^\infty(G_{T,S}) \cap L^2((0,T)\times(0,S); W_0^{1,2}(\Omega))$. *Then, we can choose the sequence* $\{\xi_l\}_{l\in\mathbb{N}} \subset C^2(\Omega)\cap C_0^1(\overline{\Omega})$ *with the limit* $\xi \in C^2(\Omega)\cap C_0(\overline{\Omega})$. *Therefore, we can assume that inequality* (39) *is valid for any* $\xi \in C^2(\Omega)\cap C_0(\overline{\Omega})$.

Let $\xi(\boldsymbol{x}) = \xi_P(\boldsymbol{x})$ in inequality (44), where ξ_P is the solution of Poisson's equation in Ω:

$$\Delta_x \xi_P(\boldsymbol{x}) = -1, \quad \xi_P(\boldsymbol{x})\big|_{\partial\Omega} = 0.$$

Finally, only for a small constant

$$C_{\varphi,\Omega} := \|\boldsymbol{\varphi}'\|_{C(-M,M)} \|\nabla_x \xi_P\|_{C(\Omega)} < 1 \tag{45}$$

it follows from inequality (44) that

$$(1 - C_{\varphi,\Omega}) \int_{G_{T,S}} \big|u_1(\boldsymbol{x},t,s) - u_2(\boldsymbol{x},t,s)\big|\, d\boldsymbol{x}dtds \leq \tag{46}$$

$$\max(\mathcal{A},\mathcal{B})\|\xi_P\|_{C(\Omega)} \Big(\int_\Omega \int_0^S \big(\big|u^\Gamma_{1,0}(\boldsymbol{x},s) - u^\Gamma_{2,0}(\boldsymbol{x},s)\big| + \big|u^\Gamma_{1,T}(\boldsymbol{x},s) - u^\Gamma_{2,T}(\boldsymbol{x},s)\big|\big)\, d\boldsymbol{x}ds$$

$$+ \int_\Omega \int_0^T \big(\big|u^\Xi_{1,0}(\boldsymbol{x},t) - u^\Xi_{2,0}(\boldsymbol{x},t)\big| + \big|u^\Xi_{1,S}(\boldsymbol{x},t) - u^\Xi_{2,S}(\boldsymbol{x},t)\big|\big)\, d\boldsymbol{x}dt\Big).$$

Inequality (46) completes the proof of Theorem 1.

Conclusion

In this paper we have proved the existence and uniqueness of entropy solutions to the Dirichlet problem for genuinely nonlinear forward-backward ultra-parabolic equations. Moreover, with the help of the kinetic formulation and scaling procedure we have proved the existence of traces $u_0^{\tau,\Gamma}$, $u_T^{\tau,\Gamma}$, $u_0^{\tau,\Xi}$ and $u_S^{\tau,\Xi}$ in the L^1 sense. The latter result is equivalent to the assertion that functions $f_0^{\tau,\Gamma}$, $f_T^{\tau,\Gamma}$, $f_0^{\tau,\Xi}$ and $f_S^{\tau,\Xi}$ are χ-functions; see Lemma 1, Definition 3 and subsection 3.3.

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