S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 14, стр. 774–793 (2017) УДК 517.95
DOI 10.17377/semi.2017.14.066 MSC 35K92

EXISTENCE OF ENTROPY MEASURE-VALUED SOLUTIONS FOR FORWARD-BACKWARD p-PARABOLIC EQUATIONS

S.N. ANTONTSEV, I.V. KUZNETSOV

Abstract. In this paper we have proved that the Dirichlet problem for the forward-backward p-parabolic equation has an entropy measure-valued solution which has been obtained as a singular limit of weak solutions and their gradients to the Dirichlet problem for the elliptic equation containing the anisotropic $(p, 2)$-Laplace operator. In order to guarantee the existence of entropy measure-valued solutions, the initial and final conditions should be formulated in the form of integral inequalities. This means that an entropy measure-valued solution can deviate from both initial and final data on the boundary. Moreover, a gradient Young measure appears in the representation of an entropy measure-valued solution. The uniqueness of entropy measure-valued solutions is still an open question.

Keywords: anisotropic Laplace operator, entropy measure-valued solution, forward-backward parabolic equation, gradient Young measure

Introduction

Entropy solutions were constructed in [16] in order to obtain unique weak solutions to the Cauchy problem for multi-dimensional scalar conservation laws. It is important to note that, initially, entropy solutions were proposed in [20] for hyperbolic systems in one space variable.

In recent years methods invented for hyperbolic conservation laws were also applied for elliptic and degenerate parabolic equations. In this paper we are interested in forward-backward parabolic equations having the p-Laplace operator. We call

---

Antontsev, S.N., Kuznetsov, I.V., Existence of entropy measure-valued solutions for forward-backward p-parabolic equations.
© 2017 Antontsev S.N., Kuznetsov I.V.
The work was supported by the Russian Science Foundation (Grant 15–11–20019).
Received May, 28, 2017, published August, 16, 2017.

these equations as forward-backward p-parabolic equations. It is important to note that the research on the well-posedness of nonlinear forward-backward parabolic equations was started in [14, 23, 31]. For linear forward-backward parabolic equations with variable coefficients we address the reader to [5, 30].

The existence and uniqueness of entropy solutions for p-parabolic equations were proved in [2, 6, 7].

It is worth to mention that the presence of the p-Laplace operator can make the problem very complicated. For example, gradient Young measures take part in the representation of weak solutions for the p-wave equation [1] when $p \neq 2$.

For forward-backward parabolic equations with non-monotonic flux the existence of entropy solutions was established in [22, 25]. Taking into account unstable phase, the non-uniqueness of entropy solutions was proved in [28, 29]. Moreover, gradient Young measures were used in the representation of weak solutions for forward-backward parabolic equations [11, 12, 32, 33].

In the present paper we deal with entropy measure-valued solutions of forward-backward p-parabolic equations. The case $p = 2$ was treated in [17, 18]. In the present paper entropy solutions have been obtained as singular limits of weak solutions to the Dirichlet problem containing the anisotropic $(p, 2)$-Laplace operator. The general case of the anisotropic (p, q)-Laplace operator ($p > 1$, $q > 1$, $q \neq 2$) was treated in [4]. Here we assume that the case $q = 2$ enables to formulate the entropy boundary conditions (16a) and (16b) in the form of inequalities.

This paper is organized as follows. In section 1 we have formulated problem Π_0 and auxiliary problem Π_ε. In section 2 we have formulated the definition of the entropy measure-valued solution to problem Π_0 which is obtained as a singular limit of regularized solutions u_ε and their gradients ∇u_ε as $\varepsilon \to 0+$. In sections 3 and 4 we have proved results announced in section 2.

1. Elliptic regularization of problem Π_0

In this section we deal with the elliptic regularization of the forward-backward p-parabolic equation where the anisotropic $(p, 2)$-Laplace operator is involved (see the right hand side of (3)). It is important to note that in one-dimensional case the elliptic regularization of the forward-backward parabolic equation was analyzed in [23].

Let the scalar function $a(z)$ and the vector function $\boldsymbol{\varphi}(z)$ satisfy the following conditions.

Conditions on $a\&\boldsymbol{\varphi}$. *Let* $a \in C^2(\mathbb{R})$, $a(0) = 0$, $\boldsymbol{\varphi}(z) = (\varphi_1(z), \dots, \varphi_d(z))$, $z \in \mathbb{R}$, $\varphi_j \in C^2(\mathbb{R})$, $j = 1, \dots, d$, $\boldsymbol{\varphi}(0) = \mathbf{0}$. *Function* $a(z)$ *is non-monotonic and* $a'(z)$ *is not equal to zero identically on intervals of positive measure.*

Under Conditions on $a\&\boldsymbol{\varphi}$, we formulate problem Π_0.

Problem Π_0**.** *For arbitrary initial and final data* $u_0, u_T \in L^\infty(\Omega) \cap C^{1,\alpha}(\Omega)$ *the unknown function* $u : G_T \to \mathbb{R}$ *satisfies*

$$\partial_t a(u) + \operatorname{div}\boldsymbol{\varphi}(u) = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad (\boldsymbol{x}, t) \in G_T, \tag{1}$$

$$u|_{\Gamma_0} \approx u_0, \quad u|_{\Gamma_T} \approx u_T, \quad u|_{\Gamma_l} = 0, \tag{2}$$

in the form given in Definition 3.

Here we assume that $G_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^d$, $\Gamma_0 = \overline{\Omega} \times \{t = 0\}$, $\Gamma_T = \overline{\Omega} \times \{t = T\}$, $\Gamma_l = \partial\Omega \times [0,T]$, $p > 1$, the sign $\approx$ means the equality only on a part of the boundary.

Remark 1. *We formulate equation* (1) *in a sense of distributions. Since function* $a(z)$ *is non-monotonic on* $\mathbb{R}$, *equation* (1) *is a forward-backward p-parabolic equation. Moreover, a weak solution* $u \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega))$ *can deviate from initial and final data* u_0, $u_T \in W_0^{m,p}(\Omega)$. *Therefore, the difficulty of problem* Π_0 *is that equation* (1) *and boundary conditions* (2) *must be reformulated in the form of integral inequalities given in Definition 3.*

We are going to construct entropy measure-valued solutions as singular limits of weak solutions u_ε (and their gradients) to problem Π_ε as $\varepsilon \to 0+$.

Problem Π_ε. *For arbitrary initial and final data* u_0, $u_T \in W_0^{m,p}(\Omega)$ *the unknown function* u_ε *satisfies the Dirichlet problem*

$$\partial_t a(u_\varepsilon) + \operatorname{div}\boldsymbol{\varphi}(u_\varepsilon) = \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) + \varepsilon \partial_t^2 u_\varepsilon, \quad (\boldsymbol{x}, t) \in G_T, \tag{3}$$

$$u_\varepsilon|_{\Gamma_0} = u_0(\boldsymbol{x}), \quad u_\varepsilon|_{\Gamma_T} = u_T(\boldsymbol{x}), \quad u_\varepsilon|_{\Gamma_l} = 0, \tag{4}$$

in a weak sense, see Definition 1.

We assume here that $\varepsilon \in (0,1]$.

Definition 1. *Function* $u_\varepsilon \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$ *is called a weak solution to problem* Π_ε *if the following assertions hold:*

(EL.1) *Let* $\widehat{u} \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$ *be an arbitrary extension of functions* u_0 *and* u_T *into* G_T. *Therefore,* $u_\varepsilon - \widehat{u} \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W_0^{1,2}(0,T; L^2(\Omega))$.

(EL.2) *The equality*

$$\int_{G_T} \big(- a(u_\varepsilon)\partial_t \phi - \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla\phi + |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla\phi + \varepsilon \partial_t u_\varepsilon \partial_t \phi \big)\, d\boldsymbol{x}dt = 0 \tag{5a}$$

holds for every $\phi \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W_0^{1,2}(0,T; L^2(\Omega))$.

Remark 2. *We can reformulate* (5a) *in the equivalent way:*

$$\int_{G_T} \big(\partial_t a(u_\varepsilon)\phi + \operatorname{div}\boldsymbol{\varphi}(u_\varepsilon)\phi + |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla\phi + \varepsilon \partial_t u_\varepsilon \partial_t \phi \big)\, d\boldsymbol{x}dt = 0. \tag{5b}$$

Remark 3. *The extension* $\widehat{u} \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$ *of* u_0 *and* u_T *into* G_T *can be obtained if we assume that* u_0, $u_T \in W_0^{m,p}(\Omega)$, $m - \frac{d}{p} > 1 + \alpha$, $\alpha \in (0,1)$. *Here we use the continuous embedding* $W_0^{m,p}(\Omega) \subset C_0^{1,\alpha}(\Omega)$.

Proposition 1. *Under Conditions on* $a\&\boldsymbol{\varphi}$, *problem* Π_ε *has at least one weak solution* u_ε *for all* u_0, $u_T \in W_0^{m,p}(\Omega)$. *Moreover, the maximum principle*

$$\|u_\varepsilon\|_{L^\infty(G_T)} \leq M = \max\Big(\|u_0\|_{L^\infty(\Omega)}, \|u_T\|_{L^\infty(\Omega)} \Big) \tag{6}$$

and the energy inequality

$$\int_{G_T} \Big(|\nabla u_\varepsilon|^p + \varepsilon \, |\partial_t u_\varepsilon|^2 \Big) \; d\boldsymbol{x}dt < C_{(11)} \tag{7}$$

hold. The constant $C_{(11)}$ *is defined in inequality* (11) *and does not depend on* $\varepsilon \in (0,1]$.

It is important to note that the case $a(z) \equiv 0$ and $\boldsymbol{\varphi}(z) \equiv 0$ was studied in [9].

Proof of Proposition 1.

To establish the existence of a weak solution u_ε to problem Π_ε, we use energy methods [3] and well-known results on elliptic differential equations [8, 13, 19].

We are going to deduce a priori estimates (6) and (7). Let us introduce the function

$$u_\varepsilon^M = \max(u_\varepsilon - M, 0) = \begin{cases} u_\varepsilon - M & \text{if } u_\varepsilon > M, \\ 0 & \text{if } u_\varepsilon \le M, \end{cases} \quad u_\varepsilon^M\big|_{\partial G_T} = 0,$$

$$\nabla u_\varepsilon^M = \begin{cases} \nabla u_\varepsilon & \text{if } u_\varepsilon > M, \\ 0 & \text{if } u_\varepsilon \le M, \end{cases} \quad \partial_t u_\varepsilon^M = \begin{cases} \partial_t u_\varepsilon & \text{if } u_\varepsilon > M, \\ 0 & \text{if } u_\varepsilon \le M. \end{cases}$$

Putting $\phi = u_\varepsilon^M$ in (5a), we derive

$$\int_{G_T} \left(\left|\nabla u_\varepsilon^M\right|^p + \varepsilon \left|\partial_t u_\varepsilon^M\right|^2 \right) d\boldsymbol{x}dt = I_1 + I_2, \tag{8}$$

where

$$I_1 := \int_{G_T} a(u_\varepsilon)\partial_t u_\varepsilon^M \; d\boldsymbol{x}dt, \;\; I_2 := \int_{G_T} \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla u_\varepsilon^M \; d\boldsymbol{x}dt.$$

Taking into account the properties of u_ε^M, ∇u_ε^M and $\partial_t u_\varepsilon^M$, we have

$$I_1 = \int_{G_T} a(u_\varepsilon^M + M)\partial_t u_\varepsilon^M \; d\boldsymbol{x}dt = \int_{G_T} \partial_t \left(\int_0^{u_\varepsilon^M} a(\lambda + M) \; d\lambda \right) \; d\boldsymbol{x}dt$$

$$= \int_\Omega \left(\int_0^{u_\varepsilon^M} a(\lambda + M) \; d\lambda \right) \; d\boldsymbol{x} \Bigg|_{t=0}^{t=T} = 0,$$

$$I_2 = \int_{G_T} \boldsymbol{\varphi}(u_\varepsilon^M + M) \cdot \nabla u_\varepsilon^M \; d\boldsymbol{x}dt = \int_{G_T} \operatorname{div} \left(\int_0^{u_\varepsilon^M} \boldsymbol{\varphi}(\lambda + M) \; d\lambda \right) \; d\boldsymbol{x}dt = 0.$$

Hence, according to (8) we get

$$\int_{G_T} \left(\left|\nabla u_\varepsilon^M\right|^p + \varepsilon \left|\partial_t u_\varepsilon^M\right|^2 \right) \; d\boldsymbol{x}dt = 0$$

and

$$u_\varepsilon^M = 0 \Longrightarrow u_\varepsilon \le M.$$

Analogously, we obtain

$$-u_\varepsilon \le M \text{ and } |u_\varepsilon| \le M.$$

To deduce the energy inequality (7), we consider an extension $\widehat{u}$ of u_0, $u_T \in W_0^{m,p}(\Omega)$ into G_T such that (see Remark 3)

$$\widehat{u} \in L^\infty(G_T) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)), \quad u_\varepsilon - \widehat{u}|_{\partial G_T} = 0.$$

In equation (5a) we take $\phi = u_\varepsilon - \widehat{u}$:

$$- \int_{G_T} \partial_t(u_\varepsilon - \widehat{u})a(u_\varepsilon) \; d\boldsymbol{x}dt + \int_{G_T} \nabla(u_\varepsilon - \widehat{u}) \cdot \nabla u_\varepsilon \, |\nabla u_\varepsilon|^{p-2} \; d\boldsymbol{x}dt$$

$$= \int_{G_T} \nabla(u_\varepsilon - \widehat{u}) \cdot \boldsymbol{\varphi}(u_\varepsilon) \; d\boldsymbol{x}dt - \varepsilon \int_{G_T} \partial_t(u_\varepsilon - \widehat{u})\partial_t u_\varepsilon \; d\boldsymbol{x}dt.$$

This reads in the following way

$$\int_{G_T} \left(|\nabla u_\varepsilon|^p + \varepsilon \left| \partial_t u_\varepsilon \right|^2 \right) \, d\boldsymbol{x}dt = J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 := \int_{G_T} \left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \widehat{u} + \varepsilon \partial_t u_\varepsilon \partial_t \widehat{u} \right) \, d\boldsymbol{x}dt, \; J_2 := \int_{G_T} a(u_\varepsilon) \partial_t u_\varepsilon \, d\boldsymbol{x}dt, \tag{9}$$

$$J_3 := -\int_{G_T} a(u_\varepsilon) \partial_t \widehat{u} \, d\boldsymbol{x}dt, \; J_4 := \int_{G_T} \boldsymbol{\varphi}(u_\varepsilon) \cdot (\nabla u_\varepsilon - \nabla \widehat{u}) \, d\boldsymbol{x}dt. \tag{10}$$

Applying the Young inequality

$$yz \le \frac{\epsilon^{r'}}{r'} y^{r'} + \frac{\epsilon^{-r}}{r} z^r, \quad 1 < r < \infty, \; r' = \frac{r}{r-1}, \; y, z \ge 0, \; \epsilon \in (0, 1],$$

we derive

$$|J_1| \le \int_{G_T} \left(\frac{\delta^{p'}}{p'} |\nabla u_\varepsilon|^p + \frac{\delta^{-p}}{p} |\nabla \widehat{u}|^p + \frac{\varepsilon \delta_1^2}{2} |\partial_t u_\varepsilon|^2 + \frac{\varepsilon \delta_1^{-2}}{2} |\partial_t \widehat{u}|^2 \right) \, d\boldsymbol{x}dt,$$

$$|J_3| \le \int_{G_T} \left(\frac{1}{2} |\partial_t \widehat{u}|^2 + \frac{1}{2} |a(u_\varepsilon)|^2 \right) \, d\boldsymbol{x}dt,$$

$$|J_4| \le \int_{G_T} \left(\frac{\delta_2^p}{p} |\nabla u_\varepsilon|^p + \frac{\delta_2^{-p'}}{p'} |\boldsymbol{\varphi}(u_\varepsilon)|^{p'} + \frac{1}{p} |\nabla \widehat{u}|^p + \frac{1}{p'} |\boldsymbol{\varphi}(u_\varepsilon)|^{p'} \right) \, d\boldsymbol{x}dt.$$

We evaluate the term J_2 in the following way

$$|J_2| = \left| \int_\Omega \left(\int_0^{u_\varepsilon(\boldsymbol{x},t)} a(\lambda) \, d\lambda \right) \, d\boldsymbol{x} \Bigg|_{t=0}^{t=T} \right| \le 2 \sup_{|z| \le M} \left| \int_0^z a(\lambda) \, d\lambda \right| |\Omega| \, .$$

Gathering the last estimates and choosing δ, δ_1 and δ_2 appropriately small, we find that

$$\begin{aligned} &\int_{G_T} \left(|\nabla u_\varepsilon|^p + \varepsilon |\partial_t u_\varepsilon|^2 \right) \, d\boldsymbol{x}dt \le C(p) \int_{G_T} \left(|\partial_t \widehat{u}|^2 + |\nabla \widehat{u}|^p + |a(u_\varepsilon)|^2 \right. \\ &\quad \left. + |\boldsymbol{\varphi}(u_\varepsilon)|^{p'} \right) \, d\boldsymbol{x}dt + \sup_{|z| \le M} \left| \int_0^z a(\lambda) \, d\lambda \right| |\Omega| \le C(p) \int_{G_T} \left(|\partial_t \widehat{u}|^2 + |\nabla \widehat{u}|^p \right) \, d\boldsymbol{x}dt \\ &\quad + C(p, |\Omega|, T) \Big(\sup_{|z| \le M} |a(z)|^2 + \sup_{|z| \le M} |\boldsymbol{\varphi}(z)|^{p'} + \sup_{|z| \le M} \left| \int_0^z a(\lambda) \, d\lambda \right| \Big) =: C_{(11)}. \end{aligned} \tag{11}$$

The maximum principle (6) and the energy inequality (7) imply that the operator L defined by the formula

$$\begin{aligned} L(u_\varepsilon, \vartheta) := \int_{G_T} \big(&- a(u_\varepsilon) \partial_t (\vartheta - \widehat{u}) - \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla (\vartheta - \widehat{u}) + |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla (\vartheta - \widehat{u}) \\ &+ \varepsilon \partial_t u_\varepsilon \partial_t (\vartheta - \widehat{u}) \big) \, d\boldsymbol{x}dt \end{aligned}$$

is coercive in the space $L^\infty(G_T)\cap L^p(0,T;W_0^{1,p}(\Omega))\cap W^{1,2}(0,T;L^2(\Omega))$, that is

$$L(u_\varepsilon,u_\varepsilon)=\int_{G_T}(|\nabla u_\varepsilon|^p+\varepsilon\,|\partial_t u_\varepsilon|^2)\;d\boldsymbol{x}dt-J_1-J_2-J_3-J_4>$$
$$\min\left((1-\frac{\delta^{p'}}{p'}-\frac{\delta_2^p}{p}),\varepsilon(1-\frac{\delta_1^2}{2})\right)\left(\int_{G_T}|\nabla u_\varepsilon|^p\;d\boldsymbol{x}dt+\int_{G_T}|\partial_t u_\varepsilon|^2\;d\boldsymbol{x}dt\right)-C_{(12)},$$

where J_1, J_2, J_3 and J_4 are defined in (9)–(10) and

$$C_{(12)}=\frac{1+\delta^{-p}}{p}\int_{G_T}|\nabla\widehat{u}|^p\;d\boldsymbol{x}dt+\frac{1+\varepsilon\delta_1^{-2}}{2}\int_{G_T}|\partial_t\widehat{u}|^2\;d\boldsymbol{x}dt$$
$$+|G_T|\Big(\frac{1}{2}\sup_{|z|\le M}|a(z)|^2+\frac{1+\delta_2^{-p'}}{p'}\sup_{|z|\le M}|\boldsymbol{\varphi}(z)|^p\Big)+2\,|\Omega|\sup_{|z|\le M}\left|\int_0^z a(\lambda)\;d\lambda\right|. \tag{12}$$

According to well known results (see [8, Theorem 8.5], [13], [19, Theorem 9.2, Ch. IV]), we conclude that problem Π_ε has at least one weak solution u_ε.

□

2. Definition of entropy measure-valued solution of problem Π_0. Existence Theorem 1

Here we need to introduce auxiliary functions.

Definition 2. *A triple of functions $(H,Q,\boldsymbol{\Phi})$ is called a boundary entropy-entropy flux triple if H, $Q\in C^2(\mathbb{R}^2)$, $\boldsymbol{\Phi}\in C^2(\mathbb{R}^2;\mathbb{R}^d)$ and for every $(z,k)\in\mathbb{R}^2$:*
$\partial_1 Q(z,k)=a'(z)\partial_1 H(z,k)$, $\partial_1\Phi_j(z,k)=\varphi_j'(z)\partial_1 H(z,k)$, $\partial_1^2 H(z,k)\ge 0$, $Q(z,z)=\partial_1 Q(z,z)=\Phi_j(z,z)=\partial_1\Phi_j(z,z)=H(z,z)=\partial_1 H(z,z)=0$, $j=1,\dots,d$, $\boldsymbol{\Phi}(z,k)=(\Phi_1(z,k),\dots,\Phi_d(z,k))$,
where ∂_1 means differentiation with respect to the first variable.

These functions were defined in [24]. Suppose that $\mathcal{A}_1=\|a'\|_{C[-M,M]}$, $M=\max(\|u_0\|_{L^\infty(\Omega)},\|u_T\|_{L^\infty(\Omega)})$ and u_0, $u_T\in W_0^{m,p}(\Omega)$.

Definition 3. *A measurable function $u:G_T\to\mathbb{R}$ and a gradient Young measure $\nu_{\boldsymbol{x},t}$ are called an entropy measure-valued solution if the following assertions hold:*

(FB.1) *(Regularity and maximum principle)* $u\in L^\infty(G_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$,

$$\|u\|_{L^\infty(G_T)}\le M. \tag{13}$$

(FB.2) *(Gradient Young measure) Here $\nu=\nu_{\boldsymbol{x},t}$ is a probability measure for a.e. $(\boldsymbol{x},t)\in G_T$, and $\langle\nu_{\boldsymbol{x},t},\boldsymbol{S}_p(\cdot)\rangle$ is defined as a dual pairing of the vector function $\boldsymbol{S}_p(\boldsymbol{\xi})=|\boldsymbol{\xi}|^{p-2}\,\boldsymbol{\xi}$ with the gradient Young measure $\nu_{\boldsymbol{x},t}$, i.e.*

$$\langle\nu_{\boldsymbol{x},t},\boldsymbol{S}_p(\cdot)\rangle:=\int_{\mathbb{R}^d}\boldsymbol{S}_p(\boldsymbol{\xi})\,d\nu_{\boldsymbol{x},t}(\boldsymbol{\xi}). \tag{14}$$

(FB.3) *(Entropy condition) The integral inequality*

$$(15)\quad -\int_{G_T}\Big(Q(u,k)\partial_t\gamma+\boldsymbol{\Phi}(u,k)\cdot\nabla\gamma-\partial_1 H(u,k)\langle\nu_{\boldsymbol{x},t},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\gamma - \partial_1^2 H(u,k)\,|\nabla u|^p\,\gamma\Big)\,d\boldsymbol{x}dt \le \mathcal{A}_1\int_\Omega\big(H(u_0(\boldsymbol{x}),k)\gamma(\boldsymbol{x},0) + H(u_T(\boldsymbol{x}),k)\gamma(\boldsymbol{x},T)\big)\,d\boldsymbol{x}$$

holds for every constant $k\in\mathbb{R}$, a nonnegative test function $\gamma\in\mathcal{D}(\Omega\times\mathbb{R})$ and a boundary entropy-entropy flux triple $(H,Q,\boldsymbol{\Phi})$.

In the following theorem it is asserted that an entropy measure-valued solution of problem Π_0 is the singular limit of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to problem Π_ε and the associated gradient Young measure.

Theorem 1. *Under Conditions on a&$\boldsymbol{\varphi}$, for given initial and final conditions u_0, $u_T\in W_0^{m,p}(\Omega)$ there exists an entropy measure-valued solution $u(\boldsymbol{x},t)$ to problem Π_0 such that*

$$u_\varepsilon(\boldsymbol{x},t)\to u(\boldsymbol{x},t)\quad in\quad L^p(G_T),$$

$$|\nabla u_\varepsilon(\boldsymbol{x},t)|^p\rightharpoonup\int_{\mathbb{R}^d}|\boldsymbol{\xi}|^p\;d\nu_{\boldsymbol{x},t}\quad in\quad L^1(G_T),$$

$$|\nabla u_\varepsilon(\boldsymbol{x},t)|^{p-2}\,\nabla u_\varepsilon(\boldsymbol{x},t)\rightharpoonup\int_{\mathbb{R}^d}|\boldsymbol{\xi}|^{p-2}\,\boldsymbol{\xi}\,d\nu_{\boldsymbol{x},t}\quad in\quad (L^{p'}(G_T))^d$$

as $\varepsilon\to 0+$.

The following definition was formulated in [21].

Definition 4. *Families of functions $\{\alpha_{0,\epsilon}(t)\}_{\epsilon>0}$ and $\{\alpha_{T,\epsilon}(t)\}_{\epsilon>0}$ are called boundary-layer sequences if $\alpha_{0,\epsilon}$, $\alpha_{T,\epsilon}\in C^2[0,T]$ and*

$$0\le\alpha_{0,\epsilon}(t)\le 1\quad\forall t\in[0,T];\quad \alpha_{0,\epsilon}(0)=0;\quad \lim_{\epsilon\to0+}\alpha_{0,\epsilon}(t)=1\quad\forall t\in(0,T];$$
$$0\le\alpha'_{0,\epsilon}(t)\le\frac{c}{\epsilon}\quad\forall t\in[0,\epsilon],\quad c>0;\quad \alpha_{0,\epsilon}(t)\equiv1\quad\forall t\in[\epsilon,T];$$

$$0\le\alpha_{T,\epsilon}(t)\le 1\quad\forall t\in[0,T];\quad \alpha_{T,\epsilon}(T)=0;\quad \lim_{\epsilon\to0+}\alpha_{T,\epsilon}(t)=1\quad\forall t\in[0,T);$$
$$-\frac{c}{\epsilon}\le\alpha'_{T,\epsilon}(t)\le 0\quad\forall t\in[T-\epsilon,T];\quad \alpha_{T,\epsilon}(t)\equiv1\quad\forall t\in[0,T-\epsilon];$$

$$\int_0^\epsilon\alpha'_{0,\epsilon}(\tau)\;d\tau=-\int_{T-\epsilon}^T\alpha'_{T,\epsilon}(\tau)\;d\tau=1.$$

Definition 5. *The following expressions are called essential one-sided limits of a function $f:(0,T)\to\mathbb{R}$:*

$$\operatorname*{esslim}_{t\to0+}f(t)=\lim_{\epsilon\to0+}\frac{1}{\epsilon}\int_0^\epsilon f(\tau)\,d\tau=\lim_{\epsilon\to0+}\int_0^\epsilon f(\tau)\alpha'_{0,\epsilon}(\tau)\,d\tau,$$

$$\operatorname*{esslim}_{t\to T-}f(t)=\lim_{\epsilon\to0+}\frac{1}{\epsilon}\int_{T-\epsilon}^T f(\tau)\,d\tau=-\lim_{\epsilon\to0+}\int_{T-\epsilon}^T f(\tau)\alpha'_{T,\epsilon}(\tau)\,d\tau.$$

Proposition 2. *If u is an entropy measure-valued solution to problem Π_0, it satisfies boundary conditions in the form of inequalities*

$$\operatorname*{esslim}_{t\to 0+}\int_\Omega Q(u(\boldsymbol{x},t),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\,d\boldsymbol{x} \le \mathcal{A}_1\int_\Omega H(u_0(\boldsymbol{x}),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\,d\boldsymbol{x}, \tag{16a}$$

$$\operatorname*{esslim}_{t\to T-}\int_\Omega Q(u(\boldsymbol{x},t),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\,d\boldsymbol{x} \ge -\mathcal{A}_1\int_\Omega H(u_T(\boldsymbol{x}),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\,d\boldsymbol{x}, \tag{16b}$$

for every boundary entropy-entropy flux triple $(H,Q,\boldsymbol{\Phi})$ and for all functions $\upsilon\in L^\infty(\Omega)$ and $\beta\in L^1(\Omega)$ where $\beta\ge 0$ a.e. in Ω.

Corollary 1. *From boundary conditions* (16a) *and* (16b) *it follows that*

$$\operatorname*{esslim}_{t\to 0+}\int_\Omega \operatorname{sgn}(u(\boldsymbol{x},t)-\upsilon(\boldsymbol{x}))(a(u(\boldsymbol{x},t))-a(\upsilon(\boldsymbol{x})))\beta(\boldsymbol{x})\,d\boldsymbol{x} \le \mathcal{A}_1\int_\Omega |u_0(\boldsymbol{x})-\upsilon(\boldsymbol{x})|\,\beta(\boldsymbol{x})\,d\boldsymbol{x}, \tag{17a}$$

$$\operatorname*{esslim}_{t\to T-}\int_\Omega \operatorname{sgn}(u(\boldsymbol{x},t)-\upsilon(\boldsymbol{x}))(a(u(\boldsymbol{x},t))-a(\upsilon(\boldsymbol{x})))\beta(\boldsymbol{x})\,d\boldsymbol{x} \ge -\mathcal{A}_1\int_\Omega |u_T(\boldsymbol{x})-\upsilon(\boldsymbol{x})|\,\beta(\boldsymbol{x})\,d\boldsymbol{x}, \tag{17b}$$

for every $\upsilon\in L^\infty(\Omega)$, $\beta\in L^1(\Omega)$, $\beta\ge 0$ a.e. in Ω.

Remark 4. *Corollary 1 can be proved with the help of Proposition 2 and Example 1 (subsection 4.2) in the limiting case as $\delta\to 0+$. If we assume that $a(z)$ in strictly increasing function on $[-M,M]$ and take $\upsilon=u_0$ in* (17a)*:*

$$\operatorname*{esslim}_{t\to 0+}\int_\Omega |a(u(\boldsymbol{x},t))-a(u_0(\boldsymbol{x}))|\,\beta(\boldsymbol{x})\,d\boldsymbol{x}\le 0,$$

it follows that

$$u|_{\Gamma_0}=u_0.$$

Moreover, (17b) *is fulfilled anyway. The opposite result holds for a strictly decreasing function $a(z)$ on $[-M,M]$:*

$$u|_{\Gamma_T}=u_T.$$

Corollary 2. *Let $u(\boldsymbol{x},t)$ be an entropy measure-valued solution for boundary value problem Π_0. Then the following conditions hold*

$$\operatorname*{esslim}_{t\to 0+}\int_\Omega \big(Q_0(u(\boldsymbol{x},t),\upsilon(\boldsymbol{x}))+Q_0(u(\boldsymbol{x},t),u_0(\boldsymbol{x})) - Q_0(\upsilon(\boldsymbol{x}),u_0(\boldsymbol{x}))\big)\beta(\boldsymbol{x})d\boldsymbol{x}\le 0, \tag{18a}$$

$$\operatorname*{esslim}_{t\to T-}\int_\Omega \big(Q_0(u(\boldsymbol{x},t),\upsilon(\boldsymbol{x}))+Q_0(u(\boldsymbol{x},t),u_T(\boldsymbol{x})) - Q_0(\upsilon(\boldsymbol{x}),u_T(\boldsymbol{x}))\big)\beta(\boldsymbol{x})d\boldsymbol{x}\ge 0, \tag{18b}$$

for every $\upsilon\in L^\infty(\Omega)$, $\beta\in L^1(\Omega)$, $\beta\ge 0$ a.e. in Ω.

Remark 5. *This corollary plays important role in the proof of the uniqueness of an entropy measure-valued solution when $p = 2$. During the proof of Corollary 2 one can repeat the scheme of Proposition 2 for the boundary entropy-entropy flux triple given in Example 2 (subsection 4.2). See Remark 4.*

The following result was proved in [17].

Theorem 2. *Put $p = 2$. Let functions $u(\boldsymbol{x}, t)$ and $v(\boldsymbol{x}, t)$ be the entropy measure-valued solutions of problem Π_0 with initial and final data $(u_0, u_T) \in (L^\infty(\Omega))^2$ and, correspondingly, $(v_0, v_T) \in (L^\infty(\Omega))^2$. Under Conditions on a&$\boldsymbol{\varphi}$ and* [17, *inequality* (70)]*, an entropy measure-valued solution is stable with respect to perturbations of the initial and final data:*

$$\|u - v\|_{L^1(G_T)} \leq C\big(\|u_0 - v_0\|_{L^1(\Omega)} + \|u_T - v_T\|_{L^1(\Omega)}\big). \tag{19}$$

The authors of the present paper do not know how to prove the uniqueness of entropy measure-valued solutions in the case $p \neq 2$ when function $a(z)$ is non-monotonic on interval $(-M, M)$.

3. Proof of Theorem 1

In this section we have used methods proposed in [16], [24] and modified in [17].

3.1. Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$.

Let us prove that for all initial and final data $u_0,\ u_T \in W_0^{m,p}(\Omega)$ an entropy measure-valued solution of problem Π_0 can be represented as the singular limit in $L^p(G_T) = L^p(0, T; L^p(\Omega))$ of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to problem Π_ε and the weak limit of their gradients $\{\nabla u_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0+$.

From Definition 1 it follows that

$$\int_{G_T} \partial_t w_\varepsilon \phi \, d\boldsymbol{x}dt = \int_{G_T} \big(\boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla\phi - |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla\phi\big) \, d\boldsymbol{x}dt,$$

where $w_\varepsilon = a(u_\varepsilon) - \varepsilon\partial_t u_\varepsilon$, $\phi \in L^p(0, T; W_0^{1,p}(\Omega))$. Therefore, $\partial_t w_\varepsilon \in L^{p'}(0, T; W_0^{-1,p}(\Omega))$ and

$$\left| \int_{G_T} \partial_t w_\varepsilon(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t) \, d\boldsymbol{x}dt \right| \leq C_{(20)} \|\phi\|_{L^p(0,T;W_0^{1,p}(\Omega))}, \tag{20}$$

where

$$C_{(20)} = \sup_{|z|\leq M} |\boldsymbol{\varphi}(z)| \, |G_T|^{\frac{p-1}{p}} + \big(C_{(11)}\big)^{\frac{p-1}{p}}.$$

Let $g(z) = \int_0^z (a'(\tau))^2 \, d\tau$ and an arbitrary function $\psi \in W_0^{s,p}(\Omega)$, $s \geq [\frac{d}{p}] + 1$, where $\|\psi\|_{L^\infty(\Omega)} \leq C(\Omega) \|\psi\|_{W_0^{s,p}(\Omega)}$.

We have

(21)
$$\begin{aligned}\int_\Omega \big(g(u_\varepsilon(\boldsymbol{x},t+h)) - g(u_\varepsilon(\boldsymbol{x},t))\big)\psi(\boldsymbol{x})\,d\boldsymbol{x} &= \int_\Omega\int_t^{t+h} \partial_s g(u_\varepsilon(\boldsymbol{x},s))\psi(\boldsymbol{x})\,d\boldsymbol{x}ds\\ &= \int_\Omega\int_t^{t+h} a'(u_\varepsilon(\boldsymbol{x},s))\partial_s w_\varepsilon(\boldsymbol{x},s)\psi(\boldsymbol{x})\,d\boldsymbol{x}ds\\ &- \int_\Omega\int_t^{t+h} \varepsilon a''(u_\varepsilon(\boldsymbol{x},s))\left|\partial_s u_\varepsilon(\boldsymbol{x},s)\right|^2\psi(\boldsymbol{x})\,d\boldsymbol{x}ds\\ &+ \int_\Omega \varepsilon\big(\partial_t a(u_\varepsilon(\boldsymbol{x},t+h)) - \partial_t a(u_\varepsilon(\boldsymbol{x},t))\big)\psi(\boldsymbol{x})\,d\boldsymbol{x}.\end{aligned}$$

We can use here [26, Theorem 5] for the set $\{g(u_\varepsilon)\}_{\varepsilon>0}$. We are going to estimate the right-hand side of (21). It follows from (20) that

(22)
$$\begin{aligned}&\left|\int_\Omega\int_t^{t+h} a'(u_\varepsilon(\boldsymbol{x},s))\partial_s w_\varepsilon(\boldsymbol{x},s)\psi(\boldsymbol{x})\,d\boldsymbol{x}ds\right|\\ &\qquad\le C_{(20)}\left\|a'(u_\varepsilon)\psi\right\|_{L^p(t,t+h;W_0^{1,p}(\Omega))}\\ &\le C_{(20)}\max(\mathcal{A}_1,\mathcal{A}_2)\left\|u_\varepsilon\right\|_{L^p(t,t+h;W_0^{1,p}(\Omega))} C(\Omega)\left\|\psi\right\|_{W_0^{s,p}(\Omega)} \to 0+ \ \text{ as } h\to 0+.\end{aligned}$$

Moreover,

(23)
$$\begin{aligned}&\left|\int_\Omega\int_t^{t+h} \varepsilon a''(u_\varepsilon(\boldsymbol{x},s))\left|\partial_s u_\varepsilon(\boldsymbol{x},s)\right|^2\psi(\boldsymbol{x})\,d\boldsymbol{x}ds\right|\\ &\qquad\le \mathcal{A}_2\int_\Omega\int_t^{t+h}\varepsilon\left|\partial_s u_\varepsilon(\boldsymbol{x},s)\right|^2\,d\boldsymbol{x}ds C(\Omega)\left\|\psi\right\|_{W_0^{s,p}(\Omega)} \to 0+ \ \text{ as } h\to 0+,\end{aligned}$$

(24)
$$\begin{aligned}&\left|\int_\Omega \varepsilon\big(\partial_t a(u_\varepsilon(\boldsymbol{x},t+h)) - \partial_t a(u_\varepsilon(\boldsymbol{x},t))\big)\psi(\boldsymbol{x})\,d\boldsymbol{x}\right| \le\\ &\left\|\varepsilon\partial_t a(u_\varepsilon(\cdot,t+h)) - \varepsilon\partial_t a(u_\varepsilon(\cdot,t))\right\|_{L^2(\Omega)}\left|\Omega\right|^{1/2} C(\Omega)\left\|\psi(\boldsymbol{x})\right\|_{W_0^{s,p}(\Omega)} \to 0+ \text{ as } h\to 0+,\end{aligned}$$

where $\mathcal{A}_1 = \sup\limits_{|z|\le M}|a'(z)|$, $\mathcal{A}_2 = \sup\limits_{|z|\le M}|a''(z)|$. The latter assertion is valid since the set $\left\{\varepsilon\partial_t a(u_\varepsilon)\right\}_{\varepsilon>0}$ is a relatively compact set in $L^2(G_T)$.

From assertions (21)–(24) follows the one-sided limit

(25)
$$\lim_{h\to 0+}\int_0^{T-h}\left\|g(u_\varepsilon(\cdot,t+h)) - g(u_\varepsilon(\cdot,t))\right\|_{W^{-s,p}(\Omega)}\,dt = 0.$$

From the energy inequality (7), $|\Omega| < \infty$, the maximum principle (6) and Conditions on $a\&\boldsymbol{\varphi}$ it follows that

(26)
$$\{g(u_\varepsilon)\}_{\varepsilon>0} \subset L^p(0,T;W_0^{1,p}(\Omega)) \cap L^p(0,T;L^p(\Omega)).$$

From (25) and (26) (see [26, theorem 5]) it follows that the set $\{g(u_\varepsilon)\}_{\varepsilon>0}$ is a relatively compact set in $L^p(0,T;L^p(\Omega)) = L^p(G_T)$. By the strict monotonicity of the function g the set $\{u_\varepsilon\}_{\varepsilon>0}$ is a relatively compact set in $L^p(G_T)$.

3.2. **Proof of inequality** (15)**.**

Consider the finite functions s and $\xi_K \in C_0[0,T]$ introduced in [24]:

$$s(t) = \min\{t, \frac{T}{10}, T-t\}, \quad \xi_K(t) = 1 - \exp\big(-\frac{s(t)}{K}\big), \quad t \in [0,T].$$

Lemma 1. *For any value of the positive parameter $\varepsilon \in (0,1]$ a weak solution u_ε to problem Π_ε satisfies the inequality:*

$$(27) \quad -\int_{G_T} \big(Q(u_\varepsilon,k)\partial_t\gamma + \mathbf{\Phi}(u_\varepsilon,k)\cdot\nabla\gamma + \varepsilon H(u_\varepsilon,k)\partial_t^2\gamma$$
$$- \partial_1 H(u_\varepsilon,k)\,|\nabla u_\varepsilon|^{p-2}\,\nabla u_\varepsilon\cdot\nabla\gamma - \partial_1^2 H(u_\varepsilon,k)(|\nabla u_\varepsilon|^p + \varepsilon\,|\partial_t u_\varepsilon|^2)\gamma\big)\xi_{K_\varepsilon}\, d\boldsymbol{x}dt$$
$$\leq 2\varepsilon\int_{G_T} H(u_\varepsilon,k)\xi'_{K_\varepsilon}\partial_t\gamma\, d\boldsymbol{x}dt + \mathcal{A}_1\int_\Omega \big(H(u_0(\boldsymbol{x}),k)\gamma(\boldsymbol{x},0) + H(u_T(\boldsymbol{x}),k)\gamma(\boldsymbol{x},T)\big)\, d\boldsymbol{x},$$

where $(H,Q,\mathbf{\Phi})$ is an arbitrary boundary entropy-entropy flux triple, $\gamma \in C_0^\infty(\Omega\times\mathbb{R})$ is non-negative test function, $k\in\mathbb{R}$, $\mathcal{A}_1 = \max\|a'\|_{C[-M,M]}$, $K_\varepsilon = \varepsilon/\mathcal{A}_1$, $M =$ $\max(\|u_0\|_{L^\infty(\Omega)}, \|u_T\|_{L^\infty(\Omega)})$.

Proof. Let a monotone sequence of nonnegative functions $s_n(t) \in C_0^2[0,T]$ converge uniformly to $s(t)$ on $[0,T]$. Furthermore, we assume that

$$\int_0^{\frac{1}{n}} s''_n(t)\, dt = -\int_{\frac{T}{10}-\frac{1}{n}}^{\frac{T}{10}+\frac{1}{n}} s''_n(t)\, dt = -\int_{\frac{9T}{10}-\frac{1}{n}}^{\frac{9T}{10}+\frac{1}{n}} s''_n(t)\, dt = \int_{T-\frac{1}{n}}^{T} s''_n(t)\, dt = 1,$$

$$s'_n(0) = s'_n(t) = s'_n(T) = 0 \ \text{ if } \ t \in \left[\frac{T}{10}+\frac{1}{n}, \frac{9T}{10}-\frac{1}{n}\right];$$

$$s'_n(t) = 1 \ \text{ if } \ t\in\left[\frac{1}{n}, \frac{T}{10}-\frac{1}{n}\right]; \quad s'_n(t) = -1 \ \text{ if } \ t \in \left[\frac{9T}{10}+\frac{1}{n}, T-\frac{1}{n}\right];$$

$$|s'_n(t)| \leq 1 \ \text{ if } \ t\in[0,T].$$

Let

$$\xi_{K,n}(t) = 1 - \exp\big(-\frac{s_n(t)}{K}\big).$$

Therefore, for any nonnegative function $\alpha \in C[0,T]$ the following equalities hold

$$(28) \qquad \lim_{n\to\infty}\int_0^T \alpha(t)s_n(t)\, dt = \int_0^T \alpha(t)s(t)\, dt,$$

$$(29) \qquad \lim_{n\to\infty}\int_0^T \alpha(t)s'_n(t)\, dt = \int_0^T \alpha(t)s'(t)\, dt,$$

$$(30) \qquad \lim_{n\to\infty}\int_0^T \alpha(t)s''_n(t)\, dt = \alpha(0) + \alpha(T) - \alpha(\frac{T}{10}) - \alpha(\frac{9T}{10}),$$

$$(31) \qquad \lim_{n\to\infty}\int_0^T \alpha(t)\xi_{K,n}(t)\, dt = \int_0^T \alpha(t)\xi_K(t)\, dt,$$

$$(32) \qquad \lim_{n\to\infty}\int_0^T \alpha(t)\xi'_{K,n}(t)\, dt = \int_0^T \alpha(t)\xi'_K(t)\, dt,$$

$$\lim_{n\to\infty}\int_0^T \alpha(t)\xi'_{K,n}(t)s'_n(t)\,dt = \int_0^T \alpha(t)\,|\xi'_K(t)|\,dt. \tag{33}$$

Taking into account that

$$\xi'_{K,n}(t) = \frac{1}{K}\exp(\frac{-s_n(t)}{K})s'_n(t)$$

and

$$\xi''_{K,n}(t) = -\frac{1}{K}\xi'_{K,n}(t)s'_n(t) + \frac{1}{K}\exp(-\frac{s_n(t)}{K})s''_n(t),$$

and equalities (28)–(33), we deduce the inequality in the limit as $n\to\infty$:

$$\begin{aligned}\lim_{n\to\infty}\int_0^T \alpha(t)\xi''_{K,n}(t)\,dt &= -\frac{1}{K}\int_0^T \alpha(t)\,|\xi'_K(t)|\,dt + \frac{1}{K}\big(\alpha(0)+\alpha(T)\big)\\ -\frac{1}{K}\big(\alpha(\frac{T}{10})+\alpha(\frac{9T}{10})\big)\exp(-\frac{T}{10K}) &\le -\frac{1}{K}\int_0^T \alpha(t)\,|\xi'_K(t)|\,dt + \frac{1}{K}\big(\alpha(0)+\alpha(T)\big).\end{aligned} \tag{34}$$

The latter inequality can be extended to the case $\alpha\in L^\infty(0,T)\cap W^{1,2}(0,T)$, $\alpha\ge 0$ a.e. on $(0,T)$.

In equation (5b) we put $\phi(\boldsymbol{x},t) = \partial_1 H(u_\varepsilon,k)\gamma(\boldsymbol{x},t)\xi_{K_\varepsilon,n}(t)$ and integrate by parts:

$$\begin{aligned}&-\int_{G_T}\Big(Q(u_\varepsilon,k)\partial_t\gamma + \boldsymbol{\Phi}(u_\varepsilon,k)\cdot\nabla\gamma + \varepsilon H(u_\varepsilon,k)\partial_t^2\gamma\\ &\quad -|\nabla u_\varepsilon|^{p-2}\nabla H(u_\varepsilon,k)\cdot\nabla\gamma - \partial_1^2 H(u_\varepsilon,k)(|\nabla u_\varepsilon|^p + \varepsilon\,|\partial_t u_\varepsilon|^2)\gamma\Big)\xi_{K_\varepsilon,n}\,d\boldsymbol{x}dt\\ &= \int_{G_T} Q(u_\varepsilon,k)\gamma\xi'_{K_\varepsilon,n}\,d\boldsymbol{x}dt + 2\varepsilon\int_{G_T} H(u_\varepsilon,k)\xi'_{K_\varepsilon,n}\partial_t\gamma\,d\boldsymbol{x}dt\\ &\quad + \varepsilon\int_{G_T} H(u_\varepsilon,k)\gamma\,\xi''_{K_\varepsilon,n}\,d\boldsymbol{x}dt,\end{aligned} \tag{35}$$

where $K_\varepsilon = \varepsilon/\mathcal{A}_1$, $\mathcal{A}_1 = \|a'\|_{C[-M,M]}$. In all terms of equality (35) (except the third one in the right-hand side) passing to the limit as $n\to\infty$ is provided with the help of equalities (31) and (32). In the limit the third term in the right-hand side of equality (35) can be estimated with the help of inequality (34) where

$$\alpha(t) = \varepsilon\int_\Omega H(u_\varepsilon(\boldsymbol{x},t),k)\gamma(\boldsymbol{x},t)\;d\boldsymbol{x}.$$

Therefore, we obtain the inequality

$$\begin{aligned}&-\int_{G_T}\big(Q(u_\varepsilon,k)\partial_t\gamma + \boldsymbol{\Phi}(u_\varepsilon,k)\cdot\nabla\gamma + \varepsilon H(u_\varepsilon,k)\partial_t^2\gamma\\ &\quad -\partial_1 H(u_\varepsilon,k)\,|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla\gamma - \partial_1^2 H(u_\varepsilon,k)(|\nabla u_\varepsilon|^p + \varepsilon\,|\partial_t u_\varepsilon|^2)\gamma\big)\xi_{K_\varepsilon}\,d\boldsymbol{x}dt\\ &\le 2\varepsilon\int_{G_T} H(u_\varepsilon,k)\xi'_{K_\varepsilon}\partial_t\gamma\,d\boldsymbol{x}dt + \mathcal{A}_1\int_\Omega\big(H(u_0(\boldsymbol{x}),k)\gamma(\boldsymbol{x},0) + H(u_T(\boldsymbol{x}),k)\gamma(\boldsymbol{x},T)\big)\,d\boldsymbol{x}\\ &\quad + \int_{G_T}\big(Q(u_\varepsilon,k)\xi'_{K_\varepsilon} - \mathcal{A}_1 H(u_\varepsilon,k)\,\big|\xi'_{K_\varepsilon}\big|\big)\gamma\,d\boldsymbol{x}dt.\end{aligned} \tag{36}$$

It is obvious that the third term in the right-hand side of (36) is negative:

$$Q(u_\varepsilon,k)\xi'_{K_\varepsilon} \le |Q(u_\varepsilon,k)|\,\left|\xi'_{K_\varepsilon}\right| = \left|\int_k^{u_\varepsilon} a'(\lambda)\partial_1 H(\lambda,k)\,d\lambda\right|\left|\xi'_{K_\varepsilon}\right|$$
$$\le \mathcal{A}_1\left|\int_k^{u_\varepsilon} |\partial_1 H(\lambda,k)|\,d\lambda\right|\left|\xi'_{K_\varepsilon}\right| = \mathcal{A}_1 H(u_\varepsilon,k)\left|\xi'_{K_\varepsilon}\right|.$$

This inequality and (36) lead to inequality (27). □

Since the set $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(G_T)$ and it is a relatively compact set in $L^p(0,T;L^p(\Omega))$, from this set we choose a subsequence $\{u_{\varepsilon_l}\}_{l\in\mathbb{N}}$ which has a limit $u\in L^\infty(G_T)\cap L^p(0,T;L^p(\Omega))$. By Lemma 1 function u_{ε_l} satisfies inequality (27) when $\varepsilon=\varepsilon_l$.

Almost everywhere on $(0,T)$ we have:

$$\lim_{l\to\infty}\xi_{K_{\varepsilon_l}}(t) = 1 - \lim_{l\to\infty}\exp\bigl(-\frac{\mathcal{A}_1 s(t)}{\varepsilon_l}\bigr) = 1, \tag{37}$$

$$\lim_{l\to\infty}\varepsilon_l\,\xi'_{K_{\varepsilon_l}}(t) = \mathcal{A}_1\lim_{l\to\infty}s'(t)\exp\bigl(-\frac{\mathcal{A}_1 s(t)}{\varepsilon_l}\bigr) = 0. \tag{38}$$

Because of the maximum principle (6) and the energy inequality (7) the set $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and the sequence ∇u_{ε_l} converges weakly to ∇u in $(L^p(G_T))^d$ as $l\to\infty$. Moreover, there exists the Young measure $\{\nu_{\boldsymbol{x},t}\}$ and we can express the following weak limit with the help of results from [15] and [27]:

$$\partial_1 H(u_{\varepsilon_l}(\boldsymbol{x},t),k)\,|\nabla u_{\varepsilon_l}(\boldsymbol{x},t)|^{p-2}\,\nabla u_{\varepsilon_l}(\boldsymbol{x},t) \rightharpoonup \partial_1 H(u(\boldsymbol{x},t),k)\langle\nu_{\boldsymbol{x},t},\boldsymbol{S}_p(\cdot)\rangle =$$
$$\partial_1 H(u(\boldsymbol{x},t),k)\int_{\mathbb{R}^d}|\boldsymbol{\xi}|^{p-2}\,\boldsymbol{\xi}\,d\nu_{\boldsymbol{x},t} \quad\text{in}\quad (L^{p'}(G_T))^d \quad\text{as}\quad l\to\infty. \tag{39}$$

Let $\nabla u = \int_{\mathbb{R}^d}\boldsymbol{\xi}\,d\nu_{\boldsymbol{x},t}$. Since

$$L(t,\boldsymbol{x},\boldsymbol{\xi}) = \partial_1^2 H(u(\boldsymbol{x},t),k)\,|\boldsymbol{\xi}|^p\,\gamma(\boldsymbol{x},t)$$

is convex in $\boldsymbol{\xi}\in\mathbb{R}^d$, we have the following result (see [27, Theorem 4.3])

$$\int_{G_T}\partial_1^2 H(u,k)\,|\nabla u|^p\,\gamma\,d\boldsymbol{x}dt \le \int_{G_T}\partial_1^2 H(u,k)\langle\nu_{\boldsymbol{x},t},|\cdot|^p\rangle\gamma\,d\boldsymbol{x}dt$$
$$\le \liminf_{\varepsilon_l\to 0+}\int_{G_T}\partial_1^2 H(u_{\varepsilon_l},k)\,|\nabla u_{\varepsilon_l}|^p\,\gamma\xi_{K_{\varepsilon_l}}\,d\boldsymbol{x}dt. \tag{40}$$

It is easy to show that function u satisfies the inequality $\|u\|_{L^\infty(G_T)}\le M$ and inequality (15). The latter assertion follows from (37)–(40) and Lemma 1.

Therefore, the limit u of a subsequence $\{u_{\varepsilon_l}\}_{l\in\mathbb{N}}$ and the associated gradient Young measure correspond to the entropy measure-valued solution of problem Π_0.

4. Proof of Proposition 2 and examples of boundary entropy-entropy flux triples

In this section we deduce boundary conditions on Γ_0 and Γ_T in the form of inequalities (16a) and (16b). To prove Corollaries 1 and 2, we use Examples 1 and 2 (see subsection 4.2).

4.1. **Proof of Proposition 2.**

Here we repeat the method proposed in [24] for hyperbolic equations and modified in [17]. Let $(u, \nu_{\boldsymbol{x},t})$ be an entropy measure-valued solution to problem Π_0. For an arbitrary function Q satisfying Definition 2 and a function $\beta \in C_0^2(\Omega)$, $\beta \geq 0$, let's define the function

$$f_{k,\beta}(t) = \int_\Omega Q(u(\boldsymbol{x},t),k)\beta(\boldsymbol{x})\,d\boldsymbol{x},$$

where $k \in \mathbb{R}$, $t \in (0,T)$.

Lemma 2. *Under the above conditions the following inequalities hold*

$$\underset{t\to 0+}{\operatorname{esslim}} f_{k,\beta}(t) \leq \mathcal{A}_1 \int_\Omega H(u_0(\boldsymbol{x}),k)\beta(\boldsymbol{x})\,d\boldsymbol{x}, \tag{41a}$$

$$\underset{t\to T-}{\operatorname{esslim}} f_{k,\beta}(t) \geq -\mathcal{A}_1 \int_\Omega H(u_T(\boldsymbol{x}),k)\beta(\boldsymbol{x})\,d\boldsymbol{x}. \tag{41b}$$

Proof. We are going to show that function $f_{k,\beta}(t)$ is equivalent to a function of bounded variation on $(0,T)$. For an arbitrary finite function $\alpha \in C_0^2[0,T]$, $0 \leq \alpha(t) \leq 1$, $t \in [0,T]$, put $\gamma = \alpha(t)\beta(\boldsymbol{x})$ in inequality (15):

$$-\int_0^T f_{k,\beta}(t)\alpha'(t)\,dt \leq \int_\Omega \int_0^T \big(\boldsymbol{\Phi}(u,k)\cdot\nabla\beta - \partial_1 H(u,k)\langle \nu_{\boldsymbol{x},t}, \boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta \\ - \partial_1^2 H(u,k)\,|\nabla u|^p\,\beta\big)\alpha(t)\,d\boldsymbol{x}dt. \tag{42}$$

Note that the variation $V(f,(0,T))$ of an arbitrary function $f \in BV(0,T)$ has the form

$$V(f,(0,T)) = \sup\Big\{\int_0^T \alpha'(t)f(t)\,dt : \alpha \in C_0^1[0,T], \|\alpha\|_{L^\infty(0,T)} \leq 1\Big\}. \tag{43}$$

Without limiting the generality of the foregoing, in the formula (43) we can replace the condition $\|\alpha\|_{L^\infty(0,T)} \leq 1$ by the condition $0 \leq \alpha(t) \leq 1$, $t \in (0,T)$. From (42) and Definition 3 it follows that $V(f_{k,\beta},(0,T)) < \infty$ and function $f_{k,\beta}$ has one-sided limits at points 0 and T:

$$\underset{t\to 0+}{\operatorname{esslim}} f_{k,\beta}(t), \quad \underset{t\to T-}{\operatorname{esslim}} f_{k,\beta}(t). \tag{44}$$

It is important to show that these one-sided limits satisfy the inequalities (41a) and, respectively, (41b).

Since entropy measure-valued solution u belongs to $L^\infty(G_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$, it can be concluded that

$$\lim_{t\to 0+} \int_\Omega \int_0^t \partial_1^2 H(u,k)\,|\nabla u|^p\,\beta(\boldsymbol{x})\,d\boldsymbol{x}d\tau = 0, \tag{45}$$

$$\lim_{t\to T-} \int_\Omega \int_t^T \partial_1^2 H(u,k)\,|\nabla u|^p\,\beta(\boldsymbol{x})\,d\boldsymbol{x}d\tau = 0, \tag{46}$$

$$\lim_{t\to 0+} \int_\Omega \int_0^t \partial_1 H(u,k)\langle \nu_{\boldsymbol{x},\tau}, \boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\,d\boldsymbol{x}d\tau = 0, \tag{47}$$

$$\lim_{t\to T-} \int_\Omega \int_t^T \partial_1 H(u,k)\langle \nu_{\boldsymbol{x},\tau}, \boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\,d\boldsymbol{x}d\tau = 0. \tag{48}$$

Put in (15) $\gamma(\boldsymbol{x},t)=\alpha(t)\beta(\boldsymbol{x})$, $\alpha\in C^2[0,T]$, $\alpha\ge 0$, $\beta\in C_0^2(\Omega)$, $\beta\ge 0$. Therefore, integration by parts gives two different inequalities:

$$(49)\quad 0\le \int_0^T \alpha'(t)\Big(\int_\Omega Q(u,k)\beta\,d\boldsymbol{x}+\int_\Omega\int_0^t \Big(\partial_1^2 H(u,k)\,|\nabla u|^p\,\beta + \partial_1 H(u,k)\langle \nu_{\boldsymbol{x},\tau},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\Big)\,d\boldsymbol{x}d\tau\Big)dt - \alpha(T)\int_{G_T}\Big(\partial_1^2 H(u,k)\,|\nabla u|^p\,\beta+\partial_1 H(u,k)\langle\nu_{\boldsymbol{x},t},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\Big)\,d\boldsymbol{x}dt + \int_{G_T}\boldsymbol{\Phi}(u,k)\cdot\nabla\beta(\boldsymbol{x})\alpha(t)\,d\boldsymbol{x}dt+\mathcal{A}_1\int_\Omega\big(\alpha(0)H(u_0,k)+\alpha(T)H(u_T,k)\big)\beta(\boldsymbol{x})\,d\boldsymbol{x},$$

and

$$(50)\quad 0\le \int_0^T \alpha'(t)\Big(\int_\Omega Q(u,k)\beta\,d\boldsymbol{x}-\int_\Omega\int_t^T \Big(\partial_1^2 H(u,k)\,|\nabla u|^p\,\beta + \partial_1 H(u,k)\langle \nu_{\boldsymbol{x},\tau},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\Big)\,d\boldsymbol{x}d\tau\Big)\,dt - \alpha(0)\int_{G_T}\big(\partial_1^2 H(u,k)\,|\nabla u|^p\,\beta+\partial_1 H(u,k)\langle\nu_{\boldsymbol{x},t},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\big)\,d\boldsymbol{x}dt + \int_{G_T}\boldsymbol{\Phi}(u,k)\cdot\nabla\beta\ \alpha(t)\,d\boldsymbol{x}dt+\mathcal{A}_1\int_\Omega\big(\alpha(0)H(u_0,k)+\alpha(T)H(u_T,k)\big)\beta\,d\boldsymbol{x}.$$

Note that the functions

$$(51)\quad h_{0,k,\beta}(t)=f_{k,\beta}(t)+\int_\Omega\int_0^t\big(\partial_1^2 H(u,k)\,|\nabla u|^p\,\beta + \partial_1 H(u,k)\langle\nu_{\boldsymbol{x},\tau},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\big)\,d\boldsymbol{x}d\tau,$$

$$(52)\quad h_{T,k,\beta}(t)=f_{k,\beta}(t)-\int_\Omega\int_t^T\big(\partial_1^2 H(u,k)\,|\nabla u|^p\,\beta + \partial_1 H(u,k)\langle\nu_{\boldsymbol{x},\tau},\boldsymbol{S}_p(\cdot)\rangle\cdot\nabla\beta\big)\,d\boldsymbol{x}d\tau,$$

are bounded almost everywhere on $(0,T)$. From (44)–(48), (51) and (52), it follows that there exist one-sided limits almost everywhere

$$(53)\qquad \operatorname*{esslim}_{t\to0+} h_{0,k,\beta}(t)=\operatorname*{esslim}_{t\to0+} f_{k,\beta}(t),\quad \operatorname*{esslim}_{t\to T-} h_{T,k,\beta}(t)=\operatorname*{esslim}_{t\to T-} f_{k,\beta}(t).$$

In (49) put $\alpha(t)=1-\alpha_{0,\epsilon}(t)$, where $\{\alpha_{0,\epsilon}(t)\}_{\epsilon>0}$ is a boundary-layer sequence, given in Definition 4. Since $\alpha(0)=1$ and $\alpha(T)=0$, inequality (49) reads

$$\int_0^\epsilon \alpha'_{0,\epsilon}(t)h_{0,k,\beta}(t)\,dt\le\int_\Omega\int_0^\epsilon \boldsymbol{\Phi}(u(\boldsymbol{x},t),k)\cdot\nabla\beta(\boldsymbol{x})(1-\alpha_{0,\epsilon}(t))\,d\boldsymbol{x}dt + \mathcal{A}_1\int_\Omega H(u_0(\boldsymbol{x}),k)\beta(\boldsymbol{x})\,d\boldsymbol{x}.$$

Taking into account (53) and passing to the limit as $\epsilon\to0+$, we get inequality (41a).

Similarly, we take $\alpha(t)=1-\alpha_{T,\epsilon}(t)$ in inequality (50) where $\{\alpha_{T,\epsilon}(t)\}_{\epsilon>0}$ is a boundary-layer sequence. Using equality (52), $\alpha(0)=0$ and $\alpha(T)=1$, we get the inequality

$$-\int_{T-\epsilon}^{T} \alpha'_{T,\epsilon}(t) h_{T,k,\beta}(t)\, dt \geq -\int_{\Omega}\int_{T-\epsilon}^{T} \mathbf{\Phi}(u,k)\cdot\nabla\beta(\boldsymbol{x})(1-\alpha_{T,\epsilon}(t))\, d\boldsymbol{x}dt - \mathcal{A}_1\int_{\Omega} H(u_T(\boldsymbol{x}),k)\beta(\boldsymbol{x})\, d\boldsymbol{x}.$$

Taking into account (53) and passing to the limit as $\epsilon \to 0+$, we get (41b). □

Here we introduce the following function

$$f_{\upsilon,\beta}(t) = \int_{\Omega} Q(u(\boldsymbol{x},t),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\, d\boldsymbol{x},$$

where $\beta \in L^1(\Omega)$, $\beta \geq 0$ a.e. in Ω and $\upsilon \in L^\infty(\Omega)$.

Inequalities (16a) and (16b) formulated in Proposition 2 can be reformulated as (54a) and (54b).

Lemma 3. *If u is an entropy measure-valued solution to problem Π_0, it satisfies boundary conditions in the form of inequalities*

$$\operatorname*{esslim}_{t\to 0+} f_{\upsilon,\beta}(t) \leq \mathcal{A}_1\int_{\Omega} H(u_0(\boldsymbol{x}),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\, d\boldsymbol{x}, \tag{54a}$$

$$\operatorname*{esslim}_{t\to T-} f_{\upsilon,\beta}(t) \geq -\mathcal{A}_1\int_{\Omega} H(u_T(\boldsymbol{x}),\upsilon(\boldsymbol{x}))\beta(\boldsymbol{x})\, d\boldsymbol{x}. \tag{54b}$$

Proof. Consider a sequence of nonnegative functions $\beta_l \in C_0^2(\Omega)$ such that $\lim\limits_{l\to\infty}\beta_l = \beta$ in $L^1(\Omega)$. Then there is convergence almost everywhere of f_{k,β_l} on $(0,T)$:

$$\lim_{l\to\infty} f_{k,\beta_l}(t) = f_{k,\beta}(t). \tag{55}$$

Moreover, we have

$$\lim_{l\to\infty}\int_{\Omega} H(u_0(\boldsymbol{x}),k)\beta_l(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega} H(u_0(\boldsymbol{x}),k)\beta(\boldsymbol{x})\, d\boldsymbol{x}, \tag{56}$$

$$\lim_{l\to\infty}\int_{\Omega} H(u_T(\boldsymbol{x}),k)\beta_l(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega} H(u_T(\boldsymbol{x}),k)\beta(\boldsymbol{x})\, d\boldsymbol{x}. \tag{57}$$

From (55)–(57) and Definition 3 we get the validity of the inequalities (41a) and (41b) if $\beta \in L^1(\Omega)$, $\beta \geq 0$ almost everywhere in Ω.

For an arbitrary simple function

$$\omega_n(\boldsymbol{x}) = \sum_{j=1}^{n} k_j \chi_{B_j}(\boldsymbol{x}),$$

where $\chi_{B_j}(\boldsymbol{x})$ is the characteristic function of B_j, $\bigcup\limits_{j=1}^{n} B_j = \Omega$, we construct the function $f_{\omega_n,\beta}(t)$ by the rule

$$f_{\omega_n,\beta}(t) := \int_{\Omega} Q(u(\boldsymbol{x},t),\omega_n(\boldsymbol{x}))\beta(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{j=1}^{n}\int_{\Omega} Q(u(\boldsymbol{x},t),k_j)\mathbf{1}_{B_j}(\boldsymbol{x})\beta(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{j=1}^{n} f_{k_j,\chi_{B_j}\beta}(t).$$

From (53)–(57) and Definition 3 we obtain the existence of the limits $\operatorname*{esslim}_{t\to 0+} f_{k_j,\chi_{B_j}\beta}(t)$ and $\operatorname*{esslim}_{t\to T-} f_{k_j,\chi_{B_j}\beta}(t)$ which satisfy the inequalities

$$\operatorname*{esslim}_{t\to 0+} f_{k_j,\chi_{B_j}\beta}(t) \le \mathcal{A}_1 \int_{B_j} H(u_0(\boldsymbol{x}), k_j)\beta(\boldsymbol{x})\, d\boldsymbol{x}, \tag{58}$$

$$\operatorname*{esslim}_{t\to T-} f_{k_j,\chi_{B_j}\beta}(t) \ge -\mathcal{A}_1 \int_{B_j} H(u_T(\boldsymbol{x}), k_j)\beta(\boldsymbol{x})\, d\boldsymbol{x}. \tag{59}$$

By summing inequalities (58) and, respectively, (59) over $j = 1, \ldots, n$, we obtain

$$\operatorname*{esslim}_{t\to 0+} f_{\omega_n,\beta}(t) \le \mathcal{A}_1 \int_{\Omega} H(u_0(\boldsymbol{x}), \omega_n(\boldsymbol{x}))\beta(\boldsymbol{x})\, d\boldsymbol{x}, \tag{60}$$

$$\operatorname*{esslim}_{t\to T-} f_{\omega_n,\beta}(t) \ge -\mathcal{A}_1 \int_{\Omega} H(u_T(\boldsymbol{x}), \omega_n(\boldsymbol{x}))\beta(\boldsymbol{x})\, d\boldsymbol{x}. \tag{61}$$

For any function $v \in L^\infty(\Omega)$ there exists a sequence of simple functions ω_n which converges uniformly to v on Ω. Therefore, for a.e. $t \in (0, T)$ we get

$$\lim_{n\to\infty} f_{\omega_n,\beta}(t) = f_{v,\beta}(t). \tag{62}$$

It is easy to show that there are limits

$$\lim_{n\to\infty} \int_{\Omega} H(u_0, \omega_n)\beta\, d\boldsymbol{x} = \int_{\Omega} H(u_0, v)\beta\, d\boldsymbol{x}, \tag{63}$$

$$\lim_{n\to\infty} \int_{\Omega} H(u_T, \omega_n)\beta\, d\boldsymbol{x} = \int_{\Omega} H(u_T, v)\beta\, d\boldsymbol{x}. \tag{64}$$

Therefore, due to (60)–(64) and Definition 3 we obtain inequalities (54a) and (54b). □

4.2. **Examples of boundary entropy-entropy flux triples.**

It is important to note that, namely, boundary entropy-entropy flux pairs were defined in [24]. Because of the presence of vector function $\boldsymbol{\varphi}$ we have to deal with boundary entropy-entropy flux triples.

Example 1 reflects the idea proposed in [16].

Example 1. *We consider the following class of boundary entropy-entropy flux triples* $(H_\delta, Q_\delta, \boldsymbol{\Phi}_\delta)$*:*

$$H_\delta(z,k) = \sqrt{(z-k)^2 + \delta^2} - \delta,$$
$$Q_\delta(z,k) = \int_k^z a'(\lambda)\partial_1 H_\delta(\lambda,k)\, d\lambda,$$
$$\boldsymbol{\Phi}_\delta(z,k) = \int_k^z \boldsymbol{\varphi}'(\lambda)\partial_1 H_\delta(\lambda,k)\ d\lambda.$$

By passing to the limit as $\delta \to 0+$ *we obtain*

$$H_0(z,k) = |z-k|, \quad Q_0(z,k) = \operatorname{sign}(z-k)(a(z)-a(k)),$$
$$\boldsymbol{\Phi}_0(z,k) = \operatorname{sign}(z-k)(\boldsymbol{\varphi}(z) - \boldsymbol{\varphi}(k)).$$

The idea of Example 2 was proposed in [24]. Moreover, it is used in the proof of the uniqueness of the entropy measure-valued solution when $p = 2$ [17].

Example 2. *We introduce the boundary entropy-entropy flux triple function*

$$(H_{\delta,w}, Q_{\delta,w}, \mathbf{\Phi}_{\delta,w})$$

by the rule:

$$\begin{aligned} H_{\delta,w}(z,k) &= \big((H_0(z,k) + H_0(z,w) - H_0(k,w))^2 + \delta^2\big)^{1/2} - \delta, \\ Q_{\delta,w}(z,k) &= \int_k^z a'(\lambda)\partial_\lambda H_{\delta,w}(\lambda,k)\,d\lambda, \\ \mathbf{\Phi}_{\delta,w}(z,k) &= \int_k^z \boldsymbol{\varphi}'(\lambda)\partial_\lambda H_{\delta,w}(\lambda,k)\,d\lambda, \quad \forall\ (k,z,w) \in \mathbb{R}^3, \end{aligned}$$

where $H_0(z,k) + H_0(z,w) - H_0(k,w) = 2\mathrm{dist}(z, \mathcal{I}[w,k])$*,* $\mathcal{I}[w,k]$ *is an interval with the endpoints* w *and* k*. It is easy to check that the triple of functions* $(H_{\delta,w}, Q_{\delta,w}, \mathbf{\Phi}_{\delta,w})$ *is a boundary entropy-entropy flux triple. In the limit as* $\delta \to 0+$ *we get*

$$\begin{aligned} H_{0,w}(z,k) &= H_0(z,k) + H_0(z,w) - H_0(k,w), \\ Q_{0,w}(z,k) &= Q_0(z,k) + Q_0(z,w) - Q_0(k,w), \\ \mathbf{\Phi}_{0,w}(z,k) &= \mathbf{\Phi}_0(z,k) + \mathbf{\Phi}_0(z,w) - \mathbf{\Phi}_0(k,w). \end{aligned}$$

It is important to note that

$$Q_{0,w}(z,k) = \begin{cases} 2(a(k) - a(z)) & \text{if} \quad z \le k \le w, \\ 2(a(w) - a(z)) & \text{if} \quad z \le w \le k, \\ 0 & \text{if} \quad w \le z \le k, \\ 0 & \text{if} \quad k \le z \le w, \\ 2(a(z) - a(w)) & \text{if} \quad k \le w \le z, \\ 2(a(z) - a(k)) & \text{if} \quad w \le k \le z. \end{cases}$$

Example 3 was given in [10, The proof of Theorem 4.1.1].

Example 3. *Here we construct the boundary entropy-entropy flux triple by the rule:*

$$H(z,k) = \eta(z) - \eta(k) - \eta'(k)(z-k),$$

$$Q(z,k) = q_a(z) - q_a(k) - \eta'(k)(a(z) - a(k)),$$

$$\mathbf{\Phi}(z,k) = \boldsymbol{q}_\varphi(z) - \boldsymbol{q}_\varphi(k) - \eta'(k)(\boldsymbol{\varphi}(z) - \boldsymbol{\varphi}(k)),$$

where smooth functions q_a*,* $\boldsymbol{q}_\varphi$ *and* η *satisfy the following conditions:*

$$\eta''(z) \ge 0, \quad q_a'(z) = a'(z)\eta'(z), \quad \boldsymbol{q}_\varphi'(z) = \boldsymbol{\varphi}'(z)\eta'(z), \quad \forall z \in \mathbb{R}.$$

Remark 6. *In the case* $p = 2$ *Example 3 enables to link Proposition 2 with the boundary entropy conditions* (6b) *and* (6c) *formulated in* [18] *(here we need to put* $v = u_0$ *and, correspondingly,* $v = u_T$*). Therefore, in our future research we are also going to deal with a kinetic formulation of problem* Π_0 *for forward-backward* $p(\boldsymbol{x},t)$*-parabolic equations.*

Conclusion

In this paper we have proved the existence of entropy measure-valued solutions to problem Π_0. When $p \ne 2$, the uniqueness is an open question.

References

[1] P. Amorim, S. Antontsev, *Young measure solutions for the wave equation with $p(x,t)$-Laplacian: existence and blow-up*, Nonlinear Analysis: Theory, Methods & Applications, **92** (2013), 153–167. MR3091116

[2] B. Andreianov, M. Bendahmane, K.H. Karlsen, S. Ouaro, *Well-posedness results for triply nonlinear degenerate parabolic equations*, Journal of Differential Equations, **247** (2009), 277–302. MR2510137

[3] S.N. Antontsev, J.I. Diaz, S. Shmarev, *Energy Methods for Free Boundary Problems. Applications to Nonlinear PDEs and Fluid Mechanics*, Boston-Birkhäuser, 2002. MR1858749

[4] S.N. Antontsev, I.V. Kuznetsov, *Singular perturbations of forward-backward p-parabolic equations*, Journal of Elliptic and Parabolic Equations, **2**:1–2 (2016), 357–370. MR3645952

[5] A.K. Aziz, D.A. French, S. Jensen, B. Kellogg, *Origins, analysis, numerical analysis, and numerical approximation of a forward-backward parabolic problem*, Mathematical Modelling and Numerical Analysis M2AN, **33** (1999), 895–922. MR1726715

[6] M. Bendahmane, K.H. Karlsen, *Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations*, SIAM Journal on Mathematical Analysis, **36** (2004), 405–422. MR2111783

[7] M. Bendahmane, K.H. Karlsen, *Uniqueness of entropy solutions for doubly nonlinear anisotropic degenerate parabolic equations*, Contemporary Mathematics, **371** (2005), 1–27. MR2143857

[8] M. Borsuk, V. Kondratiev, *Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains*, Elsevier, 2006. MR2286361

[9] M. Chipot, S. Guesmia, A. Sengouga, *Singular perturbations of some nonlinear problems*, Journal of Mathematical Sciences, **176**:6 (2011), 828–843. MR2838978

[10] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, 2000. MR1763936

[11] S. Demoulini, *Young measure solutions for a nonlinear parabolic equation of forward-backward type*, SIAM Journal on Mathematical Analysis, **27**:2 (1996), 376–403. MR1377480

[12] S. Demoulini, *Variational methods for Young measure solutions of nonlinear parabolic evolutions of forward-backward type and of high spatial order*, Applicable Analysis, **63**:3–4 (1996), 363–373. MR1624060

[13] X. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, Journal of Differential Equations, **235** (2007), 397–417. MR2317489

[14] K. Höllig, *Existence of infinitely many solutions for a forward backward heat equation*, Transactions of the American Mathematical Society, **278**:1 (1983), 299–316. MR0697076

[15] D. Kinderlehrer, P. Pedregal, *Gradient Young measures generated by sequences in Sobolev spaces*, The Journal of Geometric Analysis, **4** (1994), 59–90. MR1274138

[16] S.N. Kruzhkov, *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.), **81**(123) (1970), 228–255 (in Russian). Translated in: Math. USSR Sbornik, **10**:2 (1970), 217–243. MR0267257

[17] I.V. Kuznetsov, *Entropy solutions to differential equations with variable parabolicity direction*, Journal of Mathematical Sciences, **202**:1 (2014), 91–112. MR3256130

[18] I.V. Kuznetsov, *Kinetic formulation of forward-backward parabolic equations*, Siberian Electronic Mathematical Reports, **13** (2016), 930–949. MR3576020

[19] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Nauka, Moscow, 1973 (in Russian). MR0509265

[20] P.D. Lax, *Hyperbolic system of conservation laws II*, Communications on Pure and Applied Mathematics, **10** (1957), 537–566. MR0093653

[21] C. Mascia, A. Porretta, A. Terracina, *Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations*, Archive for Rational Mechanics and Analysis, **163** (2002), 87–124. MR1911095

[22] C. Mascia, A. Terracina, A. Tesei, *Two-phase entropy solutions of a forward-backward parabolic equation*, Archive for Rational Mechanics and Analysis, **194** (2009), 887–925. MR2563628

[23] V.N. Monakhov, *Reciprocal flows in boundary layer*, Dinamika Sploshnoy Sredy, **113** (1998), 107–113 (in Russian). MR1769268

[24] F. Otto, *Initial-boundary value problem for a scalar conservation law*, C. R. Acad. Sci. Paris Ser. I Math., **322**:8, (1996), 729–734. MR1387428

[25] P.I. Plotnikov, *Forward-backward parabolic equations and hysteresis*, Journal of Mathematical Sciences, **93**:5 (1999), 747–766. MR1699122

[26] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Annali di Matematica Pura ed Applicata, **146**:1 (1987), 65–96. MR0916688

[27] M.A. Sychev, *A new approach to Young measure theory, relaxation and convergence in energy*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, **16**:6 (1999), 773–812. MR1720517

[28] A. Terracina, *Non-uniqueness results for entropy two-phase solutions of forward-backward parabolic problems with unstable phase*, Journal of Mathematical Analysis and Applications, **413**:2 (2014), 963–975. MR3159815

[29] A. Terracina, *Two-phase entropy solutions of forward-backward parabolic problems with unstable phase*, Interfaces and Free Boundaries, **17**:3 (2015), 289–315. MR3421908

[30] S.A. Tersenov, *On some problems for forward-backward parabolic equations*, Siberian Mathematical Journal, **51**:2 (2010), 338–345. MR2668109

[31] N.A. Larkin, V.A. Novikov, N.N. Yanenko, *Nonlinear Equations of a Variable Type*, Novosibirsk: Nauka, 1983 (in Russian). MR0743597

[32] J. Yin, C. Wang, *Young measure solutions of a class of forward-backward diffusion equations*, Journal of Mathematical Analysis and Applications, **279** (2003), 659–683. MR1974053

[33] C. Wang, Y. Nie, J. Yin, *Young measure solutions for a class of forward-backward convection-diffusion equations*, Quarterly of Applied Mathematics, **72** (2014), 177–192. MR3185137

Stanislav N. Antontsev
Novosibirsk State University,
Pirogova st., 2,
630090, Novosibirsk, Russia,
and
Lavrentyev Institute of Hydrodynamics,
Siberian Division of the Russian Academy of Sciences,
pr. Acad. Lavrentyeva 15,
630090, Novosibirsk, Russia
and
CMAF-CIO, University of Lisbon, 1749-016 Lisbon, Portugal
E-mail address: antontsevsn@mail.ru

Ivan V. Kuznetsov
Novosibirsk State University,
Pirogova st., 2,
630090, Novosibirsk, Russia
and
Lavrentyev Institute of Hydrodynamics,
Siberian Division of the Russian Academy of Sciences,
pr. Acad. Lavrentyeva 15,
630090, Novosibirsk, Russia
E-mail address: kuznetsov_i@hydro.nsc.ru