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THE STEADY PROBLEM OF THE MOTION OF A RIGID BALL IN A STOKES — POISEUILLE FLOW: DIFFERENTIABILITY OF THE SOLUTION WITH RESPECT TO THE BALL POSITION

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Abstract. This paper deals with the steady problem of the motion of a rigid body in a viscous incompressible fluid that fills a cylindrical domain. The fluid flow is governed by the Stokes equation and tends to Poiseuille flow at infinity. The body is a ball that moves according to the laws of classical mechanics. The unique solvability of this problem was proved in an earlier work of the authors. Here, the differentiability of the solution in the function space L^2 with respect to the position of the ball is established.

Keywords: viscous fluid, rigid body, cylindrical pipe, steady motion.

1. Introduction

Let Σ be a bounded domain in $\mathbb{R}^2$ with a locally Lipschitz boundary $\partial\Sigma$ and $\Omega = \Sigma \times \mathbb{R}$. The boundary of the cylindrical domain $\Omega \in \mathbb{R}^3$ will be denoted by Γ. Clearly, $\Gamma = \partial\Sigma \times \mathbb{R}$. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the orthonormal basis in $\mathbb{R}^3$ such that Σ is in the plane of the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. The coordinates of a point $\boldsymbol{x} \in \mathbb{R}^3$ with respect to this basis will be denoted by x^i, $i = 1, 2, 3$. Thus, $\boldsymbol{x} = (x^1, x^2, x^3)$ and $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 \mid (x^1, x^2) \in \Sigma\}$. Besides that, the components of a vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ will be also related to this basis and denoted by v^i: $\boldsymbol{v} = (v^1, v^2, v^3)$.

Suppose that the domain Ω is filled with a viscous incompressible fluid and a rigid body swims there. It is assumed that the fluid is governed by the Stokes equation and the body moves according to the laws of classical mechanics. In [1],

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the unique solvability of this problem was proved. Let us briefly recall the main result of that paper.

First of all, it is assumed that the flow tends to the Poiseuille flow at infinity:

$$\boldsymbol{v}(\boldsymbol{x}) \to \boldsymbol{v}_p(\boldsymbol{x}) \quad \text{as} \quad x^3 \to \pm\infty,$$

where $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ is the velocity of the fluid, $\boldsymbol{v}_p(\boldsymbol{x}) = (0, 0, V_p(x^1, x^2))$ and the function $V_p : \Sigma \to \mathbb{R}$ is the solution of the following problem:

$$\partial_1^2 V_p + \partial_2^2 V_p = \lambda = const, \quad (x^1, x^2) \in \Sigma,$$
$$V_p(x^1, x^2) = 0, \quad (x^1, x^2) \in \partial\Sigma.$$

Every value of the constant λ gives a corresponding Poiseuille flow. Instead of λ, one can use another parameter that is more meaningful from mechanical point of view, namely, the flow rate:

$$\int_\Sigma V_p(x^1, x^2)\, dx^1 dx^2 = Q \in \mathbb{R}.$$

It is not difficult to see that

$$\lambda = -\frac{1}{Q} \int_\Sigma |\nabla V_p|^2\, dx^1 dx^2.$$

Due to the flow of the fluid, the body cannot stay put and, generally speaking, the steady solution cannot exist. For this reason, the problem was reformulated in a coordinate system that moves together with the body. We refer for details to [1] and give only the final statement of the problem which will be called Problem A_{st}.

The subdomains of Ω occupied by the fluid and the body will be denoted by F and B, respectively. It is assumed that B is a ball centered in a point $\boldsymbol{x}_* \in \Omega$. So, $\Omega = F \cup \partial B \cup B$, where ∂B is the boundary of the body. After the mentioned above reformulation of the problem, the point $\boldsymbol{x}_*$ stays in the plane of the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, i.e., $x_*^3 = 0$. We suppose also that $\operatorname{dist}(B, \Gamma) > 0$.

In order to introduce the notion of weak solution of Problem A_{st}, let us describe necessary function spaces. We use the same notations for spaces of scalar and vector functions. Let us fix an arbitrary positive real number δ that is greater than the radius of the body B and denote $\Omega_\delta = \Sigma \times (-\delta, \delta)$. Let G be either Ω or Ω_δ. We will employ the Lebesgue and Sobolev spaces $L^p(G)$ and $H^1(G)$ as well as usual spaces of mathematical hydrodynamics:

$$C_\sigma^\infty(G) = \{\boldsymbol{v} \in C^\infty(G) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } G \quad \text{and} \quad \boldsymbol{v}|_{\Gamma \cap \overline{G}} = 0\},$$
$L_\sigma^2(G)$ is the closure of $C_\sigma^\infty(G) \cap L^2(G)$ in $L^2(G)$,
$H_\sigma^1(G)$ is the closure of $C_\sigma^\infty(G) \cap H^1(G)$ in $H^1(G)$.

Besides that, let us introduce the function classes which are specific for the problem of the motion of a rigid body in a fluid. Let $\mathfrak{D}(\boldsymbol{v})$ be the strain rate tensor that corresponds to a velocity field $\boldsymbol{v}$ and has the following components:

$$\mathfrak{D}^{ij}(\boldsymbol{v}) = \frac{1}{2}\Big(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\Big).$$

Sometimes we will write $\mathfrak{D}_{\boldsymbol{x}}$ in order to emphasize that the differentiation in the operator $\mathfrak{D}$ is carried out with respect to $\boldsymbol{x}$. Given a domain $G_0 \subset G$, define the following function spaces:

$$L^2_{\mathcal{R}}(G,G_0) = \big\{\boldsymbol{v} \in L^2_\sigma(G) \mid \mathfrak{D}(\boldsymbol{v}) = 0 \text{ in } G_0\big\},$$
$$H^1_{\mathcal{R}}(G,G_0) = \big\{\boldsymbol{v} \in H^1_\sigma(G) \mid \mathfrak{D}(\boldsymbol{v}) = 0 \text{ in } G_0\big\}.$$

It is well known that $\mathfrak{D}(\boldsymbol{v}) = 0$ in G_0 if and only if $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\eta} + \boldsymbol{\zeta} \times \boldsymbol{x}$ for some vectors $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$. This means that $\boldsymbol{v}$ is the velocity vector field of a rigid motion. These spaces were introduced and investigated in [2, 3] (see also [4, 5]). The norms in the spaces $L^2_{\mathcal{R}}(G,G_0)$ and $H^1_{\mathcal{R}}(G,G_0)$ will be the same as in the spaces $L^2(G)$ and $H^1(G)$ respectively. As it follows from the Korn inequality, the norm in $H^1_\sigma(G)$ and in $H^1_{\mathcal{R}}(G,G_0)$ can be equivalently defined by the following expression:

$$\Big(\int_G \big|\mathfrak{D}(\boldsymbol{v})\big|^2\,d\boldsymbol{x}\Big)^{1/2},$$

where $|\mathfrak{D}(\boldsymbol{v})|^2 = \mathfrak{D}(\boldsymbol{v}) : \mathfrak{D}(\boldsymbol{v}) = \mathfrak{D}^{ij}(\boldsymbol{v})\mathfrak{D}^{ij}(\boldsymbol{v})$. Here and below, we use the usual convention of summing over repeated indices.

Denote by $\boldsymbol{u}$ the velocity vector field in the whole domain Ω:

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{cases} \boldsymbol{v}(\boldsymbol{x}), & \boldsymbol{x} \in F, \\ \boldsymbol{\eta} + \boldsymbol{\zeta} \times (\boldsymbol{x} - \boldsymbol{x}_*), & \boldsymbol{x} \in B \cup \partial B. \end{cases}$$

Notice that if the domain G is defined as above and $\boldsymbol{u} \in H^1_\sigma(G)$, then $\boldsymbol{u} \in H^1_{\mathcal{R}}(G,B)$. We assume that the densities of the fluid and of the body are positive constants ϱ_f and ϱ_s respectively. Besides that, the acceleration of the external mass force is the vector $\boldsymbol{g} = (0,0,-g)$, where g is a positive constant.

Definition 1. *A vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and an open ball $B \subset \Omega$ form a weak solution of Problem A_{st}, if $x^3_* = 0$, where $\boldsymbol{x}_*$ is the center of B, $\boldsymbol{u} \in H^1_{\mathcal{R}}(\Omega_\delta, B)$, $\boldsymbol{u} - \boldsymbol{v}_p \in H^1_\sigma(\Omega)$, and the following integral identity*

$$2\mu \int_\Omega \mathfrak{D}(\boldsymbol{u})(\boldsymbol{x}) : \mathfrak{D}(\boldsymbol{\varphi})(\boldsymbol{x})\,d\boldsymbol{x} = \int_B (\varrho_s - \varrho_f)\,\boldsymbol{g}\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x} \tag{1}$$

holds true for an arbitrary function $\boldsymbol{\varphi} \in H^1_{\mathcal{R}}(\Omega, B)$ that has a bounded support.

This definition is standard for the problems of the considered type (see, for instance, [2, 1]). The fact that velocity field is rigid in the domain B corresponding to the body is contained in the inclusion $\boldsymbol{u} \in H^1_{\mathcal{R}}(\Omega_\delta, B)$ which implies that $\mathfrak{D}(\boldsymbol{u}) = 0$ in B. Notice also that the test function $\boldsymbol{\varphi}$ depends on the domain B.

The following theorem was proved in the paper [1].

Theorem 1. *For every position of the ball B in the domain Ω such that $x^3_* = 0$ and $\operatorname{dist}(B,\Gamma) > 0$, there exists a unique function $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ that together with B forms a weak solution of Problem A_{st}.*

In the present paper, we investigate the dependence of $\boldsymbol{u}$ on the position of B. Namely, we prove the differentiability of $\boldsymbol{u}$ in $L^2(\Omega_\delta)$ with respect to $\boldsymbol{x}_*$. Let $\boldsymbol{x}_*$ be a smooth function of a scalar parameter t and $x^3_*(t) = 0$ for all t. In other words, we move the center of the rigid ball B in the cross section Σ of the domain Ω. For every t, we denote by $\{\boldsymbol{u}_t, B_t\}$ is the unique solution of Problem A_{st}, where B_t is the ball centered at the point $\boldsymbol{x}_*(t)$. Let us fix a value t_0 of the parameter t and denote by $\{\boldsymbol{u}, B\}$ the corresponding solution of Problem A_{st}. The parameter t will be vary in an interval I_t containing t_0. Without loss of generality we can put $t_0 = 0$.

The following theorem is the main result of this work.

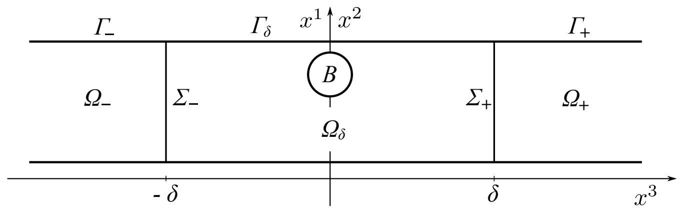

FIG. 1. The partition of the domain Ω into the subdomains Ω_-, Ω_δ, and Ω_+.

Theorem 2. *There exist vector functions* $\boldsymbol{U} : \Omega \to \mathbb{R}^3$ *and* $\boldsymbol{R} : \Omega \times I_t \to \mathbb{R}^3$ *such that*

$$\boldsymbol{U} \in L^2_{\mathcal{R}}(\Omega_\delta, B), \quad \boldsymbol{R}(\cdot, t) \in L^2(\Omega_\delta)$$

and

$$\boldsymbol{u}_t - \boldsymbol{u} = t\boldsymbol{U} + \boldsymbol{R}(\cdot, t)$$

for $t \in I_t$, *where* $\|\boldsymbol{R}(\cdot, t)\|_{L^2(\Omega_\delta)} = o(t)$ *as* $t \to 0$.

Generally speaking, the domain Ω_δ in the formulation of this theorem can be replaced by Ω. Although neither $\boldsymbol{u}$ nor $\boldsymbol{u}_t$ are in $L^2(\Omega)$, their difference $\boldsymbol{u}_t - \boldsymbol{u}$ belongs to this space.

Close results were obtained in [6]. A similar problem in a bounded domain was considered there. Besides that, only the Lipschitz continuity of the solution with respect to the position of the body was proved. It is worth noting that a technique similar to ours is often employed in various problems of shape optimization. There are a lot of works on this topic and we cannot mention all of them. We cite only [7] and [8], where shape optimization problems of fluid dynamics are considered. Another works one can find in the references to these papers. Our result differs from that obtained in the cited papers. We have not only a fluid, but a mechanical system that consists of a fluid and a rigid body. Even the unique solvability of this problem was proved quite recently (see [1]).

2. Statement of the problem in a bounded domain

The main difficulties in [1] when proving Theorem 1 were the unboundedness of the domain Ω and the inhomogeneous condition at infinity. For this reason, the problem was reduced to a problem in a bounded domain by employing a some kind of the Dirichlet-to-Neumann operators $\mathbb{P}_\pm$ associated with the Stokes operator. This formulation of the problem in a bounded domain is more convenient and will be used also in the present work. Let us recall the main steps of that approach.

We have already fixed an arbitrary positive real number δ that is greater than the radius of the body B. Let us introduce the following sets (see Fig. 1):

$$\Omega_+ = \{\boldsymbol{x} \in \Omega \mid x^3 > \delta\}, \quad \Gamma_+ = \{\boldsymbol{x} \in \Gamma \mid x^3 \geqslant \delta\}, \quad \Sigma_+ = \{\boldsymbol{x} \in \Omega \mid x^3 = \delta\},$$

$$\Omega_- = \{\boldsymbol{x} \in \Omega \mid x^3 < -\delta\}, \quad \Gamma_- = \{\boldsymbol{x} \in \Gamma \mid x^3 \leqslant -\delta\}, \quad \Sigma_- = \{\boldsymbol{x} \in \Omega \mid x^3 = -\delta\},$$

$$\Omega_\delta = \{\boldsymbol{x} \in \Omega \mid -\delta < x^3 < \delta\}, \quad \Gamma_\delta = \{\boldsymbol{x} \in \Gamma \mid -\delta \leqslant x^3 \leqslant \delta\}.$$

Consider the Stokes problems in the domains Ω_+ and Ω_-:

$$\begin{gathered}\operatorname{div} P(\boldsymbol{w}_\pm, q_\pm) = \mathbf{0}, \quad \operatorname{div} \boldsymbol{w}_\pm = 0, \quad \boldsymbol{x} \in \Omega_\pm, \\ P(\boldsymbol{w}_\pm, q_\pm) = -q_\pm I + 2\mu\,\mathfrak{D}(\boldsymbol{w}_\pm), \\ \boldsymbol{w}_\pm(\boldsymbol{x}) = \mathbf{0} \quad (\boldsymbol{x} \in \Gamma_\pm), \qquad \boldsymbol{w}_\pm(\boldsymbol{x}) = \boldsymbol{a}_\pm(\boldsymbol{x}) \quad (\boldsymbol{x} \in \Sigma_\pm), \\ \boldsymbol{w}_\pm(\boldsymbol{x}) \to \mathbf{0} \quad \text{as } x^3 \to \pm\infty,\end{gathered} \tag{2}$$

where $\boldsymbol{a}_+ : \Sigma_+ \to \mathbb{R}^3$ and $\boldsymbol{a}_- : \Sigma_- \to \mathbb{R}^3$ are arbitrary vector fields vanishing at $\partial\Sigma_+$ and $\partial\Sigma_-$ respectively and such that

$$\int_{\Sigma_+} a_+^3\, ds = \int_{\Sigma_-} a_-^3\, ds = 0.$$

Let us introduce spaces of functions defined on Σ_+ and Σ_-. We say that $\boldsymbol{w} \in H^{1/2}_{00}(\Sigma_\pm)$, if there exists a function $\widetilde{\boldsymbol{w}} \in H^{1/2}(\Sigma_\pm \cup \Gamma_\pm)$ such that $\widetilde{\boldsymbol{w}} = \boldsymbol{w}$ on $\Sigma_\pm$ and $\widetilde{\boldsymbol{w}} = \mathbf{0}$ on $\Gamma_\pm$. Define the following function spaces:

$$H^{1/2}_\theta(\Sigma_\pm) = \big\{\boldsymbol{w} \in H^{1/2}_{00}(\Sigma_\pm) \;\big|\; \int_{\Sigma_\pm} w^3\, ds = 0\big\}.$$

Denote by γ_+ and γ_- the trace operators at Σ_+ and Σ_- respectively. It is well known (see [9]) that

$$\gamma_\pm H^1_\sigma(\Omega_\pm) = H^{1/2}_\theta(\Sigma_\pm).$$

The Stokes problem is thoroughly investigated (see [9]). In particular, it is established that, for an arbitrary $\boldsymbol{a}_+ \in H^{1/2}_\theta(\Sigma_+)$, this problem in the domain Ω_+ has a weak solution such that $\boldsymbol{w}_+ \in H^1_\sigma(\Omega_+)$ and $q_+ \in L^2_{loc}(\Omega_+)$. Moreover, this solution is unique up to a constant that can be added to q_+. Therefore, $P(\boldsymbol{w}_+, q_+) \in L^2_{loc}(\Omega_+)$ and, due to the first equation in (2), the trace of the quantity $P(\boldsymbol{w}_+, q_+)(-\boldsymbol{e}_3)$ is well defined on Σ_+ as an element of the space $H^{-1/2}_\theta(\Sigma_+)$ that is topologically adjoint to $H^{1/2}_\theta(\Sigma_+)$. Thus, we have defined a bounded linear operator $\mathbb{P}_+ : H^{1/2}_\theta(\Sigma_+) \to H^{-1/2}_\theta(\Sigma_+)/\mathbb{R}$ such that $\mathbb{P}_+(\boldsymbol{a}_+)$ is the trace of $P(\boldsymbol{w}_+, q_+)(-\boldsymbol{e}_3)$ on Σ_+. If we additionally require that $\mathbb{P}_+(\mathbf{0}) = \mathbf{0}$, then $\mathbb{P}_+$ is a well defined bounded linear operator from $H^{1/2}_\theta(\Sigma_+)$ to $H^{-1/2}_\theta(\Sigma_+)$. Similarly, by changing the subscript "+" to "−", we can define a bounded linear operator $\mathbb{P}_- : H^{1/2}_\theta(\Sigma_-) \to H^{-1/2}_\theta(\Sigma_-)$.

These operators have the following properties:

$$\langle \mathbb{P}_\pm(\boldsymbol{\varphi}), \boldsymbol{\psi}\rangle = \langle \mathbb{P}_\pm(\boldsymbol{\psi}), \boldsymbol{\varphi}\rangle \quad \text{and} \quad \langle \mathbb{P}_\pm(\boldsymbol{\varphi}), \boldsymbol{\varphi}\rangle \geqslant 0$$

for arbitrary functions $\boldsymbol{\varphi}, \boldsymbol{\psi} \in H^{1/2}_\theta(\Sigma_\pm)$. Here, $\langle\cdot,\cdot\rangle$ is the duality pairing between the spaces $H^{-1/2}_\theta(\Sigma_\pm)$ and $H^{1/2}_\theta(\Sigma_\pm)$.

It was proved in [1] that (1) in definition 1 can be replaced by the following integral identity

$$\begin{aligned}2\mu \int_{\Omega_\delta} \mathfrak{D}(\boldsymbol{u} - \boldsymbol{v}_p)(\boldsymbol{x}) : \mathfrak{D}(\boldsymbol{\varphi})(\boldsymbol{x})\, d\boldsymbol{x} + \langle \mathbb{P}_-(\gamma_-(\boldsymbol{u} - \boldsymbol{v}_p)), \gamma_-\boldsymbol{\varphi}\rangle \\ + \langle \mathbb{P}_+(\gamma_+(\boldsymbol{u} - \boldsymbol{v}_p)), \gamma_+\boldsymbol{\varphi}\rangle = \int_B (\varrho_s - \varrho_f)\, \boldsymbol{g} \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}\end{aligned} \tag{3}$$

which holds true for an arbitrary function $\boldsymbol{\varphi} \in \widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, 0)$, where

$$\widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, \tau) = \big\{\boldsymbol{\varphi} \in H^1_{\mathcal{R}}(\Omega_\delta, B) \;\big|\; \int_{\Sigma_-} \boldsymbol{\varphi}\cdot\boldsymbol{e}_3\, d\Sigma = \int_{\Sigma_+} \boldsymbol{\varphi}\cdot\boldsymbol{e}_3\, d\Sigma = \tau\big\}.$$

For every $\tau \in \mathbb{R}$, the set $\widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, \tau)$ is a closed linear manifold in $H^1_{\mathcal{R}}(\Omega_\delta, B)$. The function $\boldsymbol{u}$ must belong to $\widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, Q)$, where Q is the flow rate of the Poiseuille flow with the velocity field $\boldsymbol{v}_p$.

3. Proof of the main result

In this section, we prove Theorem 2. To do this, we have to investigate the difference $\boldsymbol{u}_t - \boldsymbol{u}$. Here, we encounter the difficulty related to the fact that $\boldsymbol{u}_t$ and $\boldsymbol{u}$ lie in different function spaces: $\boldsymbol{u}_t \in H^1_{\mathcal{R}}(\Omega_\delta, B_t)$ and $\boldsymbol{u} \in H^1_{\mathcal{R}}(\Omega_\delta, B)$. Besides that, the test function $\boldsymbol{\varphi}$ in equation (3) (and in (1)) for these functions are also in different spaces. For this reason, we have to change the variable $\boldsymbol{x}$ in the problem that shifts B_t to B.

Let σ be a positive number such that 3σ-neighborhood of B lies in Ω_δ. Define in Ω_δ an arbitrary smooth vector field $\boldsymbol{h}$ such that $\operatorname{div} \boldsymbol{h} = 0$ in Ω_δ, $\mathfrak{D}(\boldsymbol{h}) = 0$ in the σ-neighborhood of B, and $\boldsymbol{h} = 0$ outside of the 2σ-neighborhood of B. Define a one-parameter group of mappings $G_t : \Omega_\delta \to \Omega_\delta$, $t \in \mathbb{R}$, such that $G_t(\boldsymbol{\xi})$ is the unique solution of the following problem:

$$\frac{dG_t(\boldsymbol{\xi})}{dt} = \boldsymbol{h}\big(G_t(\boldsymbol{\xi})\big), \quad G_0(\boldsymbol{\xi}) = \boldsymbol{\xi} \in \Omega_\delta.$$

Since $\operatorname{div} \boldsymbol{h} = 0$, the Jacobian of the transformation G_t is equal to one in Ω_δ. There is a depending on $\boldsymbol{h}$ positive number t_h such that $G_t(B)$ is in the σ-neighborhood of B if $t \in (-t_h, t_h)$. Without loss of generality, we assume that $I_t = (-t_h, t_h)$ and $B_t = G_t(B)$. Notice that B_t is an isometric image of B for $t \in I_t$ since $\mathfrak{D}(\boldsymbol{h}) = 0$ in the σ-neighborhood of B.

It is not difficult to see that

$$G_t(\boldsymbol{y}) = \boldsymbol{y} + t\,\boldsymbol{h}(\boldsymbol{y}) + \boldsymbol{r}_{0t}(\boldsymbol{y}) \tag{4}$$

for all $t \in I_t$, where

$$\boldsymbol{r}_{0t}(\boldsymbol{y}) = \int_0^t \big(\boldsymbol{h}\big(G_s(\boldsymbol{y})\big) - \boldsymbol{h}(\boldsymbol{y})\big)\, ds.$$

If $\boldsymbol{h}$ is a function from $C^3(\Omega_\delta)$, then $\|\boldsymbol{r}_{0t}\|_{C^2(\Omega_\delta)} = o(t)$ as $t \to 0$. Recall that $\boldsymbol{h}$ is as smooth as we would like.

The function $\boldsymbol{u}_t$ is the unique solution of Problem A_{st} with B replaced by B_t. This means that $\boldsymbol{u}_t \in \widehat{H}^1_R(\Omega_\delta, B_t, Q)$ and

$$\begin{aligned}(5)\quad 2\mu \int_{\Omega_\delta} \mathfrak{D}(\boldsymbol{u}_t - \boldsymbol{v}_p)(\boldsymbol{x}) : \mathfrak{D}(\boldsymbol{\varphi})(\boldsymbol{x})\, d\boldsymbol{x} + \langle \mathbb{P}_-(\gamma_-(\boldsymbol{u}_t - \boldsymbol{v}_p)), \gamma_- \boldsymbol{\varphi}\rangle \\ + \langle \mathbb{P}_+(\gamma_+(\boldsymbol{u}_t - \boldsymbol{v}_p)), \gamma_+ \boldsymbol{\varphi}\rangle = (\varrho_s - \varrho_f) \int_{B_t} \boldsymbol{g} \cdot \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}\end{aligned}$$

for an arbitrary function $\boldsymbol{\varphi} \in \widehat{H}^1_R(\Omega_\delta, B_t, 0)$. Thus, $\boldsymbol{u} = \boldsymbol{u}_t|_{t=0}$ and $B = B_t|_{t=0}$.

In order to prove Theorem 2, we make the change of the integration variable $\boldsymbol{x} = G_t(\boldsymbol{y})$ in (5). We will need some properties of this change. For every $t \in I_t$ and every vector field $\boldsymbol{v} : \Omega_\delta \to \mathbb{R}^3$, let us define a vector field $[\boldsymbol{v}]_t : \Omega_\delta \to \mathbb{R}^3$ with the following components:

$$[\boldsymbol{v}]^i_t(\boldsymbol{y}) = \Big(\frac{\partial G^i_{-t}(\boldsymbol{x})}{\partial x^j}\, v^j(\boldsymbol{x})\Big)_{\boldsymbol{x} = G_t(\boldsymbol{y})}.$$

It is not difficult to see that $\big[[\boldsymbol{v}]_t\big]_\tau = [\boldsymbol{v}]_{t+\tau}$.

Lemma 1. *Let $\boldsymbol{h}$ be as defined above. In particular,* $\operatorname{div}\boldsymbol{h} = 0$ *in* Ω_δ *and* $\mathfrak{D}(\boldsymbol{h}) = 0$ *in* B*. If* $\operatorname{div}\boldsymbol{v} = 0$ *in* Ω_δ*, then* $\operatorname{div}[\boldsymbol{v}]_t = 0$ *in* Ω_δ*. If* $\mathfrak{D}(\boldsymbol{v}) = 0$ *in* B_t*, then* $\mathfrak{D}\big([\boldsymbol{v}]_t\big) = 0$ *in* B*.*

Proof. Since $\operatorname{div}\boldsymbol{h} = 0$, the Jacobian of the mapping G_t is equal to one. Therefore, for every smooth function $\phi : \Omega_\delta \to \mathbb{R}$ with compact support, the following relations hold true:

$$\int_{\Omega_\delta} \operatorname{div}[\boldsymbol{v}]_t(\boldsymbol{y})\,\phi(\boldsymbol{y})\,d\boldsymbol{y} = \int_{\Omega_\delta} [\boldsymbol{v}]_t^i(\boldsymbol{y})\,\frac{\partial\phi(\boldsymbol{y})}{\partial y^i}\,d\boldsymbol{y}$$
$$= \int_{\Omega_\delta} \frac{\partial G^i_{-t}(\boldsymbol{x})}{\partial x^j}\,v^j(\boldsymbol{x})\,\frac{\partial\phi(\boldsymbol{y})}{\partial y^i}\Big|_{\boldsymbol{y}=G_{-t}(\boldsymbol{x})}\,d\boldsymbol{x}$$
$$= \int_{\Omega_\delta} v^j(\boldsymbol{x})\,\frac{\partial}{\partial x^j}\phi\big(G_{-t}(\boldsymbol{x})\big)\,d\boldsymbol{x} = \int_{\Omega_\delta} \operatorname{div} v(\boldsymbol{x})\,\phi\big(G_{-t}(\boldsymbol{x})\big)\,d\boldsymbol{x} = 0.$$

This means that $\operatorname{div}[\boldsymbol{v}]_t = 0$ in Ω_δ.

Before the proof of the second assertion of the lemma, let us make a few observations concerning the mapping G_t. If $\mathfrak{D}(\boldsymbol{h}) = 0$ in B, then there exists a rotation matrix $Q(t)$ and a vector $\boldsymbol{b}(t)$ such that $G_t(\boldsymbol{y}) = Q(t)\boldsymbol{y} + \boldsymbol{b}(t)$ for all $\boldsymbol{y} \in B$. Therefore,

$$\frac{\partial G^i_t(\boldsymbol{y})}{\partial y^j} = Q^i_j(t) \quad\text{and}\quad [\boldsymbol{v}]^i_t(\boldsymbol{y}) = Q^i_j(-t)\,v^j\big(G_t(\boldsymbol{y})\big) \quad\text{for all}\quad \boldsymbol{y} \in B.$$

Notice also, that $Q^k_m(t)\,Q^m_\ell(-t) = \delta^k_\ell$ and, since $Q(t)$ is an orthogonal matrix, $Q^k_m(-t) = Q^m_k(t)$. Thus,

$$\mathfrak{D}^{ik}_{\boldsymbol{y}}([\boldsymbol{v}]_t)(\boldsymbol{y}) = \frac{1}{2}\Big(Q^i_j(-t)\,\frac{\partial v^j(\boldsymbol{x})}{\partial x^m}\,Q^m_k(t) + Q^k_j(-t)\,\frac{\partial v^j(\boldsymbol{x})}{\partial x^m}\,Q^m_i(t)\Big)$$
$$= \frac{1}{2}\Big(Q^i_j(-t)\,\frac{\partial v^j(\boldsymbol{x})}{\partial x^m}\,Q^m_k(t) + Q^k_m(-t)\,\frac{\partial v^m(\boldsymbol{x})}{\partial x^j}\,Q^j_i(t)\Big)$$
$$= \frac{1}{2}\Big(Q^j_i(t)\,\frac{\partial v^j(\boldsymbol{x})}{\partial x^m}\,Q^m_k(t) + Q^m_k(t)\,\frac{\partial v^m(\boldsymbol{x})}{\partial x^j}\,Q^j_i(t)\Big) = Q^j_i(t)\,\mathfrak{D}^{jm}_{\boldsymbol{x}}(\boldsymbol{v})(\boldsymbol{x})\,Q^m_k(t)$$

and, as a consequence, $\mathfrak{D}\big([\boldsymbol{v}]_t\big) = 0$ in B provided that $\mathfrak{D}(\boldsymbol{v}) = 0$ in B_t. □

Lemma 2. *For every* $t \in I_t$ *and for every smooth vector function* $\boldsymbol{h}$*,* $[\,\cdot\,]_t$ *is a bounded linear operator in* $L^2(\Omega_\delta)$ *and in* $H^1(\Omega_\delta)$*:*

$$\big\|[\boldsymbol{v}]_t\big\|_{L^2(\Omega_\delta)} \leqslant C_0\|\boldsymbol{v}\|_{L^2(\Omega_\delta)} \quad\textit{and}\quad \big\|[\boldsymbol{v}]_t\big\|_{H^1(\Omega_\delta)} \leqslant C_1\|\boldsymbol{v}\|_{H^1(\Omega_\delta)},$$

where the constants C_0 *and* C_1 *depend on* $\|\boldsymbol{h}\|_{C^1(\Omega_\delta)}$ *and on* $\|\boldsymbol{h}\|_{C^2(\Omega_\delta)}$ *respectively.*

Moreover, if $\boldsymbol{v} \in H^1(\Omega_\delta)$*, then* $[\boldsymbol{v}]_t \to \boldsymbol{v}$ *in* $H^1(\Omega_\delta)$ *as* $t \to 0$*.*

Proof. The first part of this assertion is obvious and follows directly from the definition of the operator $[\,\cdot\,]_t$. Let us prove the second part related to the limit as $t \to 0$.

At first, we prove that $[\boldsymbol{v}]_t \to \boldsymbol{v}$ weakly in $H^1(\Omega_\delta)$ as $t \to 0$. Since the set of smooth functions is dense in $H^1(\Omega_\delta)$, it is enough to establish that $([\boldsymbol{v}]_t, \boldsymbol{\phi})_{H^1(\Omega_\delta)} \to (\boldsymbol{v}, \boldsymbol{\phi})_{H^1(\Omega_\delta)}$ as $t \to 0$ for any smooth function $\boldsymbol{\phi}$. This can be done by a direct calculation. Really, let us consider, for instance, the following integral:

$$\int_{\Omega_\delta} \frac{\partial[\boldsymbol{v}]^i_t}{\partial y^j}\,\frac{\partial\phi^i}{\partial y^j}\,d\boldsymbol{y} = \int_{\Omega_\delta} \frac{\partial}{\partial x^k}\Big(\frac{\partial G^i_{-t}(\boldsymbol{x})}{\partial x^m}\,v^m(\boldsymbol{x})\Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})}\,\frac{\partial G^k_t(\boldsymbol{y})}{\partial y^j}\,\frac{\partial\phi^i(\boldsymbol{y})}{\partial y^j}\,d\boldsymbol{y}$$
$$= \int_{\Omega_\delta} \frac{\partial}{\partial x^k}\Big(\frac{\partial G^i_{-t}(\boldsymbol{x})}{\partial x^m}\,v^m(\boldsymbol{x})\Big)\Big(\frac{\partial G^k_t(\boldsymbol{y})}{\partial y^j}\,\frac{\partial\phi^i(\boldsymbol{y})}{\partial y^j}\Big)_{\boldsymbol{y}=G_{-t}(\boldsymbol{x})}\,d\boldsymbol{x}.$$

By making use of the Lebesgue dominated convergence theorem and representation (4), we conclude that the last integral tends to $\int_{\Omega_\delta}\partial v^i/\partial y^j\,\partial\phi^i/\partial y^j\,d\boldsymbol{y}$ as $t\to 0$.

Similarly, we can deduce that $\|[\boldsymbol{v}]_t\|_{H^1(\Omega_\delta)}\to\|\boldsymbol{v}\|_{H^1(\Omega_\delta)}$ as $t\to 0$. Since $H^1(\Omega_\delta)$ is a Hilbert space, the assertion of the lemma is proved. □

Lemma 3. *For an arbitrary function $\boldsymbol{v}\in H^1(\Omega_\delta)$, there exist functions $\boldsymbol{W},\boldsymbol{S}_t\in L^2(\Omega_\delta)$ such that*

$$[\boldsymbol{v}]_t-\boldsymbol{v}=t\boldsymbol{W}+\boldsymbol{S}_t \quad \text{for all} \quad t\in I_t,$$

where $\|\boldsymbol{S}_t\|_{L^2(\Omega_\delta)}=o(t)$ as $t\to 0$. Moreover,

$$W^i(\boldsymbol{y})=h^j(\boldsymbol{y})\,\frac{\partial v^i(\boldsymbol{y})}{\partial y^j}-v^j(\boldsymbol{y})\,\frac{\partial h^i(\boldsymbol{y})}{\partial y^j}$$

and

$$\mathfrak{D}^{ik}(\boldsymbol{W})=\frac{\partial h^j}{\partial y^k}\,\mathfrak{D}^{ij}(\boldsymbol{v})+\frac{\partial h^j}{\partial y^i}\,\mathfrak{D}^{kj}(\boldsymbol{v})+h^j\,\frac{\partial}{\partial y^j}\mathfrak{D}^{ik}(\boldsymbol{v})$$
$$-\frac{\partial v^j}{\partial y^k}\,\mathfrak{D}^{ij}(\boldsymbol{h})-\frac{\partial v^j}{\partial y^i}\,\mathfrak{D}^{kj}(\boldsymbol{h})-v^j\,\frac{\partial}{\partial y^j}\mathfrak{D}^{ik}(\boldsymbol{h}).$$

The last relation is understood in the distributional sense.

Proof. Due to (4),

$$S^i_t=[\boldsymbol{v}]^i_t(\boldsymbol{y})-v^i(\boldsymbol{y})-tW^i(\boldsymbol{y})=\Big(\frac{\partial G^i_{-t}(\boldsymbol{x})}{\partial x^j}\,v^j(\boldsymbol{x})\Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})}-v^i(\boldsymbol{y})-tW^i(\boldsymbol{y})$$
$$=\Big(v^i(\boldsymbol{x})-t\,\frac{\partial h^i(\boldsymbol{x})}{\partial x^j}\,v^j(\boldsymbol{x})+\frac{\partial r^i_{0t}(\boldsymbol{x})}{\partial x^j}\,v^j(\boldsymbol{x})\Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})}-v^i(\boldsymbol{y})-tW^i(\boldsymbol{y})$$
$$=r^i_{1t}(\boldsymbol{y})+r^i_{2t}(\boldsymbol{y})+r^i_{3t}(\boldsymbol{y}),$$

where

$$r^i_{1t}(\boldsymbol{y})=v^i\big(G_t(\boldsymbol{y})\big)-v^i(\boldsymbol{y})-th^j(\boldsymbol{y})\,\frac{\partial v^i(\boldsymbol{y})}{\partial y^j},$$
$$r^i_{2t}(\boldsymbol{y})=-t\,\Big(v^j(\boldsymbol{x})\,\frac{\partial h^i(\boldsymbol{x})}{\partial x^j}\Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})}+tv^j(\boldsymbol{y})\,\frac{\partial h^i(\boldsymbol{y})}{\partial y^j},$$
$$r^i_{3t}(\boldsymbol{y})=\Big(\frac{\partial r^i_{0t}(\boldsymbol{x})}{\partial x^j}\,v^j(\boldsymbol{x})\Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})}.$$

It remains to prove that $\|\boldsymbol{r}_{kt}\|_{L^2(\Omega_\delta)}=o(t)$ as $t\to 0$ for $k=1,2,3$. This fact is obvious for $\boldsymbol{r}_{3t}$. Let us consider the other two expressions. For the first of them, we have:

$$r^i_{1t}=\int_0^t\Big(\frac{\partial v^i\big(G_s(\boldsymbol{y})\big)}{\partial s}-h^j(\boldsymbol{y})\,\frac{\partial v^i(\boldsymbol{y})}{\partial y^j}\Big)\,ds$$
$$=\int_0^t\Big(\Big(h^j(\boldsymbol{x})\,\frac{\partial v^i(\boldsymbol{x})}{\partial x^j}\Big)_{\boldsymbol{x}=G_s(\boldsymbol{y})}-h^j(\boldsymbol{y})\,\frac{\partial v^i(\boldsymbol{y})}{\partial y^j}\Big)\,ds.$$

Exactly in the same way as in the proof of Lemma 2 (by employing the Lebesgue dominated convergence theorem), we can prove that the function under the integral in the last expression tends to 0 in $L^2(\Omega_\delta)$ as $s \to 0$. This means that $\|\boldsymbol{r}_{1t}\|_{L^2(\Omega_\delta)} = o(t)$ as $t \to 0$.

As for the function $\boldsymbol{r}_{2t}$, similar arguments enable us to conclude that

$$\int_{\Omega_\delta} \Big(\Big(v^j(\boldsymbol{x}) \, \frac{\partial h^i(\boldsymbol{x})}{\partial x^j} \Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})} - v^j(\boldsymbol{y}) \, \frac{\partial h^i(\boldsymbol{y})}{\partial y^j} \Big)^2 \, d\boldsymbol{y} \to 0 \quad \text{as} \quad t \to 0$$

and, as a consequence, that $\|\boldsymbol{r}_{2t}\|_{L^2(\Omega_\delta)} = o(t)$ as $t \to 0$.

The last relation in the assertion of the lemma can be verified by a direct calculation. □

Now, we are ready to rewrite equation (5) in such a way that the unknown and the test functions will belong to spaces which do not depend on t. We will make the substitution $\boldsymbol{x} = G_t(\boldsymbol{y})$ in this equation. First, we notice that this substitution does not change functions on $\Sigma_\pm$ and, as a consequence, the operators $\mathbb{P}_\pm$. For this reason, we can leave the expressions in the angle brackets without changing. The integral on the right-hand side of (5) will be rewritten as follows:

$$\begin{aligned}
\int_{B_t} \boldsymbol{g} \cdot \boldsymbol{\varphi}(\boldsymbol{x}) \, d\boldsymbol{x} &= \int_{B_t} g^k \, \varphi^k(\boldsymbol{x}) \, d\boldsymbol{x} = \int_B g^k \, \varphi^k\big(G_t(\boldsymbol{y})\big) \, d\boldsymbol{y} \\
&= \int_B g^k \, \varphi^j\big(G_t(\boldsymbol{y})\big) \, \frac{\partial G^m_{-t}(\boldsymbol{x})}{\partial x^j}\Big|_{\boldsymbol{x}=G_t(\boldsymbol{y})} \frac{\partial G^k_t(\boldsymbol{y})}{\partial y^m} \, d\boldsymbol{y} \\
&= \int_B g^k \, \frac{\partial G^k_t(\boldsymbol{y})}{\partial y^m} \, [\boldsymbol{\varphi}]^m_t \, d\boldsymbol{y} = \int_B \tilde{\boldsymbol{g}}(\boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{y}) \, d\boldsymbol{y},
\end{aligned}$$

where $\boldsymbol{\psi} = [\boldsymbol{\varphi}]_t$ and $\tilde{g}^m(\boldsymbol{y}) = g^k \, \partial G^k_t(\boldsymbol{y})/\partial y^m$.

There remains to make the substitution in the first integral term of equation (5). To do this, we perform some auxiliary calculations. Notice that

$$\begin{aligned}
\frac{\partial v^k}{\partial x^m}\big(G_t(\boldsymbol{y})\big) &= \frac{\partial}{\partial y^\ell}\Big(\frac{\partial G^k_t(\boldsymbol{y})}{\partial y^i} \, [\boldsymbol{v}]^i_t(\boldsymbol{y}) \Big) \, \frac{\partial G^\ell_{-t}(\boldsymbol{x})}{\partial x^m}\Big|_{\boldsymbol{x}=G_t(\boldsymbol{y})} \\
&= \frac{\partial}{\partial y^\ell}\Big(\Big(\delta_{ki} + t \frac{\partial h^k(\boldsymbol{y})}{\partial y^i} + \frac{\partial r^k_{0t}(\boldsymbol{y})}{\partial y^i} \Big) [\boldsymbol{v}]^i_t(\boldsymbol{y}) \Big) \Big(\delta_{\ell m} - t \frac{\partial h^\ell(\boldsymbol{x})}{\partial x^m} + \frac{\partial r^\ell_{0t}(\boldsymbol{x})}{\partial x^m} \Big)_{\boldsymbol{x}=G_t(\boldsymbol{y})} \\
&= \frac{\partial [\boldsymbol{v}]^k_t(\boldsymbol{y})}{\partial y^m} + t \, \frac{\partial}{\partial y^m}\Big(\frac{\partial h^k(\boldsymbol{y})}{\partial y^i} \, [\boldsymbol{v}]^i_t(\boldsymbol{y}) \Big) - t \, \frac{\partial h^\ell(\boldsymbol{x})}{\partial x^m}\Big|_{\boldsymbol{x}=G_t(\boldsymbol{y})} \frac{\partial [\boldsymbol{v}]^k_t(\boldsymbol{y})}{\partial y^\ell} + r^{km}_{4t}(\boldsymbol{y}) \\
&= \frac{\partial [\boldsymbol{v}]^k_t(\boldsymbol{y})}{\partial y^m} + t \, \frac{\partial}{\partial y^m}\Big(\frac{\partial h^k(\boldsymbol{y})}{\partial y^i} \, [\boldsymbol{v}]^i_t(\boldsymbol{y}) \Big) - t \, \frac{\partial h^\ell(\boldsymbol{y})}{\partial y^m} \, \frac{\partial [\boldsymbol{v}]^k_t(\boldsymbol{y})}{\partial y^\ell} + r^{km}_{5t}(\boldsymbol{y}),
\end{aligned}$$

where $\boldsymbol{v} \in H^1(\Omega_\delta)$, $\|\boldsymbol{r}_{4t}\|_{L^2(\Omega_\delta)} = \|\boldsymbol{v}\|_{H^1(\Omega_\delta)} \, o(t)$ and $\|\boldsymbol{r}_{5t}\|_{L^2(\Omega_\delta)} = \|\boldsymbol{v}\|_{H^1(\Omega_\delta)} \, o(t)$ as $t \to 0$. We employed Lemma 2 when estimated the remainders $\boldsymbol{r}_{4t}$ and $\boldsymbol{r}_{5t}$. Therefore,

$$\begin{aligned}
\mathfrak{D}^{km}_{\boldsymbol{x}}(\boldsymbol{v})\big(G_t(\boldsymbol{y})\big) &= \mathfrak{D}^{km}_{\boldsymbol{y}}([\boldsymbol{v}]_t)(\boldsymbol{y}) + t \, [\boldsymbol{v}]^i_t(\boldsymbol{y}) \, \frac{\partial}{\partial y^i} \mathfrak{D}^{km}_{\boldsymbol{y}}(\boldsymbol{h})(\boldsymbol{y}) \\
&\quad + \frac{t}{2}\Big(\frac{\partial h^k(\boldsymbol{y})}{\partial y^i} \, \frac{\partial [\boldsymbol{v}]^i_t(\boldsymbol{y})}{\partial y^m} + \frac{\partial h^m(\boldsymbol{y})}{\partial y^i} \, \frac{\partial [\boldsymbol{v}]^i_t(\boldsymbol{y})}{\partial y^k} \Big) \\
&\quad - \frac{t}{2}\Big(\frac{\partial h^\ell(\boldsymbol{y})}{\partial y^m} \, \frac{\partial [\boldsymbol{v}]^k_t(\boldsymbol{y})}{\partial y^\ell} + \frac{\partial h^\ell(\boldsymbol{y})}{\partial y^k} \, \frac{\partial [\boldsymbol{v}]^m_t(\boldsymbol{y})}{\partial y^\ell} \Big) + r^{km}_{6t}(\boldsymbol{y}),
\end{aligned}$$

where $\|\boldsymbol{r}_{6t}\|_{L^2(\Omega_\delta)} = \|\boldsymbol{v}\|_{H^1(\Omega_\delta)}\, o(t)$ as $t \to 0$. If $\boldsymbol{\phi}$ is another vector field from $H^1(\Omega_\delta)$, then after some calculation we find that

$$\big(\mathfrak{D}_{\boldsymbol{x}}(\boldsymbol{v}) : \mathfrak{D}_{\boldsymbol{x}}(\boldsymbol{\phi})\big)\big(G_t(\boldsymbol{y})\big) = \big(\mathfrak{D}^{km}_{\boldsymbol{x}}(\boldsymbol{v})\, \mathfrak{D}^{km}_{\boldsymbol{x}}(\boldsymbol{\phi})\big)\big(G_t(\boldsymbol{y})\big)$$
$$= \big(\mathfrak{D}^{km}_{\boldsymbol{y}}([\boldsymbol{v}]_t)\, \mathfrak{D}^{km}_{\boldsymbol{y}}([\boldsymbol{\phi}]_t)\big)(\boldsymbol{y}) + t\, F\big([\boldsymbol{v}]_t, [\boldsymbol{\phi}]_t\big)(\boldsymbol{y}) + r_{7t}(\boldsymbol{y}),$$

where $\|r_{7t}\|_{L^1(\Omega_\delta)} = \|\boldsymbol{v}\|_{H^1(\Omega_\delta)}\, \|\boldsymbol{\phi}\|_{H^1(\Omega_\delta)}\, o(t)$ as $t \to 0$ and

$$F(\boldsymbol{v}, \boldsymbol{\phi}) = \Big(\mathfrak{D}^{km}_{\boldsymbol{y}}(\boldsymbol{\phi})\, v^i + \mathfrak{D}^{km}_{\boldsymbol{y}}(\boldsymbol{v})\, \phi^i\Big) \frac{\partial}{\partial y^i} \mathfrak{D}^{km}_{\boldsymbol{y}}(\boldsymbol{h})$$
$$+ 2\, \mathfrak{D}^{ik}_{\boldsymbol{y}}(\boldsymbol{h}) \Big(\mathfrak{S}^{im}_{\boldsymbol{y}}(\boldsymbol{v})\, \mathfrak{D}^{km}_{\boldsymbol{y}}(\boldsymbol{\phi}) + \mathfrak{S}^{im}_{\boldsymbol{y}}(\boldsymbol{\phi})\, \mathfrak{D}^{km}_{\boldsymbol{y}}(\boldsymbol{v})\Big),$$

$$\mathfrak{S}^{im}_{\boldsymbol{y}}(\boldsymbol{v}) = \frac{1}{2}\Big(\frac{\partial v^i}{\partial y^m} - \frac{\partial v^m}{\partial y^i}\Big).$$

Taking into account these calculations, we find, after the substitution $\boldsymbol{x} = G_t(\boldsymbol{y})$, that integral identity (5) takes the form:

$$(6)\quad 2\mu \int_{\Omega_\delta} \mathfrak{D}_{\boldsymbol{y}}\big([\boldsymbol{u}_t]_t - [\boldsymbol{v}_p]_t\big)(\boldsymbol{y}) : \mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{\psi})(\boldsymbol{y})\, d\boldsymbol{y} + 2\mu t \int_{\Omega_\delta} F\big([\boldsymbol{u}_t]_t - [\boldsymbol{v}_p]_t, \boldsymbol{\psi}\big)\, d\boldsymbol{y}$$
$$+ \langle \mathbb{P}_-(\gamma_-([\boldsymbol{u}_t]_t - \boldsymbol{v}_p)), \gamma_-\boldsymbol{\psi}\rangle + \langle \mathbb{P}_+(\gamma_+([\boldsymbol{u}_t]_t - \boldsymbol{v}_p)), \gamma_+\boldsymbol{\psi}\rangle$$
$$= (\varrho_s - \varrho_f) \int_B \tilde{\boldsymbol{g}}(\boldsymbol{y}) \cdot \boldsymbol{\psi}(\boldsymbol{y})\, d\boldsymbol{y} + \|\boldsymbol{u}_t - \boldsymbol{v}_p\|_{H^1(\Omega_\delta)}\, \|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)}\, o(t),$$

where $[\boldsymbol{u}_t]_t \in \widehat{H}^1_R(\Omega_\delta, B, Q)$ and $\boldsymbol{\psi} = [\boldsymbol{\varphi}]_t$ is an arbitrary function from the space $\widehat{H}^1_R(\Omega_\delta, B, 0)$. Notice that $\boldsymbol{u}$ is also in $\widehat{H}^1_R(\Omega_\delta, B, Q)$ and satisfies (6) with $t = 0$:

$$(7)\quad 2\mu \int_{\Omega_\delta} \mathfrak{D}_{\boldsymbol{y}}\big(\boldsymbol{u} - \boldsymbol{v}_p\big)(\boldsymbol{y}) : \mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{\psi})(\boldsymbol{y})\, d\boldsymbol{y}$$
$$+ \langle \mathbb{P}_-(\gamma_-(\boldsymbol{u} - \boldsymbol{v}_p)), \gamma_-\boldsymbol{\psi}\rangle + \langle \mathbb{P}_+(\gamma_+(\boldsymbol{u} - \boldsymbol{v}_p)), \gamma_+\boldsymbol{\psi}\rangle$$
$$= (\varrho_s - \varrho_f) \int_B \boldsymbol{g} \cdot \boldsymbol{\psi}(\boldsymbol{y})\, d\boldsymbol{y}.$$

If we take the difference of equations (6) and (7), then we find that

$$(8)\quad 2\mu \int_{\Omega_\delta} \mathfrak{D}_{\boldsymbol{y}}\big([\boldsymbol{u}_t]_t - \boldsymbol{u}\big)(\boldsymbol{y}) : \mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{\psi})(\boldsymbol{y})\, d\boldsymbol{y}$$
$$+ \langle \mathbb{P}_-(\gamma_-([\boldsymbol{u}_t]_t - \boldsymbol{u})), \gamma_-\boldsymbol{\psi}\rangle + \langle \mathbb{P}_+(\gamma_+([\boldsymbol{u}_t]_t - \boldsymbol{u})), \gamma_+\boldsymbol{\psi}\rangle$$
$$= 2\mu \int_{\Omega_\delta} \mathfrak{D}_{\boldsymbol{y}}\big([\boldsymbol{v}_p]_t - \boldsymbol{v}_p\big)(\boldsymbol{y}) : \mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{\psi})(\boldsymbol{y})\, d\boldsymbol{y} - 2\mu t \int_{\Omega_\delta} F\big([\boldsymbol{u}_t]_t - [\boldsymbol{v}_p]_t, \boldsymbol{\psi}\big)\, d\boldsymbol{y}$$
$$+ (\varrho_s - \varrho_f) \int_B \big(\tilde{\boldsymbol{g}}(\boldsymbol{y}) - \boldsymbol{g}\big) \cdot \boldsymbol{\psi}(\boldsymbol{y})\, d\boldsymbol{y} + \|\boldsymbol{u}_t - \boldsymbol{v}_p\|_{H^1(\Omega_\delta)}\, \|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)}\, o(t).$$

Exactly as in [6], from this identity, we can obtain an estimate that looks like $\|[\boldsymbol{u}_t]_t - \boldsymbol{u}\|_{H^1(\Omega_\delta)} \leqslant C\,t$ and implies that $\|\boldsymbol{u}_t - \boldsymbol{u}\|_{L^2(\Omega_\delta)} \leqslant C\,t$. However, this estimate means only the Lipschitz continuity of $\boldsymbol{u}_t$ with respect to t in $L^2(\Omega_\delta)$ at $t = 0$. We want to prove the differentiability at this point, which is a more strong result. In fact, we have to guess the derivative explicitly. To do this, we formally differentiate equation (6) with respect to t at the point $t = 0$.

At first, we notice that

$$\frac{\partial \tilde{g}^m}{\partial t}\Big|_{t=0} = -g\,\frac{\partial h^3}{\partial y^m}$$

and, due to Lemma 3,

$$\frac{\partial [\boldsymbol{v}_p]_t}{\partial t}\Big|_{t=0} = \boldsymbol{w}_p,$$

where

$$\boldsymbol{w}_p = \Big(V_p\,\frac{\partial h^1}{\partial y^3},\, V_p\,\frac{\partial h^2}{\partial y^3},\, h^1\,\frac{\partial V_p}{\partial y^1} + h^2\,\frac{\partial V_p}{\partial y^2}\Big).$$

Therefore, we obtain the following equation:

$$\text{(9)}\quad H(\boldsymbol{U}_0,\boldsymbol{\psi}) = 2\mu\int_{\Omega_\delta}\mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{w}_p):\mathfrak{D}(\boldsymbol{\psi})\,d\boldsymbol{y} - 2\mu\int_{\Omega_\delta}F(\boldsymbol{u}-\boldsymbol{v}_p,\boldsymbol{\psi})\,d\boldsymbol{y} - (\varrho_s-\varrho_f)\int_B g\,\frac{\partial h^3}{\partial y^m}\psi^m\,d\boldsymbol{y},$$

where

$$H(\boldsymbol{U}_0,\boldsymbol{\psi}) = 2\mu\int_{\Omega_\delta}\mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{U}_0):\mathfrak{D}(\boldsymbol{\psi})\,d\boldsymbol{y} + \langle\mathbb{P}_-(\gamma_-\boldsymbol{U}_0),\gamma_-\boldsymbol{\psi}\rangle + \langle\mathbb{P}_+(\gamma_+\boldsymbol{U}_0),\gamma_+\boldsymbol{\psi}\rangle$$

and

$$\text{(10)}\qquad \frac{\partial [\boldsymbol{u}_t]_t}{\partial t}\Big|_{t=0} = \boldsymbol{U}_0.$$

Since $\int_\Sigma U_0^3\,ds = 0$, the terms $\mathbb{P}_\pm(\gamma_\pm\boldsymbol{U}_0)$ are well defined provided $\boldsymbol{U}_0 \in H^1_\sigma(\Omega_\delta)$. Due to the properties of the operators $\mathbb{P}_\pm$, the function H can be taken as an inner product in $\widehat{H}^1_R(\Omega_\delta, B, 0)$. It is not difficult to see that $\boldsymbol{w}_p \in H^1_\sigma(\Omega_\delta)$ and

$$\big\|F(\boldsymbol{u},\boldsymbol{\psi})\big\|_{L^1(\Omega)} \leqslant C\,\|\boldsymbol{u}\|_{H^1(\Omega_\delta)}\,\|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)}$$

for a depending on $\boldsymbol{h}$ constant C. Thus, equation (9) is uniquely solvable in the space $\widehat{H}^1_R(\Omega_\delta, B, 0)$.

Let us forget (10) for a moment and suppose that $\boldsymbol{U}_0 \in \widehat{H}^1_R(\Omega_\delta, B, 0)$ is the unique solution of (9). Then equality (8) can be rewritten as follows:

$$\begin{aligned} 2\mu\,H\big([\boldsymbol{u}_t]_t - \boldsymbol{u} - t\,\boldsymbol{U}_0,\boldsymbol{\psi}\big) &= 2\mu\int_{\Omega_\delta}\mathfrak{D}_{\boldsymbol{y}}\big([\boldsymbol{v}_p]_t - \boldsymbol{v}_p - t\,\boldsymbol{w}_p\big)(\boldsymbol{y}):\mathfrak{D}_{\boldsymbol{y}}(\boldsymbol{\psi})(\boldsymbol{y})\,d\boldsymbol{y} \\ &- 2\mu t\int_{\Omega_\delta}F\big([\boldsymbol{u}_t]_t-\boldsymbol{u},\boldsymbol{\psi}\big)\,d\boldsymbol{y} + 2\mu t\int_{\Omega_\delta}F\big([\boldsymbol{v}_p]_t-\boldsymbol{v}_p,\boldsymbol{\psi}\big)\,d\boldsymbol{y} \\ &+ (\varrho_s-\varrho_f)\int_B\Big(\tilde{\boldsymbol{g}} - \boldsymbol{g} - t\,\frac{\partial\tilde{\boldsymbol{g}}}{\partial t}\Big|_{t=0}\Big)\cdot\boldsymbol{\psi}\,d\boldsymbol{y} + \|\boldsymbol{u}_t - \boldsymbol{v}_p\|_{H^1(\Omega_\delta)}\,\|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)}\,o(t). \end{aligned}$$

It is not difficult to see that the right-hand side of this equality is $o(t)$ as $t\to 0$ and can be estimated by $C\|\boldsymbol{\psi}\|_{H^1(\Omega_\delta)}\,o(t)$, where the constant C depends on $\boldsymbol{h}$ and on $\|\boldsymbol{u}\|_{H^1(\Omega_\delta)}$. This means that $[\boldsymbol{u}_t]_t$ is differentiable in $H^1_\sigma(\Omega_\delta)$ with respect to t at $t=0$ and equality (10) is valid.

We are ready to prove now that $\boldsymbol{u}_t$ is differentiable in $L^2(\Omega_\delta)$ with respect to t at $t=0$. According to Lemma 3,

$$[\boldsymbol{u}_t]_t - \boldsymbol{u}_t = t\,\boldsymbol{W}_t + \boldsymbol{S}_{1t},$$

where $\|\boldsymbol{S}_{1t}\|_{L^2(\Omega_\delta)} = o(t)$ as $t \to 0$ and

$$W_t^i(\boldsymbol{y}) = h^j(\boldsymbol{y})\,\frac{\partial u_t^i(\boldsymbol{y})}{\partial y^j} - u_t^j(\boldsymbol{y})\,\frac{\partial h^i(\boldsymbol{y})}{\partial y^j}.$$

Since $\boldsymbol{u} \in H^1(\Omega_\delta)$ is the unique solution of (7), we can easily conclude that $\boldsymbol{u}_t \to \boldsymbol{u}$ in $H^1(\Omega_\delta)$ as $t \to 0$. Therefore,

$$[\boldsymbol{u}_t]_t - \boldsymbol{u}_t = t\,\boldsymbol{W}_0 + \boldsymbol{S}_{2t},$$

where $\|\boldsymbol{S}_{2t}\|_{L^2(\Omega_\delta)} = o(t)$ as $t \to 0$ and

$$W_0^i(\boldsymbol{y}) = h^j(\boldsymbol{y})\,\frac{\partial u^i(\boldsymbol{y})}{\partial y^j} - u^j(\boldsymbol{y})\,\frac{\partial h^i(\boldsymbol{y})}{\partial y^j}.$$

Taking into account the last assertion of Lemma 3, we obtain that $\mathfrak{D}(\boldsymbol{W}_0) = 0$ in B and, as a consequence, that $\boldsymbol{W}_0 \in L^2_{\mathcal{R}}(\Omega_\delta, B)$.

It follows that

$$\boldsymbol{u}_t - \boldsymbol{u} = \boldsymbol{u}_t - [\boldsymbol{u}_t]_t + [\boldsymbol{u}_t]_t - \boldsymbol{u} = t\,(\boldsymbol{U}_0 - \boldsymbol{W}_0) + \boldsymbol{S}_{3t},$$

where $\|\boldsymbol{S}_{3t}\|_{L^2(\Omega_\delta)} = o(t)$ as $t \to 0$, $\boldsymbol{W}_0$ is defined above and $\boldsymbol{U}_0 \in \widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, 0)$ is the unique solution of (9). Thus, the assertion of Theorem 2 holds true with $\boldsymbol{U} = \boldsymbol{U}_0 - \boldsymbol{W}_0$ and $\boldsymbol{R}(\cdot, t) = \boldsymbol{S}_{3t}$.

The function $\boldsymbol{U}$ is the derivative in $L^2_\sigma(\Omega_\delta)$ of $\boldsymbol{u}_t$ with respect to t at $t = 0$. It depends on $\boldsymbol{h}$ which prescribe the direction of the differentiation. It may seem that $\boldsymbol{u}_t$ is differentiable also in $H^1_\sigma(\Omega_\delta)$. Really, the formal differentiation of (5) implies that $\boldsymbol{U}$ satisfies the following equation

$$H(\boldsymbol{U}, \boldsymbol{\varphi}) = (\varrho_s - \varrho_f)\,\boldsymbol{g} \cdot \int_B \frac{\partial \boldsymbol{\varphi}}{\partial x^i}\,h^i\,d\boldsymbol{x}$$

for an arbitrary function $\boldsymbol{\varphi} \in \widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, 0)$ and, as a consequence, that $\boldsymbol{U} \in \widehat{H}^1_{\mathcal{R}}(\Omega_\delta, B, 0)$. However, we cannot justify this formal differentiation.

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