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SELF-SIMILAR DENDRITES GENERATED BY POLYGONAL SYSTEMS IN THE PLANE

M. SAMUEL, A. TETENOV, D. VAULIN

Abstract. We define a class of self-similar dendrites in $\mathbb{R}^2$ generated by system $\mathcal{S}$ of similarity maps of a convex polygon P and find upper bound for the order of their ramification points, show that such dendrites are continua of bounded turning and prove Hölder continuity of their isomorphisms.

Keywords: dendrite, ramification point, self-similar set, post-critically finite sets.

1. Introduction

The study of dendrites occupies a significant place in general topology [12, 15, 19]. One can refer to the paper [5] of J.Charatonik and W.Charatonik for exhaustive overview covering more than 75-year research in this area. At the same time, in the theory of self-similar sets there are individual attempts to work out some approaches to self-similar dendrites in certain situations. In 1985, Hata [8] studied connectedness properties of attractor K of a system $\mathcal{S}$ of weak contractions in a complete metric space and showed that if K is a dendrite then it has infinite set of end points. In 1990 Ch. Bandt showed in [2] that the Jordan arcs connecting pairs of points of a post-critically finite self-similar set are self-similar, and the set of possible values for dimensions of such arcs is finite, applying these results to dendrites. He also considered factorisation of address space giving rise to dendrites in [3]. Jun Kigami in his work [9] applied the methods of harmonic calculus on fractals to dendrites. D.Croydon in his thesis [6] obtained heat kernel estimates for continuum random tree and for certain family of p.c.f. random dendrites on the

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plane. D.Dumitru and A.Mihail [7] made an attempt to get a sufficient condition for a self-similar set to be a dendrite in terms of sequences of intersection graphs for the refinements of the system $\mathcal{S}$.

There are several questions arising in the study of self-similar dendrites. What kind of topological restrictions characterise the class of dendrites generated by systems of similarities in $\mathbb{R}^d$? What are the explicit construction algorithms for self-similar dendrites? What are the metric and analytic properties of morphisms of self-similar structures on dendrites?

The aim of our work is to make clear basic topological and metric properties of self-similar dendrites in the most simple settings. For that reason we consider systems of similarities in the plane, which we call polygonal tree systems (Definition 9). We show that the attractor K of such system $\mathcal{S}$ is a dendrite (Theorem 12), that, by the construction, each such system $\mathcal{S}$ satisfies open set condition, one-point intersection property and is post-critically finite (Proposition 10, Corollary 21); for the dendrite K we define its main tree (Definition 16) and show that each cut point of K lies in some image $S_{\mathbf{j}}(\hat{\gamma})$ of the main tree (Theorem 20) and get the upper bound for the order of ramification points of K, depending only on the initial polygon P of the system $\mathcal{S}$. We show that the dendrite K is a continuum with bounded turning in the sense of P.Tukia (Theorem 26). Finally, we show that each combinatorial equivalence of polygonal tree systems $\mathcal{S}$,$\mathcal{S}'$ defines unique homeomorphism $\varphi : K \to K'$, compatible with $\mathcal{S}$ and $\mathcal{S}'$ and prove Hölder continuity of φ and φ^{-1} (Theorem 27).

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1.1. **Preliminaries. Dendrites.** A *dendrite* is a locally connected continuum containing no simple closed curve.

In the case of dendrites the order $Ord(p,X)$ of the point p with respect to X is equal to the number of components of the set $X \setminus \{p\}$. Points of order 1 in a continuum X are called *end points* of X; the set of all end points of X will be denoted by $EP(X)$. A point p of a continuum X is called a *cut point* of X provided that $X \setminus \{p\}$ is not connected; the set of all cut points of X will be denoted by $CP(X)$. Points of order at least 3 are called *ramification points* of X; the set of all ramification points of X will be denoted by $RP(X)$.

We will use the following statements selected from [5, Theorem 1.1]:

Theorem 1. *For a continuum X the following conditions are equivalent:*

- *(a) X is dendrite;*
- *(b) every two distinct points of X are separated by a third point;*
- *(c) each point of X is either a cut point or an end point of X;*
- *(d) each nondegenerate subcontinuum of X contains uncountably many cut points of X.*
- *(e) for each point $p \in X$ the number of components of the set $X \setminus \{p\} = ord(p,X)$ whenever either of these is finite;*
- *(f) the intersection of every two connected subsets of X is connected;*
- *(g) X is locally connected and uniquely arcwise connected.*

Self-similar sets. Let (X, d) be a complete metric space. A mapping $F : X \to X$ is a contraction if $\operatorname{Lip} F < 1$. The mapping $S : X \to X$ is called a similarity if

$$d(S(x), S(y)) = rd(x, y) \tag{1}$$

for all $x, y \in X$ and some fixed r.

Definition 2. *Let $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ be a system of (injective) contraction maps on the complete metric space (X, d). A nonempty compact set $K \subset X$ is said to be invariant with respect to $\mathcal{S}$, if $K = \bigcup\limits_{i=1}^{m} S_i(K)$.*

We also call the subset $K \subset X$ self-similar with respect to $\mathcal{S}$. Throughout the whole paper, the maps $S_i \in \mathcal{S}$ are supposed to be similarities and the set X to be $\mathbb{R}^2$.

Notation. $I = \{1, 2, ..., m\}$ is the set of indices, $I^* = \bigcup\limits_{n=1}^{\infty} I^n$ - is the set of all finite I-tuples, or multiindices $\mathbf{j} = j_1 j_2 ... j_n$, where $\mathbf{ij}$ is the concatenation of the corresponding multiindices;
we write $S_{\mathbf{j}} = S_{j_1 j_2 \dots j_n} = S_{j_1} S_{j_2} ... S_{j_n}$ and for the set $A \subset X$ we denote $S_{\mathbf{j}}(A)$ by $A_{\mathbf{j}}$; we also denote by $G_{\mathcal{S}} = \{S_{\mathbf{j}}, \mathbf{j} \in I^*\}$ the semigroup, generated by $\mathcal{S}$;
$I^{\infty} = \{\alpha = \alpha_1 \alpha_2 \dots, \quad \alpha_i \in I\}$ – index space; and $\pi : I^{\infty} \to K$ is the *index map* , which sends α to the point $\bigcap\limits_{n=1}^{\infty} K_{\alpha_1 \dots \alpha_n}$.

Definition 3. *The system $\mathcal{S}$ satisfies the* open set condition *(OSC) if there exists a non-empty open set $O \subset X$ such that $S_i(O), \{1 \le i \le m\}$ are pairwise disjoint and all contained in O.*

We say the self-similar set K defined by the system $\mathcal{S}$ satisfies the one-point intersection property if for any $i \ne j$, $S_i(K) \bigcap S_j(K)$ is not more than one point.

The union $\mathcal{C}$ of all $S_i(K) \cap S_j(K)$, $i, j \in I, i \ne j$ is called the critical set of the system $\mathcal{S}$. The post-critical set $\mathcal{P}$ of the system $\mathcal{S}$ is the set of all $\alpha \in I^{\infty}$ such that for some $\mathbf{j} \in I^*$, $S_{\mathbf{j}}(\pi(\alpha)) \in \mathcal{C}$. [10]

We use the following convenient criterion of connectedness of the attractor of a system $\mathcal{S}$ [8, 10].

Theorem 4. *Let K be the attractor of a system of contractions $\mathcal{S}$ in a complete metric space (X, d). Then the following statements are equivalent:*
1) for any $i, j \in I$ there are $\{i_0, i_1, \dots, i_n\} \subset I$ such that $i_0 = i, i_n = j$ and $S_{i_k}(K) \bigcap S_{i_{k+1}}(K) \ne \varnothing$ for any $k = 0, 1, \dots, n-1$.
2) K is arcwise connected.
3) K is connected.

Proposition 5. [10] *If a self-similar set K is connected, it is locally connected.*

Zippers and multizippers. The simplest way to construct a self-similar curve is to take a polygonal line and then replace each of its segments by a smaller copy of the same polygonal line; this construction is called zipper and was studied by Aseev, Tetenov and Kravchenko [1].

Definition 6. *Let X be a complete metric space. A system $\mathcal{S} = \{S_1, \dots, S_m\}$ of contraction mappings of X to itself is called a* zipper *with vertices $\{z_0, \dots, z_m\}$ and signature $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m), \varepsilon_i \in \{0, 1\}$, if for $i = 1 \dots m, S_i(z_0) = z_{i-1+\varepsilon_i}$ and $S_i(z_m) = z_{i-\varepsilon_i}$.*

More general approach for building self-similar curves and continua is provided by a graph-directed version of zipper construction, which is called multizipper [17]:

Definition 7. *Let $\{X_u, u \in V\}$ be a system of spaces, all isomorphic to $\mathbb{R}^d$. For each X_u let a finite array of points be given $\{x_0^{(u)}, \dots, x_{m_u}^{(u)}\}$. Suppose for each $u \in V$ and $0 \le k \le m_u$ we have some $v(u,k) \in V$ and $\varepsilon(u,k) \in \{0,1\}$ and a map $\mathcal{S}_k^{(u)} : X_v \to X_u$ such that*
$S_k^{(u)}(x_0^{(v)}) = x_{k-1}^{(u)}$ or $x_k^{(u)}$ and $S_k^{(u)}(x_{m_v}^{(v)}) = x_k^{(u)}$ or $x_{k-1}^{(u)}$, depending on the signature $\varepsilon(u,r)$.
The graph directed iterated function system (IFS) defined by the maps $S_k^{(u)}$ is called a multizipper $\mathcal{Z}$.

The attractor of multizipper $\mathcal{Z}$ is a system of connected and arcwise connected compact sets $K_u \subset X_u$ satisfying the system of equations

$$K_u = \bigcup_{k=1}^{m_u} S_k^{(u)}(K_{v(u,k)}), \qquad u \in V$$

We call the sets K_u the components of the attractor of $\mathcal{Z}$.

We call $\mathcal{Z}$ Jordan multizipper if the components of K_u of its attractor are Jordan arcs and if it satisfies one point intersection property. The following Theorem gives conditions under which $\mathcal{Z}$ is a Jordan multizipper:

Theorem 8. *Let $\mathcal{Z}_0 = \{S_k^{(u)}\}$ be a multizipper with node points $x_k^{(u)}$ and a signature $\varepsilon = \{(v(u,k), \varepsilon(u,k)), u \in V, k = 1, \dots, m_u\}$. If for any $u \in V$ and any $i, j \in \{1, 2, \dots, m_u\}$, the set $K_{(u,i)} \cap K_{(u,j)} = \varnothing$ if $|i-j| > 1$ and is a singleton if $|i-j| = 1$, then any linear parametrization $\{f_u : I_u \to K_u\}$ is a homeomorphism and each K_u is a Jordan arc with endpoints $x_0^{(u)}$, $x_m^{(u)}$.*

2. Polygonal tree systems.

Let P be a convex polygon in $\mathbb{R}^2$ and $V_P = \{A_1, \dots, A_{n_P}\}$ be the set of its vertices, where $n_P = \#V_P$.
Consider a system of contracting similarities $\mathcal{S} = \{S_1, \dots, S_m\}$, which possesses the following properties:
(D1) For any $k \in I$, the set $P_k = S_k(P)$ is contained in P;
(D2) For any $i \ne j$, $i, j \in I$, $P_i \bigcap P_j$ is either empty or is a common vertex of P_i and P_j;
(D3) For any $A_k \in V_P$ there is the map $S_i \in \mathcal{S}$ and a vertex $A_l \in V_P$ such that $S_i(A_l) = A_k$;
(D4) *The set $\widetilde{P} = \bigcup\limits_{i=1}^{m} P_i$ is contractible.*

Definition 9. *The system $(P, \mathcal{S})$ satisfying the conditions* **D1-D4** *is called a polygonal tree system associated with the polygon P.*

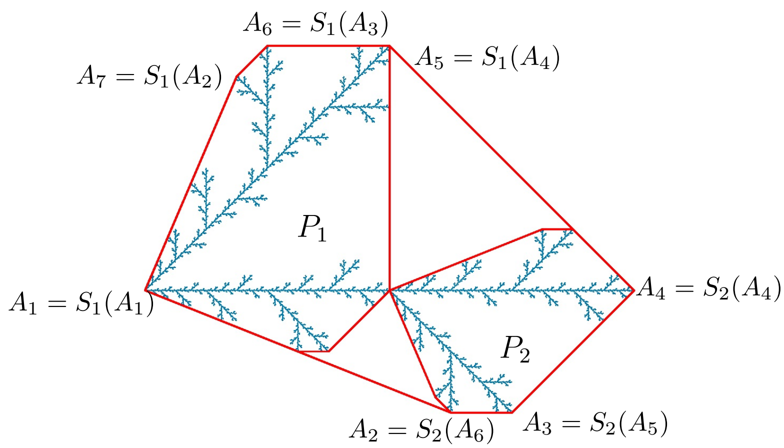
Some properties of the attractor K of a polygonal tree system $\mathcal{S}$ follow directly from its definition:

Proposition 10. *Let $\mathcal{S}$ be a polygonal tree system associated with a polygon P and let K be its attractor. Then (i) $\mathcal{S}$ satisfies open set condition; (ii) $\mathcal{S}$ satisfies one point intersection property.*

Proof. (i) Since for any $i, j = 1, \ldots, m$, $P_i \subset P$ and $\dot{P}_i \cap \dot{P}_j = \varnothing$ for $i \neq j$, $\dot{P}$ can be taken for the open set; (ii) follows directly from **(D2)**. □

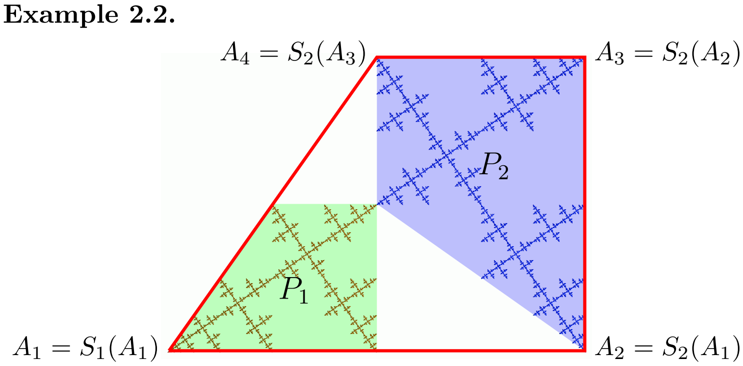
Thus, to define a polygonal tree system, we specify a polygon P, a system of its subpolygons P_i and the similarities S_i, sending P to P_i. Applying Hutchinson operator $T(A) = \bigcup_{i \in I} S_i(A)$ of the system $\mathcal{S}$ to the polyhedron P, we get the set $\widetilde{P} = \bigcup_{i \in I} P_i$. We define $\widetilde{P}^{(1)} = T(P), \widetilde{P}^{(n+1)} = T(\widetilde{P}^{(n)})$. Thus we get a nested family of sets $\widetilde{P}^{(1)} \supset \widetilde{P}^{(2)} \supset \ldots \supset \widetilde{P}^{(n)} \supset \ldots$, whose intersection is K.

Example 2.1. Hata's tree-like set.



Hata's tree-like set [8] is a widely used example of self-similar dendrite [6, 10, 11]. The picture represents it as the attractor of a polygonal tree system and shows how the condition **D3** holds. For different values of parameters the polygon P for the set has 5 to 7 vertices and the system $\mathcal{S}$ contains two maps. Here the maps are $S_1(z) = (1+i)\bar{z}/2$, $S_2(z) = (\bar{z}+1)/2$.

Example 2.2.



A dendrite, considered in R. Zeller's thesis [20, Ch.1, p.18] is contained in one-parameter family of polygonal dendrites where P is a trapezium. The picture shows the one with $S_1(z) = z/2$ and $S_2(z) = iz/\sqrt{2} + 1$.

Composition of two Hutchinson operators corresponding to two polygonal tree systems associated with the same polygon P is also an operator of the same type:

Lemma 11. *(i) Let $(P, \mathcal{S})$ and $(P, \mathcal{S}')$ be polygonal tree systems of similarities associated with P. Then the system $\mathcal{S}'' = \{S_i \circ S'_j, S_i \in \mathcal{S}, S_j \in \mathcal{S}'\}$ is a polygonal tree system of similarities associated with P.*
(ii) For any $n \in \mathbb{N}$, $\mathcal{S}^{(n)} = \{S_{\mathbf{j}}, \mathbf{j} \in I^n\}$ is also a polygonal tree system associated with P.

Proof. (i) The condition **(D1)** is obvious because $S_i \circ S'_j(P) \subset S_i(P) \subset P$.
(D2) Let $Q_1 = S_{i_1} \circ S'_{j_1}(P)$ and $Q_2 = S_{i_2} \circ S'_{j_2}(P)$ be two polygons in $\mathcal{S}''$ and consider their intersection:
if $i_1 \neq i_2$, $Q_1 \bigcap Q_2 \subset P_{i_1} \bigcap P_{i_2}$, where the left-hand side intersection contains at most one point.
if $i_1 = i_2, Q_1 \bigcap Q_2 = S_{i_1}(P'_{j_1} \bigcap P'_{j_2})$ which is either empty or a one-point set, containing $S_{i_1}(A')$ where A' is a common vertex of P'_{j_1} and P'_{j_2}.
(D3) holds because for any vertex A_k, there is a similarity $S_i \in \mathcal{S}$ and a vertex A_{k_1} such that $S_i(A_{k_1}) = A_k$ and a similarity $S'_j \in \mathcal{S}'$ and a vertex A_{k_2} such that $S'_j(A_{k_2}) = A_{k_1}$, therefore $S_i \cdot S'_j(A_{k_2}) = A_k$.
(D4) The sets $\widetilde{P} = \bigcup\limits_{i=1}^{m} P_i$ and $\widetilde{P}' = \bigcup\limits_{i=1}^{m'} P'_i$ are strong deformation retracts of the polygon P, both containing the set V_P. Let $\varphi'(X,t) : P \times [0,1] \to P$ be a strong deformation retraction from P to $\bigcup\limits_{i=1}^{m'} P'_i$. So the map φ' satisfies the conditions $\varphi'(x,0) = Id$, $\varphi'(x,1)(P) = \widetilde{P}'$ and for any $t \in [0,1]$, $\varphi'(x,t)|_{\widetilde{P}'} = Id_{\widetilde{P}'}$.
Define a map $\varphi'_i : P_i \times [0,1] \to P_i$ by the formula

$$\varphi'_i(x,t) = S_i \circ \varphi'(S_i{}^{-1}(x), t).$$

Each map φ'_i is a strong deformation retraction from P_i to $S_i(\widetilde{P}')$.
Observe that the map φ'_i keeps all the vertices $S_i(A_k)$ of the polygon P_i fixed. Therefore we can define a strong deformation retraction $\widetilde{\varphi}(x,t) : \widetilde{P} \times [0,1] \to \bigcup\limits_{i=1}^{m} S_i(\widetilde{P}') = \widetilde{P}$ by a formula

$$\widetilde{\varphi}(x,t) = \varphi'_i(x,t), \quad \text{if } x \in P_i$$

The map $\widetilde{\varphi}$ is well-defined and continuous because if $P_i \bigcap P_j = \{S_i(A_k)\} = \{S_j(A_l)\}$ for some k and l, then $\varphi'_i(S_i(A_k), t) \equiv \varphi'_j(S_j(A_l), t) \equiv S_i(A_k)$.
Moreover, $\widetilde{\varphi}(x,0) = x$ on $\widetilde{P}$, and $\widetilde{\varphi}(\widetilde{P}, 1) \equiv \bigcup\limits_{i=1}^{m} S_i(\widetilde{P}')$ and $\widetilde{\varphi}(x,t)|_{\widetilde{P}''} \equiv Id$.
So $\widetilde{\varphi}(x,t)$ is a strong deformation retraction from $\widetilde{P}$ to $\widetilde{P}''$.
Therefore, the set $\widetilde{P}'' = \bigcup S_i \circ S'_j(P)$ is contractible.

(ii) is proved by induction. Suppose $\mathcal{S}^{(n)}$ is a polygonal tree system associated with P. Then by (i), $\mathcal{S}^{(n+1)} = \mathcal{S}^{(n)} \circ \mathcal{S}$ is also such a system. □

Theorem 12. *Let S be a polygonal tree system of similarities associated with P, and let K be its attractor. Then K is a dendrite.*

Proof. By Lemma 11, the sets $\widetilde{P}^{(n)}$ are contractible compact sets, satisfying the inclusions $\widetilde{P}^{(1)}\supset\widetilde{P}^{(2)}\supset\widetilde{P}^{(3)}\ldots$. The diameter of connected components of the interior of each $\widetilde{P}^{(n)}$ does not exceed $\mathrm{diam}P\cdot q^n$, where $q=\max\mathrm{Lip}(S_i)$. Therefore the set $K=\bigcap\widetilde{P}^{(n)}$ is contractible and has empty interior. Since the system $\{P_i\}$ satisfies conditions of Theorem 4, the attractor K is connected, locally connected and arcwise connected [10, Theorem 1.6.2, Proposition 1.6.4]. Since any simple closed curve in a contractible set X on a plane bounds a disc in X which has interior points, the set K contains no simple closed curve and therefore is a dendrite. □

Corollary 13. *For any* $\mathbf{j}\in I^*$ *the set* $S_{\mathbf{j}}(V_P)$ *of vertices of the polygon* $P_{\mathbf{j}}$ *is contained in* K.

Proof. Since for any $n\in\mathbb{N}$, $\mathcal{S}^{(n)}$ is the polygonal tree system associated with P, each of the vertices $V_P\subset\widetilde{P}^{(n)}$. Therefore $V_P\subset K$. Then for any $\mathbf{j}\in I^*$, $S_{\mathbf{j}}(V_P)\subset K$. □

The dendrite K lies in the polygon P, and its intersection with the sides of P can be uncountable, or even contain the whole sides of P. This is also true for any subpolygon $S_{\mathbf{i}}(P)$. Nevertheless, the dendrite K can "penetrate"to each $P_{\mathbf{j}}$ only through its vertices, namely:

Proposition 14. *Let* $\mathbf{j}\in I^*$ *be a multiindex. For any continuum* $L\subset K$*, whose intersection with both* $P_{\mathbf{j}}$ *and* $CP_{\mathbf{j}}$ *is nonempty,* $\overline{L\backslash P_{\mathbf{j}}}\cap P_{\mathbf{j}}\subset S_{\mathbf{j}}(V_P)$.

Proof. Suppose $\mathbf{j}\in I^k$. Let $P^c_{\mathbf{j}}$ denote the closure of the set $\widetilde{P}^{(k)}\backslash P_{\mathbf{j}}$. Observe that $P^c_{\mathbf{j}}=\bigcup\limits_{\mathbf{i}\in I^k\backslash\{\mathbf{j}\}}P_{\mathbf{i}}$, therefore $P^c_{\mathbf{j}}\cap P_{\mathbf{j}}\subset S_{\mathbf{j}}(V_P)$. By the condition **D3** applied to the system $\mathcal{S}^{(k)}$, this intersection is nonempty and the set $\widetilde{P}^{(k)}\backslash S_{\mathbf{j}}(V_P)$ is disconnected, while $P_{\mathbf{j}}\backslash S_{\mathbf{j}}(V_P)$ is its component.

The continuum L lies in $\widetilde{P}^{(k)}$. If $L\cap(P_{\mathbf{j}}\backslash S_{\mathbf{j}}(V_P))=\varnothing$, then $L\cap P_{\mathbf{j}}\subset S_{\mathbf{j}}(V_P)$ and $L\cap P_{\mathbf{j}}$ consists of unique vertex A of $P_{\mathbf{j}}$, so that $\overline{L\backslash P_{\mathbf{j}}}\cap P_{\mathbf{j}}=\{A\}$.

Now, let $L\cap(P_{\mathbf{j}}\backslash S_{\mathbf{j}}(V_P))\neq\varnothing$. Since $L\backslash P_{\mathbf{j}}\subset P^c_{\mathbf{j}}$, we have

$$\overline{L\backslash P_{\mathbf{j}}}\cap P_{\mathbf{j}}\subset P^c_{\mathbf{j}}\cap P_{\mathbf{j}}\subset S_{\mathbf{j}}(V_P)$$

The set on the left side is nonempty because L intersects both $P_{\mathbf{j}}\backslash S_{\mathbf{j}}(V_P)$ and $\widetilde{P}^{(k)}\backslash P_{\mathbf{j}}$. □

2.1. The main tree and ramification points.

Let γ_{ij} be the arc in K, connecting the vertices A_i and A_j. As it was proved by C. Bandt [2] in more general situation of post-critically finite sets, these arcs are the components of the attractor of a graph-directed system of similarities. We emphasise that this system is a Jordan multizipper [17]:

Theorem 15. *The arcs* γ_{ij} *are the components of an invariant set of some multizipper* $\mathcal{Z}$.

Proof. We say that the polygons $P_{i_1},\ldots,P_{i_m}$ form a chain connecting x and y, if $P_{i_1}\ni x, P_{i_m}\ni y$ and $P_{i_k}\bigcap P_{i_l}$ is empty if $|l-k|>1$ and is a common vertex of P_{i_k} and P_{i_l} when $|l-k|=1$.

For any A_i,A_j, there is a unique chain of polygons $P_{ijk}, k=1,\ldots m_{ij}$ connecting A_i and A_j.

Let $u(i,j,k)$ и $v(i,j,k)$ be the indices of vertices P, for which

$S'_{ijk}(A_u) = P'_{ij(k-1)} \bigcap P'_{ijk} = z_{ij(k-1)}$
and $S'_{ijk}(A_v) = P'_{ijk} \bigcap P'_{ij(k+1)} = z_{ijk}$, if $1 < k < m_{ij}$,
and if $k = 1$ or $k = m_{ij}$, $u(i, j, 1) = A_i = z_{ij0}$ and $v(i, j, m_{ij}) = A_j = z_{ijm_{ij}}$.

Thus for any triple (i, j, k),$1 \le k \le m_{ij}$, such $u, v \in \{1, ..., n_P\}$ are specified, that $S'_{ijk}(z_{uv0}) = z_{ij(k-1)}$ and $S'_{ijk}(z_{uvm_{uv}}) = z_{ijk}$.

Therefore the system $\{S'_{ijk}\}$ is a multizipper $\mathcal{Z}$ with node points z_{ijk}.

Since the relations:

$$\gamma_{ij} = \bigcup_{i=1}^{m_{ij}} S'_{ijk}(\gamma_{u(i,j,k),v(i,j,k)}) = \bigcup_{i=1}^{m_{ij}} \gamma_{ijk}$$

are satisfied, the arcs γ_{ij} form a complete set of the components of the attractor of the multizipper $\mathcal{Z}$.

Since each arc γ_{ijk} lies in P_{ijk},

$$\gamma_{ijk} \bigcap \gamma_{ijl} = \varnothing,$$

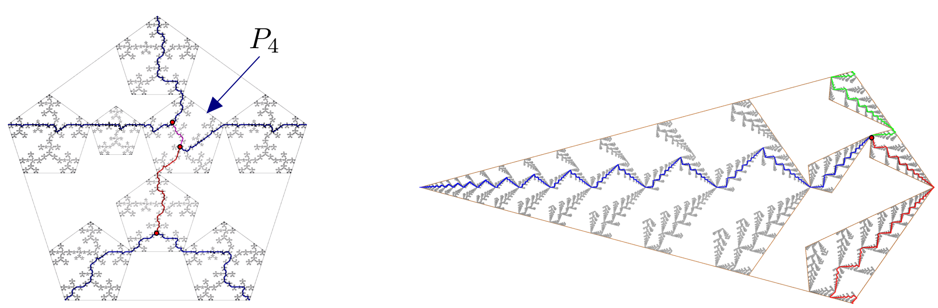
if $|k - l| > 1$ and

$$\gamma_{ijk} \bigcap \gamma_{ijl} = \{z_{ijk}\},$$

and $l = k \pm 1$. Therefore $\mathcal{Z}$ satisfies the conditions of the Theorem 8 and is a Jordan multizipper. □

Definition 16. *The union* $\hat{\gamma} = \bigcup_{i \ne j} \gamma_{ij}$ *is called* the main tree *of the dendrite* K. *The ramification points of the tree* $\hat{\gamma}$ *are called* the main ramification points *of the dendrite* K.

Example 2.3.



Two polygonal dendrites, their main trees and main ramification points. Surprisingly, instead of having a main ramification point of order 4, the polygon P_4 has two main ramification points of order 3. In second dendrite, the only main ramification point is a common vertex of 2 polygons.

There is a simple way to know whether a point $x \in K$ lies in $\hat{\gamma}$ and belongs to the set $CP(\hat{\gamma})$ of its cut points or to the set $EP(\hat{\gamma})$ of its end points:

Lemma 17. *Let* $x \in K$. *(a)* $x \in CP(\hat{\gamma})$ *iff there are vertices* A_{i_1}, A_{i_2}, *belonging to different components of* $K \backslash \{x\}$; *(b)* $x \in EP(\hat{\gamma})$ *iff* x *is a vertex and* $x \notin CP(\hat{\gamma})$.

Proof. First part of (a) is obvious. Since the union $\gamma_{xA_{i_1}} \cup \gamma_{xA_{i_2}}$ is a Jordan arc, it is equal to $\gamma_{i_1 i_2}$. So x is a cut point of $\gamma_{i_1 i_2}$, and therefore of $\hat{\gamma}$. To check (b), suppose $x \in \hat{\gamma}$ is not a vertex, then x lies in some $\gamma_{i_1 i_2}$, so it is a cut point of $\hat{\gamma}$. The second part of (b) is obvious. □

There are points in K for which their order in K and in $\hat{\gamma}$ is the same:

Lemma 18. *Let $x \in CP(K)$. If each component C_l of $K \backslash \{x\}$ contains a vertex of P, then $Ord(x, K)$ is finite and $Ord(x, K) = Ord(x, \hat{\gamma})$*

Proof. The number of components of $K\backslash\{x\}$ is not greater than n, so it's finite. Let $C_l, l = 1, ..., k, k = Ord(x, K)$ be the components of $K\backslash\{x\}$. By Lemma 17, $x \in \hat{\gamma}$. It also follows from Lemma 17 that two vertices A_{i_1} and A_{i_2} lie in the same component C_l if and only if $x \notin \gamma_{i_1 i_2}$. Therefore, all the vertices of P, belonging to the same component C_l of $K\backslash\{x\}$, belong to the same component of $\hat{\gamma}\backslash\{x\}$. Therefore $Ord(x, \hat{\gamma}) = Ord(x, K)$. □

For $x \in \mathbb{R}$, we denote by $\lceil x \rceil$ *the ceiling of* x, or the minimal integer n which is greater or equal to x.

Proposition 19. *a) For any $x \in \hat{\gamma}$, $\hat{\gamma} = \bigcup\limits_{j=1}^{n} \gamma_{A_j x}$.*
b) A_i is a cut point of $\hat{\gamma}$, if there are j_1, j_2 such that $\gamma_{j_1 i} \cap \gamma_{j_2 i} = \{A_i\}$;
c) the only end points of $\hat{\gamma}$ are the vertices A_j such that $A_j \notin CP(\hat{\gamma})$;
d) if $\#\pi^{-1}(A_i) = 1$, then $Ord(A_i, K) \le n_P - 1$, otherwise $Ord(A_i, K) \le (n_P - 1)(\left\lceil \dfrac{\theta_{max}}{\theta_{min}} \right\rceil - 1)$, where $\theta_{max}, \theta_{min}$ are maximal and minimal values of vertex angles of P.

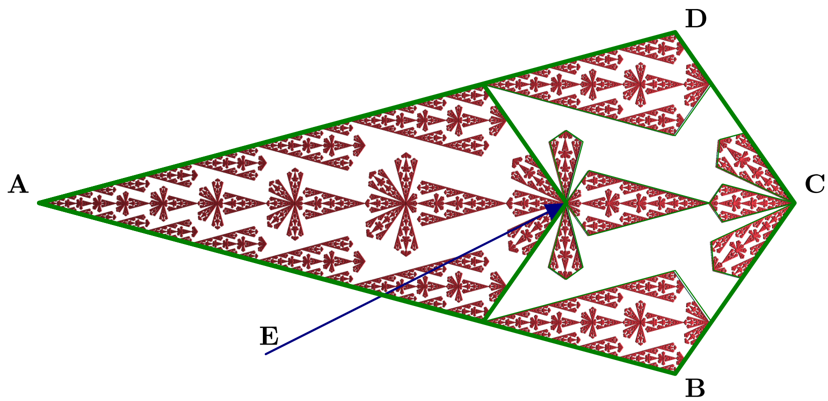
Proof. For any j_1, j_2, $\gamma_{j_1 j_2} \subset \gamma_{A_{j_1} x} \cup \gamma_{A_{j_2} x}$, which implies a). Repeating argument of Lemma 17, we see that A_i is a cut point of $\gamma_{i_1 i_2}$ and therefore of $\hat{\gamma}$, thus proving b). If $x \in \hat{\gamma}$ is not a vertex, then for some j_1, j_2, $x \in \gamma_{j_1 j_2}$, so x is a cut point of $\gamma_{j_1 j_2}$ and therefore of $\hat{\gamma}$, which implies c).

Let $\{C_l, l = 1, ..., k\}$ be some set of components of $K\backslash\{A_i\}$. Since $\{A_i\}$ is the intersection of unique nested sequence of polygons $P_{j_1} \supset P_{j_1 j_2} \supset ... \supset P_{j_1..j_s}..$, there is such s, that $\operatorname{diam} P_{j_1..j_s} < \operatorname{diam} C_i$ for any $i = 1, ..., k$. Then, by Proposition 14, each C_l contains some vertex of $P_{j_1..j_s}$, different from A_i, therefore $k \le n_P - 1$ so $Ord(A_i, K) \le n_P - 1$ is finite. So we can suppose that we took $k = Ord(A_i, K)$ initially and $\{C_1, ..., C_k\}$ was the set of all components of $K \setminus \{A_i\}$.

Let $\mathbf{j} = j_1..j_s$ and $A_i = S_\mathbf{j}(A')$. The sets $C_l \cap P_\mathbf{j}$ are the components of $K_\mathbf{j}\backslash\{A_i\}$. Since $(K \cap P_\mathbf{j})\backslash\{A_i\} = S_\mathbf{j}(K\backslash\{A'\})$, there are k components C'_l of $K\backslash\{A'\}$, such that $S_\mathbf{j}(C'_l) = C_l \cap P_\mathbf{j}$. Since each set C'_l contains the vertices of P, by Lemma 18, $Ord(A', \hat{\gamma}) = Ord(A', K) = Ord(A_i, K) \le n_P - 1$.

Suppose $\#\pi^{-1}(A_i) > 1$, and let $P_{j_1} \supset P_{j_1 j_2} \supset ..$ and $P_{j'_1} \supset P_{j'_1 j'_2} \supset ... \supset P_{j'_1..j'_s}..$ be two different nested sequences of polygons whose intersection is A_i. For any two polygons $P_\mathbf{j}, P_{\mathbf{j}'}$ either their intersection is A_i or one of these polygons contains the other. Therefore, there is some k such that $P_{j_1..j_s} = P_{j'_1..j'_s}$ for $s < k$ and $P_{j_1..j_s} \cap P_{j'_1..j'_s} = \{A_i\}$ for $s \ge k$. Since the vertex angles of respective polygons at A_i form a decreasing sequence assuming finite set of values, both sequences of these values are eventually constant. These final values are greater or equal to θ_{min}. Therefore, there is a finite number of polygons $P_{\mathbf{j}_k} \ni A_i$, whose pairwise intersections are $\{A_i\}$, such that any other polygon $P_{\mathbf{j}'}$, containing A_i, either contains one of them, or is contained in some $P_{\mathbf{j}_k}$ and has the same vertex angle at A_i. Then $Ord(A_i, K) = \sum Ord(A_i, P_{\mathbf{j}_k}) = \sum Ord(A_i, S_{\mathbf{j}^k}(\hat{\gamma}))$. The number of polygons $P_{\mathbf{j}_k}$ is not greater than $\left\lceil \dfrac{\theta_{max}}{\theta_{min}} \right\rceil - 1$, therefore $Ord(A_i, K) \le (n_P - 1) \left\lceil \dfrac{\theta_{max}}{\theta_{min}} - 1 \right\rceil$ □

Example 2.4.



A polygonal system, generated by 9 maps of a quadrilateral ABCD with vertex angles 30, 110, 110 and 110 degrees. The main tree $\hat{\gamma}$ is the union of line segments AB, AC and AD. For the vertex A, $Ord(A, K) = Ord(A, \hat{\gamma}) = 3$. The vertices B and D have order 1 both in $\hat{\gamma}$ and in the dendrite K. The vertex C has $Ord(C, \hat{\gamma}) = 1$ but $Ord(C, K) = 9$. The point E has the maximal order 24. By Theorem 20, for this type of polygon, maximal possible order may be 33.

Theorem 20. *$CP(K) \subset \bigcup\limits_{\mathbf{j}\in I^*} S_{\mathbf{j}}(\hat{\gamma})$.*
For each cut point $y \in K \setminus \bigcup\limits_{\mathbf{j}\in I^*} S_{\mathbf{j}}(\hat{\gamma})$ there is $S_{\mathbf{i}}$ and $x \in \hat{\gamma}$, such that $y = S_{\mathbf{i}}(x)$ and $Ord(y, K) = Ord(x, \hat{\gamma})$.
Otherwise, there are multiindices $\mathbf{i}_k, k = 1, .., s$ and vertices $x_1, ..., x_s$, such that for any k, $S_{\mathbf{i}_k}(x_k) = y$, for any $l \ne k$, $S_{\mathbf{i}_k}(P) \cap S_{\mathbf{i}_l}(P) = \{y\}$ and $Ord(y, K) = \sum\limits_{k=1}^{s} Ord(x_k, \hat{\gamma}) \le (n_P - 1)\left(\left\lceil \dfrac{2\pi}{\theta_{min}} \right\rceil - 1\right)$.

Proof. Let $\{C_1, ..., C_k\}, k > 1$, be some set of the components of $K \setminus \{y\}$. Take $0 < \rho < \min\limits_{i=1,...,k} \operatorname{diam}(C_i)$. Let $\mathbf{j} \in I^*$ be a multiindex such that $P_{\mathbf{j}} \ni y$ and $\operatorname{diam}(P_{\mathbf{j}}) \le \rho$ and let $y = S_{\mathbf{j}}(x)$.

Suppose the point x is not a vertex of the polygon P. Then $y \in \dot{P}_{\mathbf{j}}$ and the sets $C_i \cap P_{\mathbf{j}}$ are the components of $K_{\mathbf{j}} \setminus \{y\}$. Since $(K \cap P_{\mathbf{j}}) \setminus \{y\} = S_{\mathbf{j}}(K \setminus \{x\})$, there are k components C'_i of $K \setminus \{x\}$, such that $S_{\mathbf{j}}(C'_i) = C_i \cap P_{\mathbf{j}}$. By Proposition 14, each set C'_i contains the vertices of P, therefore $k \le n$ and $Ord(y, K) \le n$. So we can suppose that we took $k = Ord(y, K)$ initially and $\{C_1, ..., C_k\}$ was the set of all components of $K \setminus \{y\}$. Since each set C'_i contains the vertices of P, by Lemma 18, $Ord(x, \hat{\gamma}) = Ord(x, K) = Ord(y, K)$.

The proof of the last part repeats the proof of d) in Proposition 19. Suppose $y \in G_{\mathcal{S}}(V_P)$. Considering nested sequences of polygons $P_{j_1} \supset P_{j_1 j_2} \supset ..$ whose intersection is y, we see that there is a finite number of polygons $P_{\mathbf{j}_k} \ni y$, whose pairwise intersections are $\{y\}$, such that any other polygon $P_{\mathbf{j}'}$, containing y, either contains one of them, or is contained in some $P_{\mathbf{j}_k}$ and has the same vertex angle at y. Then the number of polygons $P_{\mathbf{j}_k}$ is not greater than $\left\lceil \frac{2\pi}{\theta_{min}} \right\rceil - 1$, therefore $Ord(y, K) \leq (n_P - 1)\left\lceil \frac{2\pi}{\theta_{min}} - 1 \right\rceil$. □

Corollary 21. *Let $(P, \mathcal{S})$ be a polygonal tree system and K be its attractor. (i) For any $x \in K$, $\pi^{-1}(x)$ contains at most $(n_P - 1)\left(\left\lceil \frac{2\pi}{\theta_{min}} \right\rceil - 1\right)$ elements;*
(ii) The system $\mathcal{S}$ is post-critically finite.

Proof. (i) was proved in previous Theorem. Since post-critical set is contained in $\pi^{-1}(V_P)$, it is finite. □

2.2. **Metric properties of polygonal dendrites.** Following [18], we remind that for $c \geq 1$, a set $A \subset R^n$ is of *c-bounded turning* if each pair of points $a, b \in A$ can be joined by a continuum $F \subset A$ with diameter $\operatorname{diam}(F) \leq c|a - b|$. In this subsection we prove that a dendrite K, defined by a polygonal tree system, is of c-bounded turning for some $c \geq 1$.

Lemma 22. *Let $\{P, \mathcal{S}\}$ be a polygonal tree system. There is such ρ that*
(i) for any vertex A, $V_\rho(A) \bigcap P_k \neq \varnothing \Rightarrow P_k \ni A$;
(ii) for any $x, y \in P$ such that there are $P_k, P_l : x \in P_k, y \in P_l$ and $P_k \bigcap P_l = \varnothing, d(x, y) \geq \rho$. □

Let α denote the minimal angle between the sides of polygons P_i, P_j, having common vertex.

Lemma 23. *For any vertex A of P and for any $x \in K \setminus \{A\}$,*

$$\frac{\operatorname{diam} \gamma_{Ax}}{d(x, A)} \leq \frac{\operatorname{diam} P}{\rho}$$

Proof. There are such $i_1, \dots, i_{k+1}$ that $A \in S_{i_1 \dots i_{k+1}}(P)$ and $x \in S_{i_1 \dots i_k}(P) \setminus S_{i_1 \dots i_{k+1}}(P)$. Let $x' = S^{-1}_{i_1 \dots i_k}(x)$ and $A' = S^{-1}_{i_1 \dots i_k}(A)$. Then $x' \in P \setminus P_{i_{k+1}}$ and $A' \in P_{i_{k+1}}$, so $d(x', A') \geq \rho$, and $\frac{\operatorname{diam} \gamma_{x'A'}}{d(x', A')} \leq \frac{\operatorname{diam} P}{\rho}$. Since $S_{i_1 \dots i_k}(\gamma_{x'A'}) = \gamma_{xA}$, we get $\frac{\operatorname{diam} \gamma_{xA}}{d(x, A)} \leq \frac{\operatorname{diam} P}{\rho}$. □

Lemma 24. *If $x \in S_k(K), y \in S_l(K), P_k \cap P_l = A$ and $x \neq y$, then*

$$\frac{\operatorname{diam} \gamma_{xy}}{d(x, y)} \leq \frac{\operatorname{diam} P}{\rho \sin(\alpha/2)}$$

.

Proof. $\frac{d(x, y)}{d(x, A) + d(A, y)} \geq \frac{\sqrt{d(x, A)^2 + d(A, y)^2 - 2d(x, A)d(A, y)\cos\alpha}}{d(x, A) + d(A, y)}$.

The minimum value for the right side of equation over all $d(x,A), d(y,A)$ is $\sin\alpha/2$, while, by Lemma 23,

$$\frac{d(x,A)+d(A,y)}{\operatorname{diam}\gamma_{xy}} \geq \frac{\rho}{\operatorname{diam} P} \tag{2}$$

Therefore we have $\dfrac{\operatorname{diam}\gamma_{xy}}{d(x,y)} \leq \dfrac{\operatorname{diam} P}{\rho\sin(\alpha/2)}$. □

Lemma 25. *For any $x,y \in K$, $\dfrac{\operatorname{diam}\gamma_{xy}}{d(x,y)} \leq \dfrac{\operatorname{diam} P}{\rho\sin(\alpha/2)}$.*

Proof. There are such $i_1,\dots,i_k,i_{k+1}$ that $x \in S_{i_1\dots i_{k+1}}(P)$ and $y \in S_{i_1\dots i_k}(P \setminus P_{i_{k+1}})$. Let $x' = S^{-1}_{i_1\dots i_k}(x), y' = S^{-1}_{i_1\dots i_k}(y)$. Suppose $y' \in P_l$.

If $P_l \bigcap P_{i_{k+1}} = \varnothing$, then $\dfrac{\operatorname{diam}\gamma_{x'y'}}{d(x',y')} \leq \dfrac{\operatorname{diam} P}{\rho}$.

If P_l and $P_{i_{k+1}}$ have a common vertex, then $\dfrac{\operatorname{diam}\gamma_{x'y'}}{d(x',y')} \leq \dfrac{\operatorname{diam} P}{\rho\sin\alpha/2}$.

Thus we have,

$$\frac{\operatorname{diam}\gamma_{xy}}{d(x,y)} \leq \frac{\operatorname{diam} P}{\rho\sin\alpha/2}.$$

□

From previous three Lemmas we immediately get the following

Theorem 26. *The attractor K of a polygonal tree system $\mathcal{S}$ is a continuum with bounded turning.* □

2.3. **Morphisms of polygonal dendrites.** In the following Theorem we admit that the enumeration of the vertices of the polygons P and P' needs not follow any order, and all permutations of indices are allowed.

Theorem 27. *Let dendrites K, K' be the attractors of polygonal tree systems $\mathcal{S} = \{S_1, S_2, \dots S_m\}$ and $\mathcal{S}' = \{S'_1, S'_2, \dots S'_m\}$ associated with polygons P, P' whose vertices $A_1, ..., A_n$ and $A'_1, ..., A'_n$ satisfy the conditions*

(i) For any $i,j = 1,...,n$, $S_k(A_i) = A_j$ iff $S'_k(A'_i) = A'_j$;

(ii) For any $i,j = 1,...,n$ $S_{k_1}(A_i) = S_{k_2}(A_j)$ iff $S'_{k_1}(A'_i) = S'_{k_2}(A'_j)$.

Then there is a bi-Hölder homeomorphism $\psi : K \to K'$ such that for any $i = 1,...,m$, $\psi \circ S_i = S'_i \circ \psi$.

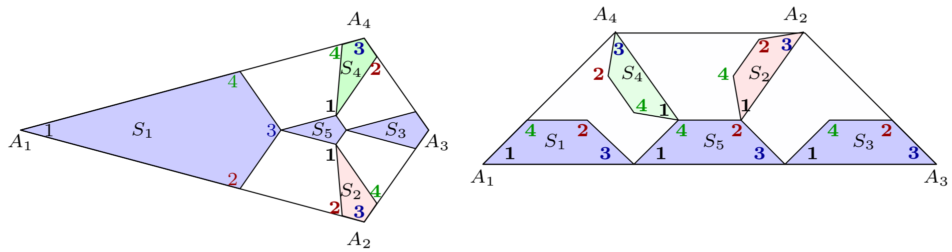
Proof. **1.** *The condition (i) implies that for any multiindex $\mathbf{k} = k_1k_2...k_l \in I^*$ the equality $S_{\mathbf{k}}(A_i) = A_j$ holds iff $S'_{\mathbf{k}}(A'_i) = A'_j$.*

Indeed, it's true for $l = 1$; proceeding by induction, let the condition (i) be true for any $k_1k_2...k_l \in I^l$ and $i,j \in \{1,\dots,n\}$, i.e.

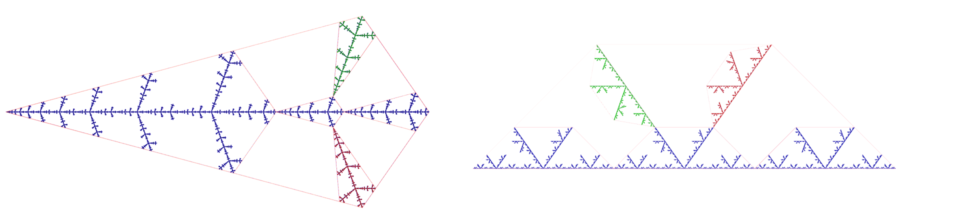
$$S_{k_1\dots k_l}(A_i) = A_j \iff S'_{k_1\dots k_l}(A'_i) = A'_j$$

Suppose for some $k_1k_2...k_{l+1} \in I^{l+1}$ and some vertices A_i, A_j we have $S_{k_1k_2...k_{l+1}}(A_i) = A_j$.

Consider the point $S_{k_2...k_lk_{l+1}}(A_i) = S_{k_1}^{-1}(A_j)$. This point is some vertex A_{i_1} of P. Since the multiindex $k_2, \dots, k_l, k_{l+1}$ is of length l, $S'_{k_2...k_lk_{l+1}}(A'_i) = A'_{i_1}$ by induction hypothesis. At the same time, $S'_{k_1}(A'_{i_1}) = A'_j$. Therefore $S'_{k_1k_2...k_lk_{l+1}}(A'_i) = A'_j$.



Permutation of the vertices defining an isomorphism of two polygonal tree systems. The respective attractors are shown below.



2. *The condition (ii) implies that for any multiindices $p_1...p_k$ and $q_1...q_l$ the equality $S_{p_1...p_k}(A_i) = S_{q_1...q_l}(A_j)$ holds iff $S'_{p_1...p_k}(A'_i) = S'_{q_1...q_l}(A'_j)$.*

Suppose for some multiindices $p_1...p_k$ and $q_1...q_l$ and vertices A_i, A_j, $S_{p_1...p_k}(A_i) = S_{q_1...q_l}(A_j)$.
Rewrite it as $S_{p_1}(S_{p_2...p_k}(A_i)) = S_{q_1}(S_{q_2...q_l}(A_j))$.
Since $S_{p_2...p_k}(A_i) = S_{p_1}^{-1}(A_j)$, this point must be some vertex A_{i_1} of P. Similarly, we also have $S_{q_2...q_l}(A_j) = A_{j_1}$.

From (i) it follows that $S'_{p_2...p_l}(A'_i) = A'_{i_1}$ and $S'_{q_2...q_l}(A'_j) = A'_{j_1}$ and from $S_{p_1}(A_{i_1}) = S_{q_1}(A_{j_1})$ by (ii) it follows that $S'_{p_1}(A'_{i_1}) = S'_{q_1}(A'_{j_1})$.
Therefore, we have $S'_{p_1...p_k}(A'_i) = S'_{q_1...q_l}(A'_j)$.

3. *There is a bijection $\varphi : K \to K'$, such that for any $i \in I$, $\varphi \cdot S_i = S'_i \cdot \varphi$.*

Consider the index maps $\pi : I^\infty \to K$ and $\pi' : I^\infty \to K'$.
Suppose for some $\mathfrak{p} = p_1p_2p_3.... \in I^\infty$ and $\mathfrak{q} = q_1q_2q_3.... \in I^\infty$, $\pi(\mathfrak{p}) = \pi(\mathfrak{q}) = \{x\}, x \in K$.
Then for any $k, l \in \mathbb{N}$, $P_{p_1...p_k} \cap P_{q_1...q_l} = \{x\}$, so there are such vertices A_{i_k}, A_{j_l} that $S_{p_1...p_k}(A_{i_k}) = S_{q_1...q_l}(A_{j_l}) = x$. Then, for any k, l, $S'_{p_1...p_k}(A'_{i_k}) = S'_{q_1...q_l}(A'_{j_l})$. These equations imply the points $S'_{p_1...p_k}(A'_{i_k})$ and $S'_{q_1...q_l}(A'_{j_l})$ coincide for all k, l

and therefore $\bigcap\limits_{k=1}^{\infty} P'_{p_1\dots p_k} = \bigcap\limits_{l=1}^{\infty} P'_{q_1\dots q_l}$. Applying this to all possible sequences $\mathfrak{p} \in \pi^{-1}(x)$, we obtain that $\pi'(\pi^{-1}(x))$ is a unique point, which we denote as x'.

Denote the map $\pi' \cdot \pi^{-1} : K \to K'$ by φ. Since the same argument shows that $\pi \cdot \pi'^{-1} : K' \to K$ is the inverse map to φ, the map φ is a bijection.

Since π and π' are compatible with the self-similar structure on I^∞, K and K', the same is true for $\varphi = \pi' \cdot \pi^{-1}$.

4. *The maps φ and φ^{-1} are Hölder continuous.*

Denote $r_i = \operatorname{Lip} S_i, r'_i = \operatorname{Lip} S'_i, \beta = \min\limits_{i=1,\dots,m} \dfrac{\log r'_i}{\log r_i}$, $\beta' = \min\limits_{i=1,\dots,m} \dfrac{\log r_i}{\log r'_i}$. Let also $|P|, |P'|$ be the diameters of P and P' respectively. Let ρ and ρ' denote the minimal distances specified by Lemma 22 for the systems $\mathcal{S}$ and $\mathcal{S}'$ respectively and let α, α' be respective minimal angles.

Observe that for any multiindex $\mathbf{i} = i_1, \dots, i_k, r'_{\mathbf{i}} \le r^\beta_{\mathbf{i}}$

Take some $x, y \in K$. There is a multiindex $i_1 \dots i_k$ such that $\{x, y\} \subset P_{i_1\dots i_k}$ and for any i_{k+1}, $\{x, y\} \not\subset P_{i_1\dots i_k i_{k+1}}$. Then there are two possibilities:

a) For some pair of multiindices, $i_1 \dots i_k j$ and $i_1 \dots i_k l$,
$P_{i_1\dots i_k j} \cap P_{i_1\dots i_k l} = \varnothing$, $\quad x \in P_{i_1\dots i_k j}$ $\quad$ and $y \in P_{i_1\dots i_k l}$.

Then $d(x, y) \le r_{i_1\dots i_k}|P|$, while by Lemma 22, $d(x, y) \ge r_{i_1\dots i_k}\rho$.
In this case, $r_{i_1\dots i_k}\rho < d(x, y) \le r_{i_1\dots i_k}|P|$.
The same way, for the system $\mathcal{S}'$ we have $r'_{i_1\dots i_k}\rho'_1 < d(x', y') \le r'_{i_1\dots i_k}|P'|$.

But $r'_{i_1\dots i_k} \le r^\beta_{i_1\dots i_k}$, therefore $d(x', y') \le r^\beta_{i_1\dots i_k}|P'| \le \left(\dfrac{d(x, y)}{\rho}\right)^\beta |P'|$.

b) There are $i_1 \dots i_k i_{k+1}$ and $j_1 \dots j_l j_{l+1}$, such that $x \in P_{i_1\dots i_k} \setminus P_{i_1\dots, i_k i_{k+1}}$, $y \in P_{j_1\dots, j_l} \setminus P_{j_1\dots j_l j_{l+1}}$ and $P_{i_1\dots i_k i_{k+1}} \bigcap P_{j_1\dots j_l j_{l+1}} = S_{i_1\dots i_k}(A)$, where A is some vertex of P.

In this case $d(x, y) \le \{r_{i_1\dots i_k} + r_{j_1\dots j_l}\}|P|$.
By Lemma 23, $d(x, A) \ge r_{i_1\dots i_k}\rho$ and $d(A, y) \ge r_{j_1\dots j_l}\rho$.
Therefore, by Lemma 24, $d(x, y) \ge \rho \cdot \sin(\alpha/2)(r_{i_1\dots i_k} + r_{j_1\dots j_l})$, thus

$$(r_{i_1\dots i_k} + r_{j_1\dots j_l})\rho \cdot \sin(\alpha/2) \le d(x, y) \le (r_{i_1\dots i_k} + r_{j_1\dots j_l})|P|.$$

Similarly, for the system $\mathcal{S}'$ we have

$$(r'_{i_1\dots i_k} + r'_{j_1\dots j_l})\rho' \cdot \sin(\alpha'/2) \le d(x', y') \le (r'_{i_1\dots i_k} + r'_{j_1\dots j_l})|P'|.$$

Suppose $r_{i_1\dots i_k} \ge r_{j_1\dots j_l}$. Then, $(r_{i_1\dots i_k})\rho \cdot \sin(\alpha/2) \le d(x, y) \le 2(r_{i_1\dots i_k})|P|$.

So, $d(x', y') \le 2(r_{i_1\dots i_k})^\beta |P'| \le 2\left(\dfrac{d(x, y)}{\rho \cdot \sin(\alpha/2)}\right)^\beta |P'|$. □

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Mary SAMUEL,
Department of Mathematics,
Bharatha Mata College, Kochi, India
E-mail address: marysamuel2000@gmail.com

Andrey V. Tetenov,
Novosibirsk State University &
Gorno-Altaisk State University
E-mail address: atet@mail.ru

Dmitry A. Vaulin,
Gorno-Altaisk State University
E-mail address: d_warrant@mail.ru