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ON $2m$-TH ORDER PARABOLIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS IN NON-RECTANGULAR DOMAINS

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Abstract. This article is devoted to the analysis of a one-space dimensional high-order parabolic equation, subject to mixed boundary conditions. The problem is set in a (possibly non-regular) non-rectangular domain and the right hand side term of the equation is taken in a Lebesgue space.

Keywords: high-order parabolic equations, anisotropic Sobolev spaces, mixed conditions, non-rectangular domains.

1. Introduction

Let Ω be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\right\},$$

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined on $[0,T]$, Lipschitz continuous on $[0,T]$, and such that

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0,$$

for every $t \in \left]0,T\right]$ and with the fundamental hypothesis $\varphi(0) = 0$. The lateral boundary of Ω is defined by

$$\Gamma_i = \left\{(t, \varphi_i(t)) \in \mathbb{R}^2 : 0 < t < T\right\},\ i = 1,\ 2.$$

We will then assume that

$$(-1)^{i+1} \varphi_i'(t) \geq 0 \quad \text{a.e. } t \in \left]0,T\right[,\ i = 1,\ 2, \tag{1}$$

$$\varphi_i'(t)\varphi^m(t) \quad \to \quad 0 \qquad \text{as } t \to 0,\ m \in \mathbb{N}^*,\ i = 1,\ 2, \tag{2}$$

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where $\mathbb{N}^*$ is the set of all nonzero natural numbers. In Ω, we consider the boundary value problems

$$\left\{\begin{array}{l} \partial_t u + (-1)^m \partial_x^{2m} u = f_1 \in L^2_\omega(\Omega), \\ \partial_x^k u\big|_{\Gamma_1} = 0, \, k = 0, 1, ..., m-1, \\ \partial_x^l u\big|_{\Gamma_2} = 0, \; l = m, m+1, ..., 2m-1, \end{array}\right. \tag{3}$$

and

$$\left\{\begin{array}{l} \partial_t v + (-1)^m \partial_x^{2m} v = f_2 \in L^2_\omega(\Omega), \\ \partial_x^k v\big|_{\Gamma_2} = 0, \, k = 0, 1, ..., m-1, \\ \partial_x^l v\big|_{\Gamma_1} = 0, \; l = m, m+1, ..., 2m-1, \end{array}\right. \tag{4}$$

where $L^2_\omega(\Omega)$ stands for the space of square-integrable functions on Ω with the measure $\omega \; dt \; dx$. Here the weight ω is a real-valued function defined on $[0, T]$, differentiable on $]0, T]$ and such that

$$\forall t \in [0, T] : \omega(t) > 0. \tag{5}$$

We also assume that

$$\omega \text{ is a decreasing function on }]0, T]. \tag{6}$$

Observe that in the case $m = 1$, Problems (3) and (4) correspond to second-order parabolic equations with Dirichlet-Neumann conditions and we can find studies of such kind of problems in [20] and [9] both in bounded and unbounded non-cylindrical domains. Note that the mixed type conditions

$$\partial_x^k u\big|_{\Gamma_1} = \partial_x^l u\big|_{\Gamma_2} = 0, \, \partial_x^k v\big|_{\Gamma_2} = \partial_x^l v\big|_{\Gamma_1} = 0, \; k = 0, 1, \ldots, m-1; \, l = m, \ldots, 2m-1,$$

can be found in the case $m = 2$ in [3], where the existence of multiple positive solutions for a nonlinear fourth-order two-point boundary value problem was proved. In the case $m = 3$ corresponding to a sixth-order problem, we can find such kind of boundary conditions in Dugundji [4] and in Shi *et al.* [21]. These specific boundary conditions are important for the originality of this work. Indeed, to our knowledge, results concerning higher-order parabolic equations on time-varying domains, subject to such kind of boundary conditions, have not appeared in the literature to date.

Another difficulty related to this kind of problems comes from the fact that the domain Ω considered here is nonstandard since it shrinks at $t = 0$ ($\varphi(0) = 0$), which prevents the domain Ω to be transformed into a regular domain without the appearance of some degenerate terms in the parabolic equations, see for example Sadallah [18]. On the other hand, the semi group generating the solution cannot be defined since the initial condition is defined on a measure zero set.

It is well known that there are two main approaches for the study of boundary value problems in such non-smooth domains. We can work directly in the non-regular domains and we obtain singular solutions (see, for example [13] and [19]), or we impose conditions on the non-regular domains to obtain regular solutions (see, for example [11] and [18]). It is the second approach that we follow in this work. So, under the above mentioned conditions on the functions of parametrization $\varphi_i, \; i = 1, \; 2$ and on the weight function ω, we will prove that Problem (3) (respectively, (4)) admits a unique solution with optimal regularity, that is a solution u (respectively,

v) belonging to the anisotropic weighted Sobolev space

$$\mathcal{H}_{\gamma,\omega}^{1,2m}(\Omega)=\left\{u\in\mathcal{H}_{\omega}^{1,2m}(\Omega):\begin{array}{l}\partial_x^k u\big|_{\Gamma_1}=\partial_x^l u\big|_{\Gamma_2}=0,\\ k=0,1,...,m-1;\ l=m,...,2m-1\end{array}\right\},$$

(respectively,

$$\mathcal{H}_{\delta,\omega}^{1,2m}(\Omega)=\left\{v\in\mathcal{H}_{\omega}^{1,2m}(\Omega):\begin{array}{l}\partial_x^k v\big|_{\Gamma_2}=\partial_x^l v\big|_{\Gamma_1}=0,\\ k=0,1,\ldots,m-1;\ l=m,\ldots,2m-1\end{array}\right\}),$$

with

$$\mathcal{H}_{\omega}^{1,2m}(\Omega)=\left\{w\in L_{\omega}^{2}(\Omega):\partial_t w,\partial_x^j w\in L_{\omega}^{2}(\Omega),j=1,2,\ldots,2m\right\}.$$

The space $\mathcal{H}_{\omega}^{1,2m}(\Omega)$ is equipped with the natural norm, that is

$$\|w\|_{\mathcal{H}_{\omega}^{1,2m}(\Omega)}=\left(\|\partial_t w\|_{L_{\omega}^{2}(\Omega)}^{2}+\sum_{j=0}^{2m}\left\|\partial_x^j w\right\|_{L_{\omega}^{2}(\Omega)}^{2}\right)^{1/2}.$$

Whereas second-order parabolic equations in non-smooth domains are well studied, the literature concerning higher-order parabolic problems in non-cylindrical domains does not seem to be very rich. The solvability of the first boundary-value problem for higher-order parabolic equations in non-cylindrical domains in Sobolev spaces was considered in Mikhailov [16] and [17] both in one-dimensional and multidimensional cases. The author considered a class of "backward"paraboloid for which the parabolic boundary lies below the characteristic plane $t=0$. In the case of Hölder spaces functional framework, in [1] and [5], we can find solvability results of boundary value problems for a 2m-th order parabolic equation for non-cylindrical domains (of the same kind but which can not include our domain) with a non-smooth (in t) “lateral” boundary. Further references on the analysis of high-order parabolic problems in non-cylindrical domains are: Galaktionov [6], Grimaldi Piro [7], Kheloufi [8], [10], [12] and Labbas *et al.* [14].

The organization of this paper is as follows. In Section 2, we begin by deriving some preliminary results we need to develop further arguments. In Section 3, first we prove that Problems (3) and (4) admit (unique) solutions in the case of truncated domains. Then, we approximate Ω by a sequence (Ω_n) of such domains and we establish (for T small enough) uniform estimates of the type

$$\|u_n\|_{\mathcal{H}_{\omega}^{1,2m}(\Omega_n)}\leq K\|f_1\|_{L_{\omega}^{2}(\Omega_n)},\quad \|v_n\|_{\mathcal{H}_{\omega}^{1,2m}(\Omega_n)}\leq K\|f_2\|_{L_{\omega}^{2}(\Omega_n)},$$

where u_n and v_n are the solutions of Problems (3) and (4), respectively, in Ω_n and K is a constant independent of n. These estimates will allow us to pass to the limit and we will prove a local in time result. Finally, by using a trace result, we show in Section 4 that the obtained local in time result can be extended to a global in time one.

2. Preliminaries

Definition 1. *The function $u(t,x)\in\mathcal{H}_{\omega}^{1,2m}(\Omega)$ (respectively, $v(t,x)\in\mathcal{H}_{\omega}^{1,2m}(\Omega)$) is called solution of the mixed boundary value problem (3) (respectively, (4)), if it satisfies the equation of Problem (3) (respectively, (4)) almost everywhere in Ω.*

2.1. **Uniqueness of solutions.**

Proposition 1. *Problem (3) (respectively, (4)) admits at most one solution.*

Proof. Let us consider $u \in \mathcal{H}^{1,2m}_{\gamma,\omega}(\Omega)$ (respectively, $v \in \mathcal{H}^{1,2m}_{\delta,\omega}(\Omega)$) a solution of Problem (3) (respectively, (4)) with a null right-hand side term. So,

$$\partial_t u + (-1)^m \partial_x^{2m} u = \partial_t v + (-1)^m \partial_x^{2m} v = 0 \text{ in } \Omega.$$

In addition, u and v fulfil the boundary conditions

$$\partial_x^k u\big|_{\Gamma_1} = \partial_x^l u\big|_{\Gamma_2} = 0;\ \partial_x^k v\big|_{\Gamma_2} = \partial_x^l v\big|_{\Gamma_1} = 0,\ k = 0, 1, ..., m-1,\ l = m, ..., 2m-1.$$

Using Green formula, we have

$$\int_\Omega \left[\left(\partial_t u + (-1)^m \partial_x^{2m} u\right) u + \left(\partial_t v + (-1)^m \partial_x^{2m} v\right) v\right] \omega(t)\, dt\ dx$$

$$= \int_{\partial\Omega} \sum_{k=0}^{m-1} (\partial_x^{2m-k-1} u . \partial_x^k u + \partial_x^{2m-k-1} v . \partial_x^k v)(-1)^{k+m} \nu_x \omega(t)\, d\sigma$$

$$+ \int_{\partial\Omega} \tfrac{1}{2}(|u|^2 + |v|^2)\nu_t \omega(t)\, d\sigma + \int_\Omega (|\partial_x^m u|^2 + |\partial_x^m v|^2)\omega(t)\, dtdx$$

$$- \int_\Omega \tfrac{1}{2}(|u|^2 + |v|^2)\omega'(t)\, dt\ dx,$$

where ν_t, ν_x are the components of the unit outward normal vector at $\partial\Omega$. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Ω where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly, the corresponding boundary integral

$$\int_{\varphi_1(T)}^{\varphi_2(T)} [\frac{1}{2}(|u|^2 + |v|^2]\omega(T)\, dx,$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1,\ 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + \left(\varphi_i'\right)^2(t)}},\ \nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1 + \left(\varphi_i'\right)^2(t)}},$$

and

$$\partial_x^k u(t, \varphi_1(t)) = \partial_x^l u(t, \varphi_2(t)) = \partial_x^k v(t, \varphi_2(t)) = \partial_x^l v(t, \varphi_1(t)) = 0,$$

$k = 0, 1, ..., m-1$; $l = m, m+1, ..., 2m-1$. Consequently, the corresponding boundary integral is

$$\int_0^T -\frac{\varphi_2'(t)}{2} u^2(t, \varphi_2(t))\, \omega(t)\, dt + \int_0^T \frac{\varphi_1'(t)}{2} v^2(t, \varphi_1(t))\, \omega(t)\, dt.$$

Then, we obtain

$$\int_\Omega [\left(\partial_t u + (-1)^m \partial_x^{2m} u\right) u + \left(\partial_t v + (-1)^m \partial_x^{2m} v\right) v]\ \omega(t)\, dt\ dx$$

$$= \int_0^T -\frac{\varphi_2'(t)}{2} u^2(t, \varphi_2(t))\, \omega(t)\, dt + \int_0^T \frac{\varphi_1'(t)}{2} v^2(t, \varphi_1(t))\, \omega(t)\, dt$$

$$+ \frac{1}{2} \int_{\varphi_1(T)}^{\varphi_2(T)} [u^2(T, x) + v^2(T, x)]\ \omega(T)\, dx + \int_\Omega [|\partial_x^m u|^2 + |\partial_x^m v|^2]\omega(t)\, dt\ dx$$

$$- \int_\Omega \tfrac{1}{2}[|u|^2 + |v|^2]\omega'(t)\, dt\ dx.$$

Consequently

$$\int_{\Omega}\left(\partial_t u+(-1)^m\partial_x^{2m}u\right) u\ \omega(t)\ dt\ dx=\int_{\Omega}\left(\partial_t v+(-1)^m\partial_x^{2m}v\right) v\ \omega(t)\ dt\ dx=0,$$

yields

$$\int_{\Omega}\left(|\partial_x^m u|^2\right)\omega(t)\,dt\ dx=\int_{\Omega}\left(|\partial_x^m v|^2\right)\omega(t)\,dt\ dx=0,$$

because

$$\int_{\varphi_1(T)}^{\varphi_2(T)}\frac{1}{2}[|u|^2+|v|^2]\omega(T)\,dx-\int_{\Omega}\frac{1}{2}[|u|^2+|v|^2]\omega'(t)\,dt\ dx\geq 0,$$

thanks to the conditions (5) and (6) and

$$\int_0^T-\frac{\varphi_2'(t)}{2}u^2(t,\varphi_2(t))\,\omega(t)\,dt+\int_0^T\frac{\varphi_1'(t)}{2}v^2(t,\varphi_1(t))\,\omega(t)\,dt\geq 0,$$

thanks to the condition (1). This implies that $|\partial_x^m u|^2=|\partial_x^m v|^2=0$ and consequently $\partial_x^{2m}u=\partial_x^{2m}v=0$. Then, the hypothesis $\partial_t u+(-1)^m\partial_x^{2m}u=\partial_t v+(-1)^m\partial_x^{2m}v=0$ gives $\partial_t u=\partial_t v=0$. Thus, $u=\sum_{k=0}^{m-1}a_k x^k,\ v=\sum_{k=0}^{m-1}b_k x^k,\ a_k,b_k\in\mathbb{R},\ k=0,1,...,m-1$. The boundary conditions imply that $u=v=0$ in Ω. This proves the uniqueness of the solutions of Problems (3) and (4). □

Remark 1. *In the sequel, we will be interested only by the question of the existence of solutions of Problems (3) and (4).*

2.2. **Technical Lemmas.**

Lemma 1. *There exists a positive constant K_1 such that for each $(u,v)\in H_\gamma^{2m}(0,1)\times H_\delta^{2m}(0,1)$*

$$\left\|u^{(j)}\right\|_{L^2(0,1)}\leq K_1\left\|u^{(2m)}\right\|_{L^2(0,1)}\ \text{and}\ \left\|v^{(j)}\right\|_{L^2(0,1)}\leq K_1\left\|v^{(2m)}\right\|_{L^2(0,1)},$$

$j=0,1,...,2m-1$, where

$$H_\gamma^{2m}(0,1)=\left\{u\in H^{2m}(0,1):\ \begin{array}{l}u^{(k)}(0)=u^{(l)}(1)=0,\\ k=0,1,...,m-1;\ l=m,...,2m-1\end{array}\right\},$$

and

$$H_\delta^{2m}(0,1)=\left\{v\in H^{2m}(0,1):\ \begin{array}{l}v^{(k)}(1)=v^{(l)}(0)=0,\\ k=0,1,...,m-1;\ l=m,...,2m-1\end{array}\right\}.$$

Here, $w^{(j)},\ j=1,2,...,2m$ is the derivative of order j of w on $(0,1)$ and $w^{(0)}=w$.

Proof. Let h_1,h_2 be arbitrary fixed elements of $L^2(0,1)$. Every solution of the ordinary differential equation $u^{(2m)}=h_1$, (respectively, $v^{(2m)}=h_2$,) is of the form

$$u(x)=\int_0^x\int_0^{x_{2m-1}}\int_0^{x_{2m-2}}...\int_0^{x_1}h_1(s)\,dsdx_1...dx_{2m-2}dx_{2m-1}+\sum_{j=0}^{2m-1}\frac{u^{(j)}(0)}{j!}x^j,$$

$x\in[0,1]$, (respectively,

$$v(x)=\int_0^x\int_0^{x_{2m-1}}\int_0^{x_{2m-2}}...\int_0^{x_1}h_2(s)\,dsdx_1....dx_{2m-2}dx_{2m-1}+\sum_{j=0}^{2m-1}\frac{v^{(j)}(0)}{j!}x^j,$$

$x\in[0,1]$). The variables $u^{(j)}(0)$, $j=0,1,...,2m-1$, (respectively, $v^{(j)}(0)$, $j=0,1,...,2m-1$,) are to be determined in a unique way such that the boundary conditions $u^{(k)}(0) = u^{(l)}(1) = 0$, (respectively, $v^{(k)}(1) = v^{(l)}(0) = 0$) $k = 0, 1, ..., m-1;\ l = m, m+1, ..., 2m-1$, are satisfied.

From the preceding representation of the solution and thus also its derivatives

$$u^{(2m-p)}(x) = \int_0^x \int_0^{x_{p-1}} \dots \int_0^{x_1} h_1(s)\, ds dx_1 \dots dx_{p-1} + \sum_{q=0}^{p-1} \frac{1}{q!} u^{(2m+q-p)}(0)\, x^q,$$

$x \in [0,1]$, (respectively,

$$v^{(2m-p)}(x) = \int_0^x \int_0^{x_{p-1}} \dots \int_0^{x_1} h_2(s)\, ds dx_1 \dots dx_{p-1} + \sum_{q=0}^{p-1} \frac{1}{q!} v^{(2m+q-p)}(0)\, x^q,$$

$x \in [0,1]$) for $p = 1, 2, \dots, 2m$, and from the required boundary conditions, we obtain the following system to be solved :

$$AX = b$$

with $A = (a_{ij})$ is an upper triangular matrix, $X = (X_i), b = (b_i)$ for $i, j = 0, 1, \dots, 2m-1$, where

$$a_{ij} = \begin{cases} 0 & \text{if } i < j;\ i, j = 1, \dots, m-1 \\ \frac{1}{(j-i)!} & \text{if } i < j;\ i = 0, \dots, 2m-2;\ j = m, \dots, 2m-1 \\ -1 & \text{if } i = j;\ i = 0, ..., m-1 \\ 1 & \text{if } i = j;\ i = m, ..., 2m-1 \\ 0 & \text{if } i > j, \end{cases}$$

$$X_i = \begin{cases} u^{(i)}(1) & \text{if } i = 0, \dots, m-1 \\ u^{(i)}(0) & \text{if } i = m, \dots, 2m-1 \end{cases}$$

and

$$b_i = -\int_0^1 \int_0^{x_{2m-i-1}} \dots \int_0^{x_1} h_1(s)\, ds dx_1 \dots dx_{2m-i-1},\ i = 0, \dots, 2m-1$$

(respectively,

$$BY = b'$$

with $B = (b_{ij})$ is a lower triangular matrix, $Y = (Y_i), b' = (b'_i)$ for $i, j = 0, 1, \dots, 2m-1$, where

$$b_{ij} = \begin{cases} 0 & \text{if } i < j \\ -1 & \text{if } i = j;\ i = 0, \dots, m-1 \\ 1 & \text{if } i = j;\ i = m, \dots, 2m-1 \\ 0 & \text{if } i > j;\ i = 1, \dots, m-1;\ j = 0, \dots, m-1 \\ \frac{1}{(i-j)!} & \text{if } i > j;\ i = m+1, \dots, 2m-1;\ j = m, \dots, 2m-2, \end{cases}$$

$$Y_i = \begin{cases} v^{(2m-i-1)}(1) & \text{if } i = 0, \dots, m-1 \\ v^{(2m-i-1)}(0) & \text{if } i = m, \dots, 2m-1 \end{cases}$$

and

$$b'_i = -\int_0^1 \int_0^{x_i} \dots \int_0^{x_1} h_2(s)\, ds dx_1 \dots dx_i,\ i = 0, \dots, 2m-1).$$

Finally, the unique solution of the problem

$$\begin{cases} u^{(2m)} = h_1, \\ u^{(k)}(0) = 0,\ k = 0, 1, \dots, m-1, \\ u^{(l)}(1) = 0,\ l = m, \dots, 2m-1, \end{cases}$$

(respectively,

$$\begin{cases} v^{(2m)} = h_2, \\ v^{(k)}(1) = 0, \ k = 0, 1, \ldots, m-1, \\ v^{(l)}(0) = 0, \ l = m, \ldots, 2m-1,) \end{cases}$$

is given by

$$u(x) = \int_0^x \int_0^{x_{2m-1}} \int_0^{x_{2m-2}} \ldots \int_0^{x_1} h_1(s)\, ds dx_1 \ldots dx_{2m-2} dx_{2m-1} + \sum_{j=m}^{2m-1} \frac{u^{(j)}(0)}{j!} x^j,$$

$x \in [0,1]$, (respectively,

$$v(x) = \int_0^x \int_0^{x_{2m-1}} \int_0^{x_{2m-2}} \ldots \int_0^{x_1} h_2(s)\, ds dx_1 \ldots dx_{2m-2} dx_{2m-1} + \sum_{j=0}^{m-1} \frac{v^{(j)}(0)}{j!} x^j,$$

$x \in [0,1]$,) where

$$u^{(2m-p)}(0) = \sum_{k=0}^{p-1} (-1)^{k+1} \int_0^1 \int_0^{x_{p-k-1}} \cdots \int_0^{x_1} h_1(s)\, ds dx_1 \ldots dx_{p-k-1},$$

for $p = 1, \ldots, m$, (respectively,

$$v^{(2m-p)}(0) = \sum_{k=0}^{p-1} (-1)^{k+1} \int_0^1 \int_0^{x_{p-k-1}} \cdots \int_0^{x_1} h_2(s)\, ds dx_1 \ldots dx_{p-k-1},$$

for $p = m+1, \ldots, 2m$). Using the Cauchy-Schwarz inequality, we obtain the following estimates

$$\begin{array}{l} \left| u^{(k)}(0) \right| \leq C \left\| h_1 \right\|_{L^2(0,1)}, \ k = m, m+1, \ldots, 2m-1 \\ \left| u^{(l)}(1) \right| \leq C \left\| h_1 \right\|_{L^2(0,1)}, \ l = 0, 1, \ldots, m-1, \end{array}$$

(respectively,

$$\begin{array}{l} \left| v^{(k)}(1) \right| \leq C \left\| h_2 \right\|_{L^2(0,1)}, \ k = m, m+1, \ldots, 2m-1 \\ \left| v^{(l)}(0) \right| \leq C \left\| h_2 \right\|_{L^2(0,1)}, \ l = 0, 1, \ldots, m-1,) \end{array}$$

where C is a positive constant, which will allow us to obtain the desired estimates. □

Lemma 2. *There exists a positive constant K_2 (independent of a and b) such that for each $(u, v) \in H_\gamma^{2m}(a,b) \times H_\delta^{2m}(a,b)$*

$$\left\| u^{(k)} \right\|_{L^2(a,b)}^2 \leq K_2 (b-a)^{2(2m-k)} \left\| u^{(2m)} \right\|_{L^2(a,b)}^2, \ k = 0, \ldots, 2m-1$$

$$\left\| v^{(k)} \right\|_{L^2(a,b)}^2 \leq K_2 (b-a)^{2(2m-k)} \left\| v^{(2m)} \right\|_{L^2(a,b)}^2, \ k = 0, \ldots, 2m-1$$

where,

$$H_\gamma^{2m}(a,b) = \left\{ u \in H^{2m}(a,b) : \begin{array}{l} u^{(k)}(a) = u^{(l)}(b) = 0, \\ k = 0, \ldots, m-1; \ l = m, \ldots, 2m-1 \end{array} \right\},$$

$$H_\delta^{2m}(a,b) = \left\{ v \in H^{2m}(a,b) : \begin{array}{l} v^{(k)}(b) = v^{(l)}(a) = 0, \\ k = 0, \ldots, m-1; \ l = m, \ldots, 2m-1 \end{array} \right\}.$$

Proof. It is a direct consequence of Lemma 1 by using the following affine change of variable

$$[0,1] \longrightarrow [a,b], \ x \mapsto (1-x)\,a + xb = y.$$

Indeed, we set $w(x) = u(y)$. then if $w \in H_\gamma^{2m}(0,1)$, u belongs to $H_\gamma^{2m}(a,b)$. We have

$$\begin{array}{rcl} \left\|w^{(k)}\right\|^2_{L^2(0,1)} & = & \int_0^1 \left(w^{(k)}\right)^2(x)\,dx \\ & = & \int_a^b \left(u^{(k)}\right)^2(y)\,(b-a)^{2k}\frac{dy}{b-a} \\ & = & \int_a^b \left(u^{(k)}\right)^2(y)\,(b-a)^{2k-1}dy \\ & = & (b-a)^{2k-1}\left\|u^{(k)}\right\|^2_{L^2(a,b)} \end{array}$$

where $k \in \{0,1,...,2m-1\}$. On the other hand, we have

$$\begin{array}{rcl} \left\|w^{(2m)}\right\|^2_{L^2(0,1)} & = & \int_0^1 \left(w^{(2m)}\right)^2(x)\,dx \\ & = & \int_a^b \left(u^{(2m)}\right)^2(y)\,(b-a)^{4m-1}dy \\ & = & (b-a)^{4m-1}\left\|u^{(2m)}\right\|^2_{L^2(a,b)}. \end{array}$$

Using the inequality

$$\left\|w^{(k)}\right\|^2_{L^2(0,1)} \ \leq \ K_2\left\|w^{(2m)}\right\|^2_{L^2(0,1)}$$

of Lemma 1, we obtain the desired inequality

$$\left\|u^{(k)}\right\|^2_{L^2(a,b)} \ \leq \ K_2\,(b-a)^{2(2m-k)}\left\|u^{(2m)}\right\|^2_{L^2(a,b)}, \ k = 0,1,\ldots,2m-1,$$

with $K_2 = K_1^2$. The inequality,

$$\left\|v^{(k)}\right\|^2_{L^2(a,b)} \leq K_2\,(b-a)^{2(2m-k)}\left\|v^{(2m)}\right\|^2_{L^2(a,b)}, \ k = 0,\ldots,2m-1$$

can be proved by a similar argument. □

Remark 2. *In Lemmas 1 and 2 we can replace* $\|.\|_{L^2}$ *by* $\|.\|_{L^2_\omega}$.

3. Local in time result

3.1. Case of a truncated domain Ω_n. In this subsection, we replace Ω by Ω_n, $n \in \mathbb{N}^*$ and $\frac{1}{n} < T$:

$$\Omega_n = \left\{(t,x) \in \Omega : \frac{1}{n} < t < T\right\}.$$

Theorem 1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the problem*

$$(7) \quad \left\{ \begin{array}{l} \partial_t u_n + (-1)^m \partial_x^{2m} u_n = f_{1,n} \in L^2_\omega(\Omega_n), \\ u_n|_{t=\frac{1}{n}} = 0, \\ \partial_x^k u_n\big|_{\Gamma_{1,n}} = \partial_x^l u_n\big|_{\Gamma_{2,n}} = 0, \ k = 0,\ldots,m-1; \ l = m,\ldots,2m-1, \end{array}\right.$$

(respectively,

$$(8) \quad \left\{ \begin{array}{l} \partial_t v_n + (-1)^m \partial_x^{2m} v_n = f_{2,n} \in L^2_\omega(\Omega_n), \\ v_n|_{t=\frac{1}{n}} = 0, \\ \partial_x^k v_n\big|_{\Gamma_{2,n}} = \partial_x^l v_n\big|_{\Gamma_{1,n}} = 0, \ k = 0,\ldots,m-1; \ l = m,\ldots,2m-1,) \end{array}\right.$$

admits a unique solution $u_n \in \mathcal{H}^{1,2m}_\omega(\Omega_n)$ (respectively, $v_n \in \mathcal{H}^{1,2m}_\omega(\Omega_n)$). Here,

$$f_{i,n} = f_i|_{\Omega_n} \ \textit{and} \ \Gamma_{i,n} = \{(t,\varphi_i(t)) \in \mathbb{R}^2 : \frac{1}{n} < t < T\}, \ i = 1,2.$$

Proof. The uniqueness of the solution is easy to check, thanks to the boundary conditions. Let us prove the existence. The change of variables

$$\Phi : (t,x) \mapsto (t,y) = \left(t, \frac{x-\varphi_1(t)}{\varphi(t)}\right),$$

transforms Ω_n into the rectangle $R_n = \left]\frac{1}{n}, T\right[\times]0,1[$. Putting

$$u_n(t,x) = w_{1,n}(t,y),\ v_n(t,x) = w_{2,n}(t,y) \text{ and } f_{i,n}(t,x) = g_{i,n}(t,y),\ i = 1,2,$$

then Problems (7) and (8) become

$$\begin{cases} \partial_t w_{1,n} + a(t,y)\,\partial_y w_{1,n} + c(t)\,\partial_y^{2m} w_{1,n} = g_{1,n}, \\ w_{1,n}|_{t=\frac{1}{n}} = 0, \\ \partial_y^k w_{1,n}\big|_{y=0} = \partial_y^l w_{1,n}\big|_{y=1} = 0,\ k = 0,\dots,m-1;\ l = m,\dots,2m-1, \end{cases}$$

and

$$\begin{cases} \partial_t w_{2,n} + a(t,y)\,\partial_y w_{2,n} + c(t)\,\partial_y^{2m} w_{2,n} = g_{2,n}, \\ w_{2,n}|_{t=\frac{1}{n}} = 0, \\ \partial_y^k w_{2,n}\big|_{y=1} = \partial_y^l w_{2,n}\big|_{y=0} = 0,\ k = 0,\dots,m-1;\ l = m,\dots,2m-1, \end{cases}$$

where $a(t,y) = -\dfrac{y\varphi'(t) + \varphi_1'(t)}{\varphi(t)}$ and $c(t) = \dfrac{(-1)^m}{\varphi^{2m}(t)}$. The above change of variables conserves the spaces L^2_ω and $\mathcal{H}^{1,2m}_\omega$ because the functions a and c are bounded when $t \in \left]\frac{1}{n}, T\right[$. In other words

$$f_{i,n} \in L^2_\omega(\Omega_n) \Longleftrightarrow g_{i,n} \in L^2_\omega(R_n),\ i = 1,\ 2,$$

and

$$u_n, v_n \in \mathcal{H}^{1,2m}_\omega(\Omega_n) \Longleftrightarrow w_{1,n}, w_{2,n} \in \mathcal{H}^{1,2m}_\omega(R_n).$$

Lemma 3. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the following operator is compact*

$$\mathcal{H}^{1,2m}_{\gamma,\omega}(R_n) \longrightarrow L^2_\omega(R_n),\ w_{1,n} \mapsto a(t,y)\,\partial_y w_{1,n},$$

(respectively, $\mathcal{H}^{1,2m}_{\delta,\omega}(R_n) \longrightarrow L^2_\omega(R_n),\ w_{2,n} \mapsto a(t,y)\,\partial_y w_{2,n}$), where

$$\mathcal{H}^{1,2m}_{\gamma,\omega}(R_n) = \left\{ w_{1,n} \in \mathcal{H}^{1,2m}_\omega(R_n) : \begin{array}{l} w_{1,n}|_{t=\frac{1}{n}} = \partial_y^k w_{1,n}\big|_{y=0} = \partial_y^l w_{1,n}\big|_{y=1} = 0, \\ k = 0,\dots,m-1; l = m,\dots,2m-1 \end{array} \right\}$$

(respectively,

$$\left.\mathcal{H}^{1,2m}_{\delta,\omega}(R_n) = \left\{ w_{2,n} \in \mathcal{H}^{1,2m}_\omega(R_n) : \begin{array}{l} w_{2,n}|_{t=\frac{1}{n}} = \partial_y^k w_{2,n}\big|_{y=1} = \partial_y^l w_{2,n}\big|_{y=0} = 0, \\ k = 0,\dots,m-1; l = m,\dots,2m-1 \end{array} \right\}\right).$$

Proof. R_n has the "horn property"of Besov [2], so

$$\partial_y : \mathcal{H}^{1,2m}_{\gamma,\omega}(R_n) \longrightarrow \mathcal{H}^{1-\frac{1}{2m},2m-1}_\omega(R_n),\ w_{1,n} \mapsto \partial_y w_{1,n},$$

(respectively,

$$\partial_y : \mathcal{H}^{1,2m}_{\delta,\omega}(R_n) \longrightarrow \mathcal{H}^{1-\frac{1}{2m},2m-1}_\omega(R_n),\ w_{2,n} \mapsto \partial_y w_{2,n}),$$

is continuous. Since R_n is bounded, the canonical injection is compact from $\mathcal{H}^{1-\frac{1}{2m},2m-1}_\omega(R_n)$ into $L^2_\omega(R_n)$, (see for instance [2]), where

$$\mathcal{H}^{1-\frac{1}{2m},2m-1}_\omega(R_n) = L^2_\omega\left(\frac{1}{n}, T; H^{2m-1}_\omega\,]0,1[\right) \cap H^{1-\frac{1}{2m}}_\omega\left(\frac{1}{n}, T; L^2_\omega\,]0,1[\right).$$

For the complete definitions of the $\mathcal{H}^{r,s}$ Hilbertian Sobolev spaces, see for instance [15]. Then, ∂_y is a compact operator from $\mathcal{H}^{1,2m}_{\gamma,\omega}(R_n)$ into $L^2_\omega(R_n)$, (respectively, from $\mathcal{H}^{1,2m}_{\delta,\omega}(R_n)$ into $L^2_\omega(R_n)$). Furthermore, since $a(.,.)$ is a bounded function, the operator $a\partial_y$ is then compact from $\mathcal{H}^{1,2m}_{\gamma,\omega}(R_n)$ into $L^2_\omega(R_n)$, (respectively, from $\mathcal{H}^{1,2m}_{\delta,\omega}(R_n)$ into $L^2_\omega(R_n)$). □

So, it is sufficient to show that the operator $\partial_t + \dfrac{(-1)^m}{\varphi^{2m}}\partial_y^{2m}$ is an isomorphism from $\mathcal{H}^{1,2m}_{\gamma,\omega}(R_n)$ into $L^2_\omega(R_n)$, (respectively, from $\mathcal{H}^{1,2m}_{\delta,\omega}(R_n)$ into $L^2_\omega(R_n)$). A simple change of variable $t = h(s)$ with $h'(s) = \varphi^{2m}(t)$, transforms the problems

$$\begin{cases} \partial_t w_{1,n} + \dfrac{(-1)^m}{\varphi^{2m}(t)}\partial_y^{2m} w_{1,n} = g_{1,n}, \\ w_{1,n}\big|_{t=\frac{1}{n}} = 0, \\ \partial_y^k w_{1,n}\big|_{y=0} = \partial_y^l w_{1,n}\big|_{y=1} = 0,\ k = 0,\dots,m-1;\ l = m,\dots,2m-1, \end{cases}$$

and

$$\begin{cases} \partial_t w_{2,n} + \dfrac{(-1)^m}{\varphi^{2m}(t)}\partial_y^{2m} w_{2,n} = g_{2,n}, \\ w_{2,n}\big|_{t=\frac{1}{n}} = 0, \\ \partial_y^k w_{2,n}\big|_{y=1} = \partial_y^l w_{2,n}\big|_{y=0} = 0,\ k = 0,\dots,m-1;\ l = m\dots,2m-1, \end{cases}$$

into the following problems

$$\begin{cases} \partial_s w_n^1(s,y) + (-1)^m\partial_y^{2m} w_n^1(s,y) = \xi_n^1(s,y), \\ w_n^1\big|_{s=h^{-1}(\frac{1}{n})} = 0, \\ \partial_y^k w_n^1\big|_{y=0} = \partial_y^l w_n^1\big|_{y=1} = 0,\ k = 0,\dots,m-1;\ l = m,\dots,2m-1, \end{cases} \tag{9}$$

and

$$\begin{cases} \partial_s w_n^2(s,y) + (-1)^m\partial_y^{2m} w_n^2(s,y) = \xi_n^2(s,y), \\ w_n^2\big|_{s=h^{-1}(\frac{1}{n})} = 0, \\ \partial_y^k w_n^2\big|_{y=1} = \partial_y^l w_n^2\big|_{y=0} = 0,\ k = 0,\dots,m-1;\ l = m,\dots,2m-1, \end{cases} \tag{10}$$

with

$$\xi_n^i(s,y) = \frac{g_{i,n}(t,y)}{h'(s)} \text{ and } w_n^i(s,y) = w_{i,n}(t,y),\ i = 1,2.$$

Note that this change of variables preserves the spaces L^2_ω and $\mathcal{H}^{1,2m}_\omega$. It follows that there exists a unique $w_n^1 \in \mathcal{H}^{1,2m}_\omega$ (respectively, $w_n^2 \in \mathcal{H}^{1,2m}_\omega$) solution of the problem (9) (respectively, (10)). This implies that Problem (7) (respectively, (8)) admits a unique solution $u_n \in \mathcal{H}^{1,2m}_\omega(\Omega_n)$ (respectively, $v_n \in \mathcal{H}^{1,2m}_\omega(\Omega_n)$). We obtain the functions u_n and v_n by setting

$$u_n(t,x) = w_{1,n}(t,y) = w_n^1\left(h^{-1}(t),y\right) \text{ and } v_n(t,x) = w_{2,n}(t,y) = w_n^2(h^{-1}(t),y).$$

□

We shall need the following result in order to justify the calculus of the next section.

Lemma 4. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space*

$$\left\{ u_n \in D\left([\frac{1}{n},T]; H^{2m}_\omega(0,1)\right) : \begin{array}{c} u_n\big|_{t=\frac{1}{n}} = \partial_y^k u_n\big|_{y=0} = \partial_y^l u_n\big|_{y=1} = 0, \\ k = 0,\dots,m-1;\ l = m,\dots,2m-1 \end{array} \right\},$$

(respectively,

$$\left\{ v_n \in D\left([\frac{1}{n},T];H^{2m}_{\omega}(0,1)\right) : \begin{array}{l} v_n|_{t=\frac{1}{n}} = \partial_y^k v_n\big|_{y=1} = \partial_y^l v_n\big|_{y=0} = 0, \\ k=0,\dots,m-1;\ l=m,\dots,2m-1 \end{array} \right\}),$$

is dense in

$$\left\{ u_n \in \mathcal{H}^{1,2m}_{\omega}\left(\left]\frac{1}{n},T\right[\times\left]0,1\right[\right) : \begin{array}{l} u_n|_{t=\frac{1}{n}} = \partial_y^k u_n\big|_{y=0} = \partial_y^l u_n\big|_{y=1} = 0, \\ k=0,\dots,m-1;\ l=m,\dots,2m-1 \end{array} \right\},$$

(respectively,

$$\left\{ v_n \in \mathcal{H}^{1,2m}_{\omega}\left(\left]\frac{1}{n},T\right[\times\left]0,1\right[\right) : \begin{array}{l} v_n|_{t=\frac{1}{n}} = \partial_y^k v_n\big|_{y=1} = \partial_y^l v_n\big|_{y=0} = 0, \\ k=0,\dots,m-1;\ l=m,\dots,2m-1 \end{array} \right\}).$$

It is a particular case of Theorem 2.1 [15].

Remark 3. *We can replace in Lemma 4* $R_n = \left]\frac{1}{n},T\right[\times\left]0,1\right[$ *by* Ω_n *with the help of the change of variables defined above.*

3.2. **Case of a triangular domain.** Now, we return to the non-rectangular domain Ω and we suppose that the function φ_1 (respectively, φ_2) satisfies conditions (1) and (2) in the case of Problem (3) (respectively, (4)).
For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we denote $f_{i,n} = f_i|_{\Omega_n} \in L^2_{\omega}(\Omega_n)$, $i=1,2$ and $u_n \in \mathcal{H}^{1,2m}_{\gamma,\omega}(\Omega_n)$ (respectively, $v_n \in \mathcal{H}^{1,2m}_{\delta,\omega}(\Omega_n)$) the solution of Problem (3) (respectively, (4)) in Ω_n. Such solutions exist by Theorem 1.

Theorem 2. *For T small enough, there exists a constant $C>0$ independent of n such that*

$$\|u_n\|^2_{\mathcal{H}^{1,2m}_{\omega}(\Omega_n)} \le C\|f_{1,n}\|^2_{L^2_{\omega}(\Omega_n)} \le C\|f_1\|^2_{L^2_{\omega}(\Omega)},$$
$$\|v_n\|^2_{\mathcal{H}^{1,2m}_{\omega}(\Omega_n)} \le C\|f_{2,n}\|^2_{L^2_{\omega}(\Omega_n)} \le C\|f_2\|^2_{L^2_{\omega}(\Omega)}.$$

In order to prove Theorem 2, we need some preliminary results.

Lemma 5. *For every $\epsilon>0$ satisfying $(\varphi_2(t)-\varphi_1(t)) \le \epsilon$, there exists a constant $C_1>0$ independent of n, such that*

$$\left\|\partial_x^j u_n\right\|^2_{L^2_{\omega}(\Omega_n)} \le C_1\epsilon^{2(2m-j)}\left\|\partial_x^{2m} u_n\right\|^2_{L^2_{\omega}(\Omega_n)}, j=0,1,\dots,2m-1$$
$$\left\|\partial_x^j v_n\right\|^2_{L^2_{\omega}(\Omega_n)} \le C_1\epsilon^{2(2m-j)}\left\|\partial_x^{2m} v_n\right\|^2_{L^2_{\omega}(\Omega_n)}, j=0,1,\dots,2m-1.$$

Proof. Replacing in Lemma 2 u by u_n, v by v_n and $]a,b[$ by $]\varphi_1(t),\varphi_2(t)[$, for a fixed t, we obtain for $j=0,1,\dots,2m-1$

$$\begin{array}{rcl} \int_{\varphi_1(t)}^{\varphi_2(t)}\left(\partial_x^j u_n\right)^2 dx & \le & C_1(\varphi_2(t)-\varphi_1(t))^{2(2m-j)}\int_{\varphi_1(t)}^{\varphi_2(t)}\left(\partial_x^{2m}u_n\right)^2 dx \\ & \le & C_1\epsilon^{2(2m-j)}\int_{\varphi_1(t)}^{\varphi_2(t)}\left(\partial_x^{2m}u_n\right)^2 dx, \end{array}$$

and

$$\begin{array}{rcl} \int_{\varphi_1(t)}^{\varphi_2(t)}\left(\partial_x^j v_n\right)^2 dx & \le & C_1(\varphi_2(t)-\varphi_1(t))^{2(2m-j)}\int_{\varphi_1(t)}^{\varphi_2(t)}\left(\partial_x^{2m}v_n\right)^2 dx \\ & \le & C_1\epsilon^{2(2m-j)}\int_{\varphi_1(t)}^{\varphi_2(t)}\left(\partial_x^{2m}v_n\right)^2 dx, \end{array}$$

where C_1 is the constant of Lemma 2. Multiplying the previous inequality by $\omega(t)$ (which is positive) and integrating with respect to t, we obtain the desired estimates. $\square$

Proof of Theorem 2 : Let us denote the inner product in $L^2_\omega(\Omega_n)$ by $\langle .,. \rangle$, and set $L = \partial_t + (-1)^m \partial_x^{2m}$, then we have

$$\|f_{1,n}\|^2_{L^2_\omega(\Omega_n)} = \langle Lu_n, Lu_n \rangle = \|\partial_t u_n\|^2_{L^2_\omega(\Omega_n)} + \left\|\partial_x^{2m} u_n\right\|^2_{L^2_\omega(\Omega_n)} + 2\langle \partial_t u_n, (-1)^m \partial_x^{2m} u_n \rangle,$$

and

$$\|f_{2,n}\|^2_{L^2_\omega(\Omega_n)} = \langle Lv_n, Lv_n \rangle = \|\partial_t v_n\|^2_{L^2_\omega(\Omega_n)} + \left\|\partial_x^{2m} v_n\right\|^2_{L^2_\omega(\Omega_n)} + 2\langle \partial_t v_n, (-1)^m \partial_x^{2m} v_n \rangle.$$

Estimation of $2\langle \partial_t u_n, (-1)^m \partial_x^{2m} u_n \rangle$: We have

$$(-1)^m \partial_t u_n . \partial_x^{2m} u_n = \sum_{k=0}^{m-1} \partial_x \left(\partial_x^k \partial_t u_n . \partial_x^{2m-k-1} u_n \right) (-1)^{m+k} + \tfrac{1}{2} \partial_t \left(\partial_x^m u_n \right)^2 .$$

Then

$$\begin{aligned} 2\langle \partial_t u_n, (-1)^m \partial_x^{2m} u_n \rangle &= 2 \int_{\Omega_n} \sum_{k=0}^{m-1} \partial_x \left(\partial_x^k \partial_t u_n . \partial_x^{2m-k-1} u_n \right) (-1)^{m+k} \omega(t) dt dx \\ &\quad + \int_{\Omega_n} \partial_t \left(\partial_x^m u_n \right)^2 \omega(t) dt dx \\ &= 2 \int_{\partial\Omega_n} \sum_{k=0}^{m-1} \left(\partial_x^k \partial_t u_n . \partial_x^{2m-k-1} u_n \right) (-1)^{m+k} \nu_x \omega(t) d\sigma \\ &\quad + \int_{\partial\Omega_n} \left(\partial_x^m u_n \right)^2 \nu_t \omega(t) d\sigma - \int_{\Omega_n} \left(\partial_x^m u_n \right)^2 \omega'(t) dt dx \end{aligned}$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of Ω_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Ω_n where $t = \frac{1}{n}$, we have $u_n = 0$ and consequently $\partial_x^k u_n = 0, k = 0, \ldots, 2m-1$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\varphi_1(T)}^{\varphi_2(T)} \left(\partial_x^m u_n \right)^2 (T, x) \, \omega(T) \, dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i(t)$, $i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + (\varphi_i')^2 (t)}}, \; \nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1 + (\varphi_i')^2 (t)}},$$

and

$$\partial_x^k u_n (t, \varphi_1(t)) = \partial_x^l u_n (t, \varphi_2(t)) = 0, \; k = 0, \ldots, m-1; \; l = m, \ldots, 2m-1.$$

Differentiating $\partial_x^k u_n (t, \varphi_1(t))$, $k = 0, \ldots, m-1$ with respect to t, we obtain

$$\partial_t \partial_x^k u_n (t, \varphi_1(t)) = -\varphi_1'(t) \, \partial_x^{k+1} u_n (t, \varphi_1(t)) .$$

The boundary conditions on $\Gamma_{1,n}$ lead to

$$\partial_t \partial_x^k u_n (t, \varphi_1(t)) = \begin{cases} 0 & \text{if } k = 0, \ldots, m-2 \\ -\varphi_1'(t) \, \partial_x^m u_n (t, \varphi_1(t)) & \text{if } k = m-1. \end{cases}$$

Consequently, the corresponding integral is

$$I_{n,1} = -\int_{\frac{1}{n}}^{T} \varphi_1'(t) \left[\partial_x^m u_n (t, \varphi_1(t)) \right]^2 \omega(t) dt.$$

Then, we have

$$2\langle \partial_t u_n, (-1)^m \partial_x^{2m} u_n \rangle \geq -|I_{n,1}| . \tag{11}$$

Estimation of $2\langle \partial_t v_n, (-1)^m \partial_x^{2m} v_n \rangle$: We have

$$(-1)^m \partial_t v_n . \partial_x^{2m} v_n = \sum_{k=0}^{m-1} \partial_x \left(\partial_x^k \partial_t v_n . \partial_x^{2m-k-1} v_n \right) (-1)^{m+k} + \frac{1}{2} \partial_t \left(\partial_x^m v_n \right)^2 .$$

Then

$$\begin{aligned} 2\langle \partial_t v_n, (-1)^m \partial_x^{2m} v_n \rangle &= 2 \int_{\Omega_n} \sum_{k=0}^{m-1} \partial_x \left(\partial_x^k \partial_t v_n . \partial_x^{2m-k-1} v_n \right) (-1)^{m+k} \omega(t) dtdx \\ &\quad + \int_{\Omega_n} \partial_t \left(\partial_x^m v_n \right)^2 \omega(t) dtdx \\ &= 2 \int_{\partial\Omega_n} \sum_{k=0}^{m-1} \left(\partial_x^k \partial_t v_n . \partial_x^{2m-k-1} v_n \right) (-1)^{m+k} \nu_x \omega(t) d\sigma \\ &\quad + \int_{\partial\Omega_n} \left(\partial_x^m v_n \right)^2 \nu_t \omega(t) d\sigma - \int_{\Omega_n} \left(\partial_x^m v_n \right)^2 \omega'(t) dtdx \end{aligned}$$

where ν_t, ν_x are the components of the unit outward normal vector at the boundary of Ω_n. We shall rewrite the boundary integral making use of the boundary conditions. On the part of the boundary of Ω_n where $t = \frac{1}{n}$, we have $v_n = 0$ and consequently $\partial_x^k v_n = 0, k = 0, \ldots, 2m-1$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_x = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_{\varphi_1(T)}^{\varphi_2(T)} \left(\partial_x^m v_n \right)^2 (T, x)\, \omega\left(T\right) dx$$

is nonnegative. On the parts of the boundary where $x = \varphi_i\left(t\right), \ i = 1, 2$, we have

$$\nu_x = \frac{(-1)^i}{\sqrt{1 + \left(\varphi_i'\right)^2 (t)}}, \ \nu_t = \frac{(-1)^{i+1} \varphi_i'\left(t\right)}{\sqrt{1 + \left(\varphi_i'\right)^2 (t)}},$$

and

$$\partial_x^k v_n \left(t, \varphi_2\left(t\right)\right) = \partial_x^l v_n \left(t, \varphi_1\left(t\right)\right) = 0, \ k = 0, \ldots, m-1; \ l = m, \ldots, 2m-1.$$

Differentiating $\partial_x^k v_n \left(t, \varphi_2\left(t\right)\right), \ k = 0, \ldots, m-1$ with respect to t, we obtain

$$\partial_t \partial_x^k v_n \left(t, \varphi_2\left(t\right)\right) = -\varphi_2'\left(t\right) \partial_x^{k+1} v_n \left(t, \varphi_2\left(t\right)\right).$$

The boundary conditions on $\Gamma_{2,n}$ lead to

$$\partial_t \partial_x^k v_n \left(t, \varphi_2\left(t\right)\right) = \begin{cases} 0 & \text{if } k = 0, \ldots, m-2 \\ -\varphi_2'\left(t\right) \partial_x^m v_n \left(t, \varphi_2\left(t\right)\right) & \text{if } k = m-1. \end{cases}$$

Consequently, the corresponding integral is

$$I_{n,2} = \int_{\frac{1}{n}}^{T} \varphi_2'\left(t\right) \left[\partial_x^m v_n \left(t, \varphi_2\left(t\right)\right)\right]^2 \omega(t) dt.$$

Then, we have

$$2\langle \partial_t v_n, (-1)^m \partial_x^{2m} v_n \rangle \geq -|I_{n,2}| . \tag{12}$$

Estimation of $I_{n,k}, k=1,2$: There exists a constant $K_3>0$ independent of n such that

$$|I_{n,1}| \leq K_3\epsilon \left\|\partial_x^{2m} u_n\right\|^2_{L^2_\omega(\Omega_n)} \quad \text{and} \quad |I_{n,2}| \leq K_3\epsilon \left\|\partial_x^{2m} v_n\right\|^2_{L^2_\omega(\Omega_n)}. \tag{13}$$

Indeed, we convert the boundary integral $I_{n,1}$ into a surface integral by setting

$$\begin{array}{rcl} \left[\partial_x^m u_n\left(t,\varphi_1\left(t\right)\right)\right]^2 & = & -\frac{\varphi_2(t)-x}{\varphi(t)}\left[\partial_x^m u_n\left(t,x\right)\right]^2\Big|_{x=\varphi_1(t)}^{x=\varphi_2(t)} \\ & = & -\int_{\varphi_1(t)}^{\varphi_2(t)}\partial_x\left\{\frac{\varphi_2(t)-x}{\varphi(t)}\left[\partial_x^m u_n\right]^2\right\}dx \\ & = & -2\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\varphi_2(t)-x}{\varphi(t)}\partial_x^m u_n.\partial_x^{m+1}u_n dx+\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{1}{\varphi(t)}\left[\partial_x^m u_n\right]^2dx. \end{array}$$

Then, we have

$$\begin{array}{rcl} I_{n,1} & = & -\int_{\frac{1}{n}}^{T}\varphi_1'\left(t\right)\left[\partial_x^m u_n\left(t,\varphi_1\left(t\right)\right)\right]^2\omega\left(t\right)dt \\ & = & -\int_{\Omega_n}\frac{\varphi_1'(t)}{\varphi(t)}\left[\partial_x^m u_n\left(t,x\right)\right]^2\omega\left(t\right)dtdx \\ & & +2\int_{\Omega_n}\frac{\varphi_2(t)-x}{\varphi(t)}\varphi_1'\left(t\right)\left(\partial_x^m u_n\right)\left(\partial_x^{m+1}u_n\right)\ \omega\left(t\right)dtdx. \end{array}$$

Thanks to Lemma 5, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_x^m u_n\right]^2dx \quad \leq \quad K_2\left[\varphi\left(t\right)\right]^{2m}\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_x^{2m}u_n\right]^2dx.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_x^m u_n\right]^2\frac{|\varphi_1'|}{\varphi}\omega\left(t\right)dx \quad \leq \quad K_2\left|\varphi_1'\right|\left[\varphi\left(t\right)\right]^{2m-1}\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\partial_x^{2m}u_n\right]^2\omega\left(t\right)dx,$$

consequently

$$\begin{array}{rcl} |I_{n,1}| & \leq & K_2\int_{\Omega_n}\left|\varphi_1'\right|\left[\varphi\left(t\right)\right]^{2m-1}\left(\partial_x^{2m}u_n\right)^2\omega\left(t\right)dtdx \\ & & +2\int_{\Omega_n}\left|\varphi_1'\right|\left|\partial_x^m u_n\right|\left|\partial_x^{m+1}u_n\right|\ \omega\left(t\right)dtdx, \end{array}$$

since $\left|\frac{\varphi_2(t)-x}{\varphi(t)}\right| \leq 1$. Using the inequality

$$2\left|\varphi_1'\partial_x^m u_n\right|\left|\partial_x^{m+1}u_n\right| \quad \leq \quad \epsilon\left(\partial_x^{m+1}u_n\right)^2+\frac{1}{\epsilon}\left(\varphi_1'\right)^2\left(\partial_x^m u_n\right)^2$$

for all $\epsilon>0$, we obtain

$$\begin{array}{rcl} |I_{n,1}| & \leq & K_2\int_{\Omega_n}\left|\varphi_1'\right|\left[\varphi\right]^{2m-1}\left(\partial_x^{2m}u_n\right)^2\omega\left(t\right)dtdx \\ & & +\int_{\Omega_n}\epsilon\left(\partial_x^{m+1}u_n\right)^2\omega\left(t\right)dtdx+\frac{1}{\epsilon}\int_{\Omega_n}\left(\varphi_1'\right)^2\left(\partial_x^m u_n\right)^2\omega\left(t\right)dtdx. \end{array}$$

Lemma 5 yields

$$\frac{1}{\epsilon}\int_{\Omega_n}\left(\varphi_1'\right)^2\left(\partial_x^m u_n\right)^2\omega\left(t\right)dtdx \leq K_2\frac{1}{\epsilon}\int_{\Omega_n}\left(\varphi_1'\right)^2\left[\varphi\right]^{2m}\left(\partial_{x_1}^{2m}u_n\right)^2\omega\left(t\right)dtdx.$$

Thus,

$$\begin{array}{rcl} |I_{n,1}| & \leq & K_2\int_{\Omega_n}\left[\left|\varphi_1'\right|\left[\varphi\right]^{2m-1}+\frac{1}{\epsilon}\left(\varphi_1'\right)^2\left[\varphi\right]^{2m}\right]\left(\partial_x^{2m}u_n\right)^2\omega\left(t\right)dtdx \\ & & +\int_{\Omega_n}\epsilon\left(\partial_x^{m+1}u_n\right)^2\omega\left(t\right)dtdx \\ & \leq & \left(K_2+1\right)\epsilon\int_{\Omega_n}\left(\partial_x^{2m}u_n\right)^2\omega\left(t\right)dtdx, \end{array}$$

since $\left|\varphi_1'\varphi^m\left(\varphi^{m-1}+\varphi_1'\varphi^m\right)\right| \leq \epsilon$. Finally, taking $K_3=K_2+1$, we obtain

$$|I_{n,1}| \quad \leq \quad K_3\epsilon\left\|\partial_x^{2m}u_n\right\|_{L^2_\omega(\Omega_n)}.$$

The inequality

$$|I_{n,2}| \quad \leq \quad K_3\epsilon\left\|\partial_x^{2m}v_n\right\|_{L^2_\omega(\Omega_n)},$$

can be proved by a similar argument.

Now, we can complete the proof of Theorem 2. Summing up the estimates (11), (12) and (13), and making use of Lemma 5, we then obtain

$$\begin{aligned}\|f_{1,n}\|^2_{L^2_\omega(\Omega_n)} &\geq \|\partial_t u_n\|^2_{L^2_\omega(\Omega_n)} + \left\|\partial_x^{2m} u_n\right\|^2_{L^2_\omega(\Omega_n)} - K_3\epsilon \left\|\partial_x^{2m} u_n\right\|^2_{L^2_\omega(\Omega_n)} \\ &\geq \|\partial_t u_n\|^2_{L^2_\omega(\Omega_n)} + (1-K_3\epsilon)\left\|\partial_x^{2m} u_n\right\|^2_{L^2_\omega(\Omega_n)},\end{aligned}$$

and

$$\begin{aligned}\|f_{2,n}\|^2_{L^2_\omega(\Omega_n)} &\geq \|\partial_t v_n\|^2_{L^2_\omega(\Omega_n)} + \left\|\partial_x^{2m} v_n\right\|^2_{L^2_\omega(\Omega_n)} - K_3\epsilon \left\|\partial_x^{2m} v_n\right\|^2_{L^2_\omega(\Omega_n)} \\ &\geq \|\partial_t v_n\|^2_{L^2_\omega(\Omega_n)} + (1-K_3\epsilon)\left\|\partial_x^{2m} v_n\right\|^2_{L^2_\omega(\Omega_n)},\end{aligned}$$

where K_3 is a positive number. Then, it is sufficient to choose ϵ such that $(1-K_3\epsilon) > 0$, to get a constant $K_4 > 0$ independent of n such that

$$\|f_{1,n}\|^2_{L^2_\omega(\Omega_n)} \geq K_4 \|u_n\|^2_{\mathcal{H}^{1,2m}_\omega(\Omega_n)} \text{ and } \|f_{2,n}\|^2_{L^2_\omega(\Omega_n)} \geq K_4 \|v_n\|^2_{\mathcal{H}^{1,2m}_\omega(\Omega_n)}.$$

But

$$\|f_{1,n}\|_{L^2_\omega(\Omega_n)} \leq \|f_1\|_{L^2_\omega(\Omega)} \text{ and } \|f_{2,n}\|_{L^2_\omega(\Omega_n)} \leq \|f_2\|_{L^2_\omega(\Omega)},$$

then, there exists a constant $C > 0$, independent of n satisfying

$$\|u_n\|^2_{\mathcal{H}^{1,2m}_\omega(\Omega_n)} \leq C \|f_{1,n}\|^2_{L^2_\omega(\Omega_n)} \leq C \|f_1\|^2_{L^2_\omega(\Omega)}$$

and

$$\|v_n\|^2_{\mathcal{H}^{1,2m}_\omega(\Omega_n)} \leq C \|f_{2,n}\|^2_{L^2_\omega(\Omega_n)} \leq C \|f_2\|^2_{L^2_\omega(\Omega)}.$$

This ends the proof of Theorem 2.

Passage to the limit

We are now in position to prove the first main result of this work.

Theorem 3. *Let us assume that the function of parametrization φ_1, (respectively, φ_2) fulfils assumptions (1) and (2) and the weight function ω verifies conditions (5) and (6). Then, for T small enough, Problem (3) (respectively, Problem (4)) admits a unique solution u (respectively, v) belonging to $\mathcal{H}^{1,2m}_{\gamma,\omega}(\Omega)$ (respectively, $\mathcal{H}^{1,2m}_{\delta,\omega}(\Omega)$).*

Proof. Choose a sequence $(\Omega_n)_{n\in\mathbb{N}^*}$ of the domains defined above. Then, we have $\Omega_n \to \Omega$, as $n \to +\infty$. Consider the solutions $u_n, v_n \in \mathcal{H}^{1,2m}_\omega(\Omega_n)$ of the boundary value problems

$$\begin{cases} \partial_t u_n + (-1)^m \partial_x^{2m} u_n = f_{1,n} \in L^2_\omega(\Omega_n), \\ u_n\big|_{t=\frac{1}{n}} = 0, \\ \partial_x^k u_n\big|_{\Gamma_{1,n}} = \partial_x^l u_n\big|_{\Gamma_{2,n}} = 0, \ k = 0,\dots,m-1; \ l = m,\dots,2m-1, \end{cases}$$

and

$$\begin{cases} \partial_t v_n + (-1)^m \partial_x^{2m} v_n = f_{2,n} \in L^2_\omega(\Omega_n), \\ v_n\big|_{t=\frac{1}{n}} = 0, \\ \partial_x^k v_n\big|_{\Gamma_{2,n}} = \partial_x^l v_n\big|_{\Gamma_{1,n}} = 0, \ k = 0,\dots,m-1; \ l = m,\dots,2m-1, \end{cases}$$

$\Gamma_{i,n}$ are the parts of the boundary of Ω_n where $x = \varphi_i(t)$, $i = 1,2$. Such solutions u_n, v_n exist by Theorem 1. Let $\widetilde{u_n}$ and $\widetilde{v_n}$ the 0−extensions of u_n and v_n, respectively, to Ω. In virtue of Theorem 2, we know that there exists a constant $K > 0$ independent of n such that

$$\|\widetilde{u_n}\|^2_{L^2_\omega(\Omega)} + \left\|\widetilde{\partial_t u_n}\right\|^2_{L^2_\omega(\Omega)} + \sum_{j=1}^{2m} \left\|\widetilde{\partial_x^j u_n}\right\|^2_{L^2_\omega(\Omega)} \leq K \|f_1\|^2_{L^2_\omega(\Omega)},$$

and

$$\|\widetilde{v_n}\|^2_{L^2_\omega(\Omega)} + \left\|\widetilde{\partial_t v_n}\right\|^2_{L^2_\omega(\Omega)} + \sum_{j=1}^{2m} \left\|\widetilde{\partial_x^j v_n}\right\|^2_{L^2_\omega(\Omega)} \leq K \|f_2\|^2_{L^2_\omega(\Omega)}.$$

This means that $\widetilde{u_n}$, $\widetilde{v_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial_t v_n}$, $\widetilde{\partial_x^j u_n}$, $\widetilde{\partial_x^j v_n}$, for $j = 1, \ldots, 2m$ are bounded functions in $L^2_\omega(\Omega)$. So, for a suitable increasing sequence of integers n_k, $k = 1, 2, ...$, there exist functions

$$u, v, w^1, w^2, w_j^1 \text{ and } w_j^2,\ j = 1, \ldots, 2m,$$

in $L^2_\omega(\Omega)$ such that

$$\widetilde{u_{n_k}} \rightharpoonup u,\ \widetilde{\partial_t u_{n_k}} \rightharpoonup w^1,\ \widetilde{\partial_x^j u_{n_k}} \rightharpoonup w_j^1,\ j = 1, \ldots, 2m,$$

and

$$\widetilde{v_{n_k}} \rightharpoonup v,\ \widetilde{\partial_t v_{n_k}} \rightharpoonup w^2,\ \widetilde{\partial_x^j v_{n_k}} \rightharpoonup w_j^2,\ j = 1, \ldots, 2m,$$

weakly in $L^2_\omega(\Omega)$, as $k \to +\infty$. Clearly

$$w^1 = \partial_t u,\ w_j^1 = \partial_x^j u, \text{ and } w^2 = \partial_t v,\ w_j^2 = \partial_x^j v,\ j = 1, \ldots, 2m,$$

in the sense of distributions in Ω. So, $u, v \in \mathcal{H}^{1,2m}(\Omega)$ and

$$\partial_t u + (-1)^m \partial_x^{2m} u = f_1 \text{ and } \partial_t v + (-1)^m \partial_x^{2m} v = f_2,$$

in Ω. On the other hand, the solutions u, v satisfy the boundary conditions

$$\partial_x^k u\big|_{\Gamma_1} = \partial_x^l u\big|_{\Gamma_2} = 0,\ k = 0, \ldots, m-1;\ l = m, \ldots, 2m-1$$

and

$$\partial_x^k v\big|_{\Gamma_2} = \partial_x^l v\big|_{\Gamma_1} = 0,\ k = 0, \ldots, m-1;\ l = m, \ldots, 2m-1,$$

since

$$\forall n \in \mathbb{N}^*,\ u|_{\Omega_n} = u_n \text{ and } v|_{\Omega_n} = v_n.$$

This proves the existence of solution to Problems (3) and (4). □

4. Global in time result

In the case where T is not in the neighborhood of zero, we set $\Omega = D_1 \cup D_2 \cup \Gamma_{T_1}$, where

$$D_1 = \{(t,x) \in \Omega : 0 < t < T_1\},\ D_2 = \{(t,x) \in \Omega : T_1 < t < T\},$$

and

$$\Gamma_{T_1} = \left\{(T_1, x) \in \mathbb{R}^2 : \varphi_1(T_1) < x < \varphi_2(T_1)\right\},$$

with T_1 small enough. In the sequel, k_1, k_2 stand for arbitrary fixed elements of $L^2_\omega(\Omega)$ and $k_{1,i} = k_1|_{D_i}$, $k_{2,i} = k_2|_{D_i}$, $i = 1, 2$.

Theorem 3 applied to the triangular domain D_1, shows that there exist unique solutions $w_1, w_2 \in \mathcal{H}^{1,2m}_\omega(D_1)$ of the problems

$$(14)\qquad \begin{cases} \partial_t w_1 + (-1)^m \partial_x^{2m} w_1 = k_{1,1} \in L^2_\omega(D_1), \\ \partial_x^k w_1\big|_{\Gamma_{1,1}} = \partial_x^l w_1\big|_{\Gamma_{2,1}} = 0,\ k = 0, \ldots, m-1;\ l = m, \ldots, 2m-1 \end{cases}$$

and

$$(15)\qquad \begin{cases} \partial_t w_2 + (-1)^m \partial_x^{2m} w_2 = k_{2,1} \in L^2_\omega(D_1), \\ \partial_x^k w_2\big|_{\Gamma_{2,1}} = \partial_x^l w_2\big|_{\Gamma_{1,1}} = 0,\ k = 0,\dots,m-1;\ l = m,\dots,2m-1, \end{cases}$$

$\Gamma_{i,1}$ are the parts of the boundary of D_1 where $x = \varphi_i(t)$, $i = 1, 2$.

Lemma 6. *If $u \in \mathcal{H}^{1,2m}_\omega(]0,T[\times]0,1[)$, then*

$$u|_{t=0} \in H^m_\omega(\gamma_0),\ u|_{x=0} \in H_\omega^{1-\frac{1}{2m}}(\gamma_1),\ \text{and}\ u|_{x=1} \in H_\omega^{1-\frac{1}{2m}}(\gamma_2),$$

where $\gamma_0 = \{0\}\times]0,1[$, $\gamma_1 =]0,T[\times\{0\}$ and $\gamma_2 =]0,T[\times\{1\}$.

It is a particular case of Theorem 2.1 ([15], Vol.2). The transformation

$$(t,x) \longmapsto (t',x') = (t, \varphi(t)x + \varphi_1(t)),$$

leads to the following lemma :

Lemma 7. *If $u \in \mathcal{H}^{1,2m}_\omega(D_2)$, then*

$$u|_{\Gamma_{T_1}} \in H^m_\omega(\Gamma_{T_1}),\ u|_{x=\varphi_1(t)} \in H_\omega^{1-\frac{1}{2m}}(\Gamma_{1,2}),\ \text{and}\ u|_{x=\varphi_2(t)} \in H_\omega^{1-\frac{1}{2m}}(\Gamma_{2,2}),$$

where $\Gamma_{i,2}$ are the parts of the boundary of D_2 where $x = \varphi_i(t)$, $i = 1, 2$.

Hereafter, we denote the trace $w_1|_{\Gamma_{T_1}}$ by ψ_1, and $w_2|_{\Gamma_{T_1}}$ by ψ_2, which are in the Sobolev space $H^m_\omega(\Gamma_{T_1})$ because $w_1, w_2 \in \mathcal{H}^{1,2m}_\omega(D_1)$ (see Lemma 7). Now, consider the following problems in D_2

$$(16)\qquad \begin{cases} \partial_t w_3 + (-1)^m \partial_x^{2m} w_3 = k_{1,2} \in L^2_\omega(D_2), \\ w_3|_{\Gamma_{T_1}} = \psi_1, \\ \partial_x^k w_3\big|_{\Gamma_{1,2}} = 0,\ k = 0,\dots,m-1, \\ \partial_x^l w_3\big|_{\Gamma_{2,2}} = 0,\ l = m,\dots,2m-1, \end{cases}$$

and

$$(17)\qquad \begin{cases} \partial_t w_4 + (-1)^m \partial_x^{2m} w_4 = k_{2,2} \in L^2_\omega(D_2), \\ w_4|_{\Gamma_{T_1}} = \psi_2, \\ \partial_x^k w_4\big|_{\Gamma_{2,2}} = 0,\ k = 0,\dots,m-1, \\ \partial_x^l w_4\big|_{\Gamma_{1,2}} = 0,\ l = m,\dots,2m-1, \end{cases}$$

$\Gamma_{i,2}$ are the parts of the boundary of D_2 where $x = \varphi_i(t)$, $i = 1, 2$. We use the following result, which is a consequence of Theorem 4.3 ([15], Vol.2), to solve Problems (16) and (17).

Proposition 2. *Let Q be the rectangle $]0,T[\times]0,1[$, $l_1, l_2 \in L^2_\omega(Q)$ and $\phi_1, \phi_2 \in H^m_\omega(\gamma_0)$. Then, the problems*

$$\begin{cases} \partial_t u + (-1)^m \partial_x^{2m} u = l_1 \in L^2_\omega(Q), \\ u|_{\gamma_0} = \phi_1, \\ \partial_x^k u\big|_{\gamma_1} = 0,\ k = 0,\dots,m-1 \\ \partial_x^l u\big|_{\gamma_2} = 0,\ l = m,\dots,2m-1, \end{cases}$$

and

$$\begin{cases} \partial_t v + (-1)^m \partial_x^{2m} v = l_2 \in L^2_\omega(Q), \\ v|_{\gamma_0} = \phi_2, \\ \partial_x^k v\big|_{\gamma_2} = 0,\ k = 0,\dots,m-1 \\ \partial_x^l v\big|_{\gamma_1} = 0,\ l = m,\dots,2m-1, \end{cases}$$

where $\gamma_0 = \{0\} \times]0,1[\,,\ \gamma_1 =]0,T[\times\{0\}$ *and* $\gamma_2 =]0,T[\times\{1\}\,,$ *admit (unique) solutions* $u, v \in \mathcal{H}_\omega^{1,2m}(Q)\,.$

Remark 4. *In the application of Theorem 4.3 (*[15]*, Vol.2) we can observe that there are no compatibility conditions to satisfy because* $\partial_x\phi_1$ *and* $\partial_x\phi_2$ *are only in* $L^2(\gamma_0)$.

Thanks to the transformation $(t,x) \longmapsto (t,y) = (t, \varphi(t)\,x + \varphi_1(t))\,,$ we deduce the following result:

Proposition 3. *Problems (16) and (17) admit (unique) solutions* $w_3, w_4 \in \mathcal{H}_\omega^{1,2m}(D_2)\,.$

So, the function u (respectively, v) defined by

$$u := \begin{cases} w_1 \text{ in } D_1, \\ w_3 \text{ in } D_2, \end{cases}$$

(respectively

$$v := \begin{cases} w_2 \text{ in } D_1, \\ w_4 \text{ in } D_2), \end{cases}$$

is the (unique) solution of Problem (3) (respectively, (4)) for an arbitrary T. Our second main result is proved, that is,

Theorem 4. *Let us assume that the function of parametrization* φ_1*, (respectively,* φ_2*) fulfils assumptions (1) and (2) and the weight function* ω *verifies conditions (5) and (6). Then, Problem (3) (respectively, Problem (4)) admits a unique solution* u *(respectively,* v*) belonging to* $\mathcal{H}_{\gamma,\omega}^{1,2m}(\Omega)$ *(respectively,* $\mathcal{H}_{\delta,\omega}^{1,2m}(\Omega)$*).*

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References

[1] E. A. Baderko, *On the solution of boundary value problems for linear parabolic equations of arbitrary order in noncylindrical domains by the method of boundary integral equations*, PhD Thesis, Moscow, 1992.

[2] O. V. Besov, *Continuation of functions from* $L_p{}^l$ *and* $W_p{}^l$, Trudy Mat. Inst. Steklov, **89** (1967), 5–17. MR0215077

[3] A. Cabada, R. Precup, L. Saavedra, S. Tersian, *Multiple positive solutions to a fourth order boundary value problem*, Electronic Journal of Differential Equations, **2016** (2016), 1–18. Zbl 06637324

[4] J. Dugundji, *Cantilever boundary condition, deflections, and stresses of sandwich beams*, AIAA Journal, **40**:5 (2002), 987–995.

[5] M. F. Cherepova, *Regularity of solutions of boundary value problems for a second-order parabolic equation in weighted Hölder spaces*, Differ. Uravn., **49**:1 (2013), 79–87. Zbl 1273.35074

[6] V. A. Galaktionov, *On regularity of a boundary point for higher-order parabolic equations: towards Petrovskii-type criterion by blow-up approach*, Nonlinear Differential Equations Appl. **16**:5 (2009), 597–655. Zbl 1219.35123

[7] A. Grimaldi Piro, F. Ragnedda, *Parabolic operators of higher order in domains with a "non regular"boundary*, Rend. Sem. Fac.Sci. Univ. Cagliari, **54** (1984), 45–62. Zbl. 0618.35055

[8] A. Kheloufi, *On a fourth order parabolic equation in a nonregular domain of* $\mathbb{R}^3$, Mediterr. J. Math., **12** (2015), 803–820. Zbl 1321.35082

[9] A. Kheloufi, B. K. Sadallah, *Study of a parabolic equation with mixed Dirichlet-Neumann type boundary conditions in unbounded noncylindrical domains*, Journal of Advanced Research in Applied Mathematics, **7**:4 (2015), 62–77.

[10] A. Kheloufi, *On a Fourth Order Parabolic Equation with Mixed Type Boundary Conditions in a Nonrectangular Domain*, Moroccan J. Pure and Appl. Anal., **1**:2 (2015), 76–90.

[11] A. Kheloufi, *Study of a $2m$-th order parabolic equation in a non-regular type of prism of* $\mathbb{R}^{N+1}$, Georgian Math. J., **23**:2 (2016), 227–237. Zbl 1338.35199

[12] A. Kheloufi, B. K. Sadallah, *Resolution of a high-order parabolic equation in conical time-dependent domains of* $\mathbb{R}^3$, Arab J. Math. Sci., **22** (2016), 165–181. Zbl 1338.35200

[13] V. A. Kondrat'ev, *Boundary problems for parabolic equations in closed regions*, Am. Math. Soc. Providence. R I. (1966), 450–504. Zbl 0162.15203

[14] R. Labbas, B. K. Sadallah, *Smoothness of the solution of a fourth order parabolic equation in a polygonal domain*, Int. J. Appl. Math., **1** (1999), 75–90. Zbl 1171.35392

[15] J. L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications,* $1, 2$, Dunod, Paris (1968). Zbl 0165.10801

[16] V. P. Mikhailov, *The Dirichlet problem for a parabolic equation I*, Mat. Sb. (N.S.), **61** (103) (1963), 40–64. MR0151726

[17] V. P. Mikhailov, *The Dirichlet problem for a parabolic equation II*, Mat. Sb. (N.S.), **62** (104) (1963), 140–159. MR0158184

[18] B. K. Sadallah, *Etude d'un problème $2m-$parabolique dans des domaines plan non rectangulaires*, Boll. Un. Mat. Ital., **2-B**:5 (1983), 51–112. Zbl 0512.35046

[19] B. K. Sadallah, *Singularities of the solution of a $2m$-parabolic problem in a polygonal domain*, Arab J. Math. Sci., **4**:2 (1998), 31–41. Zbl 0929.35005

[20] G. Savaré, *Parabolic problems with mixed variable lateral conditions: an abstract approach*, J. Math. Pures Appl., **76**:9 (1997), 321–351. Zbl 0890.35061

[21] G. Shi, G. Z. Voyiadjis, *A sixth-order theory of shear deformable beams with variational consistent boundary conditions*, J. Appl. Mech., **78**:2 (2010), 1–11.

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