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BOOLEAN ALGEBRAS REALIZED BY C.E. EQUIVALENCE RELATIONS

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Abstract. Let E be a computably enumerable (c.e.) equivalence relation on the set of natural numbers ω. We consider countable structures where basic functions are computable and respect E. If the corresponding quotient structure is a Boolean algebra B, then we say that the c.e. relation E realizes B. In this paper we study connections between algorithmic properties of E and algebraic properties of Boolean algebras realized by E. Also we compare these connections with the corresponding results for linear orders and groups realized by c.e. equivalence relations.

Keywords: computability theory, Boolean algebras, equivalence relations, computably enumerable structures.

1. Introduction

Computably enumerable (c.e.) structures are usually given by the domain ω, computable functions representing the basic functions in the structure, plus some computably enumerable predicates (among which there is a predicate E representing the equality relation). When a relation E is fixed, the c.e. structures in which the equality relation coincides with E depend heavily on the nature of E. For example, Novikov [1] constructed a finitely generated group with undecidable word problem; in other words, there is a group which can be represented using a non-computable c.e. equivalence relation E, however, it cannot be represented by using a computable

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equivalence relation. On the other hand, when dealing with Noetherian rings [2], Baur [3] showed that every c.e. Noetherian ring is already a computable Noetherian ring, as the underlying equality E is a computable relation. Thus, only computable equality relations E can be used to represent computable Noetherian rings.

In this paper we are interested in Boolean algebras, their c.e. representations, and dependency of these representations on the underlying domains. We aim to find structural properties of Boolean algebras emphasizing the role of a c.e. equivalence relation E representing the equality. Our focus on Boolean algebras is motivated by the results in [4],[5].

2. Preliminaries

For a set X, $card(X)$ is the cardinality of X. We treat Boolean algebras as structures in the language $L_{BA} = \{\vee^2, \wedge^2, C^1, 0, 1\}$. Recall that $x \triangle y = (x \wedge C(y)) \vee (y \wedge C(x))$. If $\mathcal{L}$ is a linear order, then $Int(\mathcal{L})$ denotes the corresponding *interval algebra*. The reader is referred to [6] for further background on countable Boolean algebras.

We consider only equivalence relations with domain ω. If E is an equivalence relation and $a \in \omega$, then $[a]_E$ is the E-equivalence class of a. For equivalence relations E and F, the *direct sum* $(E \oplus F)$ is an equivalence relation such that for any $x \in \omega$, we have

$$[2x]_{E \oplus F} = \{2y : y \in [x]_E\}, \text{ and } [2x+1]_{E \oplus F} = \{2y+1 : y \in [x]_F\}.$$

We say that E is *computably reducible* to F (denoted by $E \leq_c F$) if there is a computable function $f(x)$ such that for any x and y, (xEy) iff $(f(x)Ff(y))$. The systematic study of computable reducibility was initiated by Ershov [7]. For further information on computable reducibility, we refer the reader to [8, 9, 10].

For a non-zero natural number n, the equivalence relation Id_n is defined as follows: (xId_ny) iff $(x \mod n) = (y \mod n)$. By id_ω we denote the identity relation on ω. Suppose that $X_1, \dots, X_n$ are pairwise disjoint subsets of ω. Let

$$E(X_1, \dots, X_n) := \{(x, y) : (x = y) \text{ or } (x, y \in X_i) \text{ for some } i \leq n\}.$$

Notice the following: If $X_1, \dots, X_n$ are c.e. sets, then $E(X_1, \dots, X_n)$ is a c.e. equivalence relation.

Assume that E is a c.e. equivalence relation on ω. A computable function $f: \omega^n \to \omega$ *respects* E if for all $x_1, y_1, \dots, x_n, y_n \in \omega$ such that $(x_1, y_1), \dots, (x_n, y_n) \in E$, we have $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in E$. If an n-ary function f respects E, then f induces an operation on the quotient set ω/E. Following [11], we slightly abuse the notation and denote the induced map by f itself.

The next three definitions follow the lines of [4, 5, 11].

Definition 1. *Consider a structure of the form $(\omega, \vee, \wedge, C, 0, 1)$, where $0, 1$ are elements and $\vee, \wedge, C$ are computable functions respecting E. If the induced structure $\mathcal{S} = (\omega/E, \vee, \wedge, C, 0, 1)$ is a Boolean algebra, then $\mathcal{S}$ is called an E-*Boolean algebra. *For a countable Boolean algebra $\mathcal{B}$, we say that E* realizes *$\mathcal{B}$ if there is an E-Boolean algebra isomorphic to $\mathcal{B}$.*

Definition 2. *For a c.e. equivalence relation E, we define the class*

$$K_{BA}(E) = \{\mathcal{B} : \mathcal{B} \text{ is a Boolean algebra realized by } E\}.$$

*Assume that E_1 and E_2 are c.e. equivalence relations. We say that E_1 is BA-*reducible *to E_2 (denoted by $E_1 \le_{BA} E_2$) if $K_{BA}(E_1) \subseteq K_{BA}(E_2)$.*

Definition 3. *A countable Boolean algebra $\mathcal{B}$ is* computably enumerable *(*c.e.*) if there is a c.e. equivalence relation E such that E realizes $\mathcal{B}$.*

By $\mathcal{P}_{BA}$ we denote the partial order of BA-degrees. For a c.e. equivalence relation E, $\deg_{BA}(E)$ denotes the BA-degree of E.

We mention two antecendents of our work on Boolean algebras. Feiner [12] built the first example of a c.e. Boolean algebra that does not have a computable copy. Odintsov and Selivanov [13] studied countable Boolean algebras realized by Σ^0_n equivalence relations.

3. Minimal and maximal BA-degrees

Note that for a non-zero natural number n, $Int(n)$ is a finite Boolean algebra with exactly n atoms. It is not difficult to prove the following:

Proposition 1. (1) *For $n \in \omega$, we have $K_{BA}(Id_{2^n}) = \{Int(n)\}$.*
(2) *If $k \in \omega$ is not a power of two, then $K_{BA}(Id_k) = \emptyset$.*

Corollary 1. *The partial order $\mathcal{P}_{BA}$ has the least degree* $\deg_{BA}(Id_3)$.

A c.e. structure $\mathcal{A}$ is *computably categorical* if for any c.e. structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a computable function $f\colon \omega \to \omega$ such that f induces an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Lemma 1. *Let E be a c.e. equivalence relation. If $K_{BA}(E)$ contains a computably categorical structure, then* $\deg_{BA}(E)$ *is maximal in $\mathcal{P}_{BA}$.*

Proof. The proof is essentially the same as in [4, Lemma 25].

Corollary 2. *For any $n \in \omega$, the degree* $\deg_{BA}(Id_{2^n})$ *is maximal in $\mathcal{P}_{BA}$. Hence, $\mathcal{P}_{BA}$ has infinitely many maximal degrees.*

4. m-Degrees of equivalence classes

Assume that $\mathcal{B} = (\omega/E, \vee, \wedge, C, 0, 1)$ is a c.e. Boolean algebra. First, we prove a simple useful lemma.

Lemma 2. *The following sets are m-equivalent:*
1. *the equivalence relation E;*
2. *the class $[a]_E$ for any fixed $a \in \omega$;*
3. *the set $P = \{(b, c) : [b]_E \le_{\mathcal{B}} [c]_E\}$.*

Proof. We prove the following reductions:
- (i) $E \le_m [a]_E \le_m E$, and
- (ii) $[a]_E \le_m P \le_m E$.

(i) For $x, y \in \omega$, it is easy to see that xEy iff $x \triangle y \in [0]_E$. In turn, this is equivalent to $(x \triangle y) \triangle a \in [a]_E$. Since for any fixed a, the function $g(x, y) = (x \triangle y) \triangle a$ is computable, we obtain that $E \le_m [a]_E$. It is obvious that $[a]_E \le_m E$.

(ii) Note that the condition $x \in [a]_E$ is equivalent to $[x \triangle a]_E \le_{\mathcal{B}} [0]_E$. Hence, we have $[a]_E \le_m P$. Recall that $[x]_E \le_{\mathcal{B}} [y]_E$ iff $x \vee y \in [y]_E$. This implies that $P \le_m E$.

Corollary 3. *Suppose that E is a c.e. equivalence relation.*

(1) *If there is an element $a \in \omega$ such that $[a]_E \not\equiv_m E$, then $K_{BA}(E) = \emptyset$.*
(2) *If there is $b \in \omega$ such that the class $[b]_E$ is a simple set, then $K_{BA}(E) = \emptyset$.*

Proof. The first claim is a direct consequence of Lemma 2. Now assume that the relation E realizes a Boolean algebra $\mathcal{B}$ and a class $[b]_E$ is a simple set. It is easy to see that the algebra $\mathcal{B}$ is nontrivial. Choose $c \in \omega$ with $c \notin [b]_E$. Since $[c]_E \equiv_m [b]_E$, the class $[c]_E$ is an infinite c.e. subset of $(\omega \setminus [b]_E)$; this is a contradiction.

Note that, in general, the condition

$$(\forall a \in \omega)([a]_E \equiv_m E)$$

does not imply that $K_{BA}(E)$ is not empty. In particular, if E has exactly three equivalence classes, then they are all computable, but there is no Boolean algebra realized by E.

Proposition 2. *Assume that E is a c.e. equivalence relation, and there is an element $a \in \omega$ such that the class $[a]_E$ is computable. Then E satisfies exactly one of the following conditions:*

(1) $K_{BA}(E) = \emptyset$;
(2) $K_{BA}(E) = \{Int(n)\}$ *for some $n \in \omega$;*
(3) $K_{BA}(E)$ *is the class of all infinite Boolean algebras possessing computable copies.*

Proof. Without loss of generality, we may assume that E has infinitely many equivalence classes and $K_{BA}(E) \neq \emptyset$. Then, by Lemma 2, the relation E is computable. Therefore, it is not hard to show that the class $K_{BA}(E)$ consists of all infinite Boolean algebras with computable copies.

Corollary 4. *Assume that $X_1, X_2, \ldots, X_n$ are pairwise disjoint c.e. sets such that the set $(X_1 \cup X_2 \cup \ldots \cup X_n)$ is coinfinite and at least one of X_i is not computable. Then $K_{BA}(E(X_1, X_2, \ldots, X_n)) = \emptyset$.*

Corollary 4 contrasts with the results of Gavryushkin, Khoussainov, and Stephan [11] on c.e. linear orders. In particular, they proved [11, Theorem 39] the following: If X is a semirecursive simple set, then $E(X)$ realizes the orders $\omega + n$, $n + \omega^*$, and $\omega + 1 + \omega^*$ for all n.

Corollary 5. *There are c.e. equivalence relations E and F such that $K_{BA}(E) \neq \emptyset$, $K_{BA}(F) \neq \emptyset$, and $K_{BA}(E \oplus F) = \emptyset$.*

Proof. Assume that $E = Id_2$ and F is a non-computable c.e. equivalence relation such that $K_{BA}(F) \neq \emptyset$. For a particular example of such a relation F, see Theorem 1 below. Then for any $x \in \omega$, the class $[2x]_{E\oplus F}$ is computable and $[2x+1]_{E\oplus F}$ is non-computable. Thus, by Lemma 2, the sum $E \oplus F$ does not realize any Boolean algebra.

Corollary 5 shows that, in general, the conditions $K_{BA}(E) \neq \emptyset$ and $E \leq_c E_1$ together do not imply that $K_{BA}(E_1)$ is also non-empty.

5. Weakly precomplete equivalence relations

An equivalence relation E is *weakly precomplete* if for any total computable function φ_e, there is an element x such that $(xE\varphi_e(x))$. The notion of weak precompleteness was introduced by Badaev [14], and it is closely connected with the theory of numberings [7]. For further background, see [15].

Proposition 3. *Suppose that E is a weakly precomplete c.e. equivalence relation. Then E does not realize a Boolean algebra.*

Proof. Note that any weakly precomplete c.e. equivalence relation contains infinitely many classes. Assume that E is a weakly precomplete c.e. equivalence relation, and $\mathcal{B} = (\omega/E, \vee, \wedge, C, 0, 1)$ is a c.e. Boolean algebra. Then $C(x)$ is a total computable function, and for any $x \in \omega$, we have $[x]_E \neq [C(x)]_E$. This contradicts the weak precompleteness of E.

6. Connections to c.e. linear orders

In this section, the relation $\leq_{tt}$ is the *truth-table reducibility* (see [16] for details).

Theorem 1. *Assume that E is a c.e. equivalence relation. There is a c.e. equivalence relation E^{Int} such that $E^{Int} \equiv_{tt} E$ and E^{Int} has the following property: If $\mathcal{L} = (\omega/E, \trianglelefteq)$ is a c.e. linear order with the least element, then E^{Int} realizes the interval Boolean algebra $Int(\mathcal{L})$.*

Proof. If E has finitely many equivalence classes, say n, then any linear order of the form $(\omega/E, \trianglelefteq)$ has exactly n elements. Thus, we may take E^{Int} to be the relation Id_{2^n}.

Suppose that E has infinitely many equivalence classes. We fix a computable sequence of equivalence relations $\{E^s\}_{s\in\omega}$ with the following properties: $\bigcup_s E^s = E$, $E^0 = id_\omega$, $E^s \subseteq E^{s+1}$ for all s. Moreover, we assume that for every s, the relation E^{s+1} satisfies one of the following:

(1) $E^{s+1} = E^s$, or
(2) there are elements a and b such that $[a]_{E^s} \neq [b]_{E^s}$, $[a]_{E^{s+1}} = [a]_{E^s} \cup [b]_{E^s}$, and for all $c \notin [a]_{E^{s+1}}$, we have $[c]_{E^{s+1}} = [c]_{E^s}$.

Recall that for $x \in \omega$, D_x is a finite set which has number x in the standard numbering of all finite subsets of ω. For $x, y \in \omega$, we say that $(x, y) \in E^{Int}$ iff the cardinalities of the sets D_x and D_y have the same parity and

$$\{[a]_E : card([a]_E \cap D_x) \text{ is odd}\} = \{[b]_E : card([b]_E \cap D_y) \text{ is odd}\}. \tag{1}$$

It is easy to see that E^{Int} is an equivalence relation. Moreover, it is not difficult to show that the condition (1) is equivalent to the following: There is a number s such that

$$\{[a]_{E^s} : card([a]_{E^s} \cap D_x) \text{ is odd}\} = \{[b]_{E^s} : card([b]_{E^s} \cap D_y) \text{ is odd}\}.$$

Thus, E^{Int} is a c.e. set. Furthermore, we have $E \leq_m E^{Int}$ and $E^{Int} \leq_{tt} E$.

Note that if $E = id_\omega$, then $E^{Int} = id_\omega$ and the proof is trivial. Thus, we may assume that there are two elements c_0 and c_1 such that $c_0 \neq c_1$ and $c_0 E c_1$.

Now suppose that $\mathcal{L} = (\omega/E, \trianglelefteq)$ is a c.e. linear order with the least element, and let e_0 be the least element of $\mathcal{L}$. Fix a computable enumeration $\{\trianglelefteq^s\}_{s\in\omega}$ such that

$\trianglelefteq = \bigcup_s \trianglelefteq^s$, $\trianglelefteq^s$ is a finite set, and $\trianglelefteq^s \subseteq \trianglelefteq^{s+1}$ for all s. For $a, b \in \omega$, define

$$[a,b)_s = \{x \in \omega : x \neq b \text{ and } \exists y (y E^s x \ \ \& \ \ a \trianglelefteq^s y \trianglelefteq^s b)\},$$
$$[a,\infty)_s = \{x \in \omega : \exists y (y E^s x \ \ \& \ \ a \trianglelefteq^s y)\}.$$

We introduce some auxiliary notations. Suppose that $\bar{a} = a_1, a_2, \dots, a_n$ is a tuple of natural numbers and let $D = \{a_1, a_2, \dots, a_n\}$. We find the least $s_0 \in \omega$ such that the induced structure $\mathcal{L}(\bar{a}, s_0) = (D/(E^{s_0} \cap D^2), \trianglelefteq^{s_0} \cap D^2)$ is a linear order (here D^2 is the Cartesian square of D). Denote this s_0 by $s^*(\bar{a})$. Let σ be the permutation of the set $\{1, 2, \dots, n\}$ such that $[a_{\sigma(1)}]_{E^{s_0}} \trianglelefteq^{s_0} [a_{\sigma(2)}]_{E^{s_0}} \trianglelefteq^{s_0} \dots \trianglelefteq^{s_0} [a_{\sigma(n)}]_{E^{s_0}}$ and the tuple $\sigma(\bar{a}) = a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}$ has the least Gödel number. Then for $s \geq s_0$, by $int^s(\bar{a})$ we denote the following finite union of the intervals in the order $\mathcal{L}(\bar{a}, s)$:

$$int^s(\bar{a}) = \begin{cases} [a_{\sigma(1)}, a_{\sigma(2)})_s \cup [a_{\sigma(3)}, a_{\sigma(4)})_s \cup \dots \cup [a_{\sigma(n-1)}, a_{\sigma(n)})_s, \\ \qquad \text{if } n \text{ is even}, \\ [a_{\sigma(1)}, a_{\sigma(2)})_s \cup [a_{\sigma(3)}, a_{\sigma(4)})_s \cup \dots \cup [a_{\sigma(n-2)}, a_{\sigma(n-1)})_s \cup \\ \qquad [a_{\sigma(n)}, \infty)_s, \ \text{if } n \text{ is odd}. \end{cases}$$

We also set $int^{\mathcal{L}}(\bar{a}) = \{x \in \omega : [x]_E \subseteq \bigcup_{s \geq s^*(\bar{a})} int^s(\bar{a})\}$.

Now we are ready to give a brief informal explanation of our construction. Essentially, we emulate the well-known effective transformation of a computable linear order $\mathcal{M}$ into a computable copy of the interval algebra $Int(\mathcal{M})$ (see, e.g., Proposition 3.2.1 in [6]). For $x \in \omega$ and our c.e. order $\mathcal{L}$, if $D_x = \{a_1 \trianglelefteq a_2 \trianglelefteq \dots \trianglelefteq a_n\}$, then the equivalence class $[x]_{E^{Int}}$ contains all possible codes of the element $int^{\mathcal{L}}(a_1, a_2, \dots, a_n)$ from the interval algebra $Int(\mathcal{L})$.

We define the Boolean algebra $\mathcal{A} = (\omega/E^{Int}, \vee^{\mathcal{A}}, \wedge^{\mathcal{A}}, C^{\mathcal{A}}, 0^{\mathcal{A}}, 1^{\mathcal{A}})$. Choose the numbers n_0 and n_1 such that $D_{n_0} = \{c_0, c_1\}$ and $D_{n_1} = \{e_0\}$. Set $0^{\mathcal{A}} = n_0$ and $1^{\mathcal{A}} = n_1$. Here we give a detailed definition only for the union $\vee^{\mathcal{A}}$. Assume that $x, y \in \omega$. If $D_x = \emptyset$ or $D_y = \emptyset$, then set $(x \vee^{\mathcal{A}} y) = z$, where $D_z = D_x \cup D_y$. Now suppose that $D_x = \{a_1 <_\omega a_2 <_\omega \dots <_\omega a_n\}$ and $D_y = \{b_1 <_\omega b_2 <_\omega \dots <_\omega b_m\}$, where $<_\omega$ is the standard ordering of natural numbers. Let $s_0 = s^*(e_0, \bar{a}, \bar{b})$ and consider the finite interval algebra $\mathcal{D} = Int(\mathcal{L}(e_0, \bar{a}, \bar{b}, s_0))$. We find the element $p \in \mathcal{D}$ which is equal to the union $int^{s_0}(\bar{a}) \vee^{\mathcal{D}} int^{s_0}(\bar{b})$. Choose $d_j \in D_x \cup D_y$, $1 \leq j \leq k$, such that $d_j \trianglelefteq^{s_0} d_{j+1}$ for all j and $p = int^{s_0}(d_1, d_2, \dots, d_k)$. Find the number z_0 such that $D_{z_0} = \{d_1, d_2, \dots, d_k\}$ and set $(x \vee^{\mathcal{A}} y) = z_0$. The operations $\wedge^{\mathcal{A}}$ and $C^{\mathcal{A}}$ are defined in a similar way. It is straightforward to prove that the constructed structure $\mathcal{A}$ is a c.e. Boolean algebra isomorphic to $Int(\mathcal{L})$.

Fokina, Khoussainov, Semukhin, and Turetsky [5, Theorem 24] proved that there is a non-computable c.e. equivalence relation E that realizes both $(\mathbb{Q}, \leq)$ and $(\omega^2, \leq)$. This and Theorem 1 together imply the following

Corollary 6. *There is a non-computable c.e. equivalence relation F such that it realizes both the countable atomless Boolean algebra and the interval algebra $Int(\omega^2)$.*

7. Connections to c.e. groups

Theorem 2. *There exists a c.e. equivalence relation E with the following properties:*

1. *E is non-computable and $[a]_E \equiv_m E$ for all $a \in \omega$,*
2. *E realizes a c.e. abelian group,*
3. *E does not realize any Boolean algebra.*

Proof. Without loss of generality, we may identify ω with $3^{<\omega}$. Consider the group $\mathcal{G}_0$ such that it consists of all finite ternary strings, and the group operation of $\mathcal{G}_0$ is bitwise addition modulo 3. More formally, let E_0 be the smallest equivalence relation on $3^{<\omega}$ with the following property: For any $\sigma \in 3^{<\omega}$, we have $(\sigma, \sigma\widehat{\ }0) \in E_0$. The universe of $\mathcal{G}_0$ is equal to ω/E_0, and the addition on $\mathcal{G}_0$ is induced by the bitwise addition modulo 3; for example, we have $[011221]_{E_0} + [0011]_{E_0} = [012021]_{E_0}$.

Fix a hypersimple set A. The relation E is the smallest equivalence relation on $3^{<\omega}$ such that $E \supseteq E_0$ and for any σ and τ, the following holds: If $|\sigma| = |\tau|$, then we have

$$(\sigma, \tau) \in E \text{ iff } \sigma(k) = \tau(k) \text{ for all } k \notin A.$$

It is easy to show that the relation E is c.e. and $A \leq_m E$. Therefore, the structure $\mathcal{G}_0/E$ is a c.e. abelian group.

Lemma 3. *If a c.e. equivalence relation F realizes a group, then for any $x \in \omega$, we have $[x]_F \equiv_m F$.*

Proof. It is clear that $[x]_F \leq_m F$. For $x, y \in \omega$, we have the following:

$$(yFx) \;\Leftrightarrow\; y^{-1}x \in [1]_F \;\Leftrightarrow\; y^{-1}x^2 \in [x]_F.$$

Hence, $[x]_F \equiv_m F$.

Using Lemma 3, we deduce that $[a]_E \equiv_m E$ for all $a \in \omega$.

Now we prove that E cannot realize a Boolean algebra. Assume that $\mathcal{B}$ is a c.e. Boolean algebra realized by E. Suppose that $\overline{A} = \{a_0 <_\omega a_1 <_\omega a_2 <_\omega \ldots\}$. Notice that for an element $\sigma \in \mathcal{B}$, we have $C(\sigma) \notin [\sigma]_E$. This implies that there is a number m such that $\sigma(a_m) \neq (C(\sigma))(a_m)$.

We define a computable total function $f(x)$ as follows. Let

$$f(0) = \max\{k \in \omega : (C(\Lambda))(k) \neq 0\},$$

where Λ is the empty string. It is easy to see that $f(0) \geq a_0$. Now assume that $f(n)$ is already defined and $f(n) \geq a_n$. We define

$$f(n+1) = \max\big\{k : \text{There is a string } \sigma \text{ such that } |\sigma| = f(n)+1 \\ \text{and } (C(\sigma))(k) \neq 0\big\}.$$

Suppose that $M_n = \{\sigma \in 3^{<\omega} : \sigma(a_k) = 0 \text{ for all } k \text{ with } a_{n+1} \leq a_k < |\sigma|\}$. Note that the set M_n contains exactly 3^{n+1} pairwise E-inequivalent strings. Since 3^{n+1} is an odd number and $f(n) \geq a_n$, there is a string σ_0 such that $\sigma_0 \in M_n$, $|\sigma_0| = f(n)+1$, and $C(\sigma_0) \notin M_n$. This implies that $f(n+1) \geq a_{n+1}$.

We built the computable function $f(x)$ such that $f(n) \geq a_n$ for all $n \in \omega$. This contradicts the hyperimmunity of the set $\overline{A}$.

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References

[1] P. S. Novikov, *On the algorithmic unsolvability of the word problem in group theory*, Proceedings of the Steklov Institute of Mathematics, **44** (1955), 1–143. (In Russian). MR0075197

[2] Emmy Noether. *Idealtheorie in Ringbereichen*. Mathematische Annalen, **83**:1–2 (1921), 24–66. MR1511996

[3] Walter Baur, *Rekursive Algebren mit Kettenbedingungen*. Zeitschrift für mathematische Logik und Grundlagen der Mathematik, **20** (1974), 37–46. MR0351781

[4] A. Gavruskin, S. Jain, B. Khoussainov, and F. Stephan, *Graphs realized by r.e. equivalence relations*, Ann. Pure Appl. Logic, **165**:7–8 (2014), 1263–1290. MR3201708

[5] E. Fokina, B. Khoussainov, P. Semukhin, and D. Turetsky, *Linear orders realized by c.e. equivalence relations*, J. Symb. Logic, **81**:2 (2016), 463–482. MR3519440

[6] S. S. Goncharov, *Countable Boolean algebras and decidability*, Consultants Bureau, New York, 1997. MR1444819

[7] Yu. L. Ershov, *Theory of numberings*, Nauka, Moscow, 1977. (In Russian). MR0506676

[8] S. Gao and P. Gerdes, *Computably enumerable equivalence relations*, Stud. Log., **67**:1 (2001), 27–59. MR1833652

[9] U. Andrews, S. Lempp, J. S. Miller, K. M. Ng, L. San Mauro, and A. Sorbi, *Universal computably enumerable equivalence relations*, J. Symb. Log., **79**:1 (2014), 60–88. MR3226012

[10] U. Andrews, S. Badaev, and A. Sorbi, *A survey on universal computably enumerable equivalence relations*, Lect. Notes Comput. Sci., **10010** (2016), 418–451.

[11] A. Gavryushkin, B. Khoussainov, and F. Stephan, *Reducibilities among equivalence relations induced by recursively enumerable structures*, Theoret. Comput. Sci. **612** (2016), 137–152. MR3433520

[12] L. Feiner, *Hierarchies of Boolean algebras*, J. Symb. Log., **35**:3 (1970), 365–374. MR0282805

[13] S. P. Odintsov and V. L. Selivanov, *The arithmetical hierarchy and ideals of enumerated Boolean algebras*, Siberian Math. J., **30**:6 (1990), 952–960. MR1043442

[14] S. A. Badaev, *On weakly precomplete positive equivalences*, Siberian Math. J., **32**:2 (1991), 321–323. MR1138451

[15] S. Badaev and A. Sorbi, *Weakly precomplete computably enumerable equivalence relations*, Math. Log. Q., **62**:1–2 (2016), 111–127. MR3472186

[16] R. I. Soare, *Recursively enumerable sets and degrees*, Springer, Berlin, 1987. MR0882921

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